The γ-filtration on the Witt ring of a scheme

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Abstract

The K-ring of symmetric vector bundles over a scheme, the so-called Grothendieck-Witt ring, can be endowed with the structure of a (special) λ-ring. The associated γ-filtration generalizes the fundamental filtration on the (Grothendieck-)Witt ring of a field and is closely related to the "classical" filtration by the kernels of the first two Stiefel-Whitney classes.

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Introduction

In this article, we establish a (special) λ-ring structure on the Grothendieck-Witt ring of a scheme, and some basic properties of the associated γ-filtration.

As far as Witt rings of fields are concerned, there is an unchallenged natural candidate for a good filtration: the "fundamental filtration", given by powers of the "fundamental ideal". Its claim to fame is that the associated graded ring is isomorphic to the mod-2 étale cohomology ring, as predicted by Milnor [Mil69] and verified by Voevodsky et al. [Voe03, OVV07]. For the Witt ring of a more general variety X, there is no candidate filtration of equal renown. The two most frequently encountered filtrations are:

- A short filtration which we will refer to as the **classical filtration** $F^*_{clas}\mathrm{W}(X)$, given by the whole ring, the kernel of the rank homomorphism, the kernels of the first two Stiefel-Whitney classes. This filtration is used, for example, in [FC87, Zib14].
- The **unramified filtration** $F^*_K\mathrm{W}(X)$, given by the preimage of the fundamental filtration on the Witt ring of the function field K of X under the natural homomorphism $\mathrm{W}(X) \to \mathrm{W}(K)$. Said morphism is not generally injective (e. g. [Tot03]), at least not when $\dim(X) > 3$, and its kernel will clearly be contained in every piece of the filtration. Recent computations with this filtration include [FH14].

Clearly, the unramified filtration coincides with the fundamental filtration in the case of a field, and so does the classical filtration as far as it is defined. The same will be true of the γ-filtration introduced here. It may be thought of as an attempt to extend the classical filtration to higher degrees.

In general, in order to define a "γ-filtration", we simply need to exhibit a pre-λ-structure on the ring in question. However, the natural candidates for λ-operations, the exterior powers, are not well-defined on the Witt ring $\mathrm{W}(X)$. We remedy this by passing to the Grothendieck-Witt ring $\mathrm{GW}(X)$. It is defined just like the Witt ring, except that we do not quotient out hyperbolic elements. Consequently, the two rings are related by an exact sequence

$$\mathrm{K}(X) \to \mathrm{GW}(X) \to \mathrm{W}(X) \to 0.$$

Given the (pre-)λ-structure on $\mathrm{GW}(X)$, we can formulate the following theorem concerning the associated γ-filtration. Let X be an integral scheme over a field k of characteristic not two.

Theorem 1.

(1) The γ-filtration on $\mathrm{GW}(k)$ is the fundamental filtration.

(2) The γ-filtration on $\mathrm{GW}(X)$ is related to the classical filtration as follows:

$$\begin{aligned}
F^1_\gamma\mathrm{GW}(X) &= F^1_{clas}\mathrm{GW}(X) = \ker\left(\mathrm{GW}(X) \xrightarrow{\text{rank}} \mathbb{Z}\right)\\
F^2_\gamma\mathrm{GW}(X) &= F^2_{clas}\mathrm{GW}(X) = \ker\left(F^1_\gamma\mathrm{GW}(X) \xrightarrow{w_1} H^1_{et}(X,\mathbb{Z}/2)\right)\\
F^3_\gamma\mathrm{GW}(X) &\subseteq F^3_{clas}\mathrm{GW}(X) = \ker\left(F^2_\gamma\mathrm{GW}(X) \xrightarrow{w_2} H^2_{et}(X,\mathbb{Z}/2)\right)
\end{aligned}$$

However, the inclusion at the third step is not in general an equality.

(3) The γ-filtration on $\mathrm{GW}(X)$ is finer than the unramified filtration.

We define the "γ-filtration" on the Witt ring as the image of the above filtration under the canonical projection $\mathrm{GW}(X) \to \mathrm{W}(X)$. Thus, each of the above statements easily implies an analogous statement for the Witt ring: the γ-filtration on the Witt ring of a field is the fundamental filtration, $F^i_\gamma\mathrm{W}(X)$

agrees with $F^i_{clas}\mathrm{W}(X)$ for $i < 3$ etc. The same example (Example 5.5) shows that $F^3_\gamma\mathrm{W}(X) \neq F^3_{clas}\mathrm{W}(X)$ in general.

Most statements of Theorem 1 also hold under weaker hypotheses—see *(1)* Proposition 4.1, *(2)* Propositions 4.4 and 4.8 and *(3)* Proposition 4.9. On the other hand, under some additional restrictions, the relation with the unramified filtration can be made more precise. For example, if X is a regular variety of dimension at most three and k is infinite, the unramified filtration on the Witt ring agrees with the global sections of the sheafified γ-filtration (Section 4.3).

The crucial assertion is of course the equality of $F^2_\gamma\mathrm{GW}(X)$ with the kernel of w_1—all other statements would hold similarly for the naive filtration of $\mathrm{GW}(X)$ by the powers of the "fundamental ideal" $F^1_\gamma\mathrm{GW}(X)$. The equality follows from the fact that the exterior powers make $\mathrm{GW}(X)$ not only a pre-λ-ring, but even a λ-ring:[1]

Theorem 2. *For any scheme X over a field of characteristic not two, the exterior power operations give* $\mathrm{GW}(X)$ *the structure of a λ-ring.*

In the case when X is a field, this was established in [McG02]. The underlying pre-λ-structure for affine X has also recently been established independently in [Xie14], where it is used to study sums-of-squares formulas.

Although in this article the λ-structure is used mainly as a tool in proving Theorem 1, it should be noted that λ-rings and even pre-λ-rings have strong structural properties. Much of the general structure of Witt rings of *fields*—for example, the fact they contain no p-torsion for odd p—could be (re)derived using the pre-λ-structure on the Grothendieck-Witt ring. Among the few results that generalize immediately to Grothendieck-Witt rings of schemes is the fact that torsion elements are nilpotent: this is true in any pre-λ-ring. For λ-rings, Clauwens has even found a sharp bound on the nilpotence degree [Cla10]. In our situation, Clauwens result reads:

Corollary. *Let X be as above. Suppose $x \in \mathrm{GW}(X)$ is an element satisfying $p^e x = 0$ for some prime p and some exponent e. Then $x^{p^e+p^{e-1}} = 0$.*

To put the corollary into context, recall that for a field k of characteristic not two, an element $x \in \mathrm{W}(k)$ is nilpotent if and only if it is 2^n-torsion for some n [Lam05, VIII.8]. This equivalence may be generalized at least to connected semi-local rings in which two is invertible, using the pre-λ-structure for one implication and [KRW72, Ex. 3.11] for the other. See [McG02] for further applications of the λ-ring structure on Grothendieck-Witt rings of fields and [Bal03] for nilpotence results for Witt rings of regular schemes.

From the λ-theoretic point of view, the main complication in the Grothendieck-Witt ring of a general scheme as opposed to that of a field is that not all generators can be written as sums of line elements. In K-theory, this difficulty can often be overcome by embedding $\mathrm{K}(X)$ into the K-ring of

[1] In older terminology, pre-λ-rings are called "λ-rings", while λ-rings are referred to as "special λ-rings". See also the introduction to [Zib13].

some auxiliary scheme in which a given generator does have this property, but in our situation this is impossible: there is no splitting principle for Grothendieck-Witt rings (Section 3).

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1 Generalities

1.1 λ-rings

We give a quick and informal introduction to λ-rings, treading medium ground between the traditional definition in terms of exterior power operations [SGA6, Exposé V] and the abstract definition of λ-rings as coalgebras over a comonad [Bor11, 1.17]. The main point we would like to get across is that a λ-ring is "a ring equipped with all possible symmetric operations", not just "a ring with exterior powers". This observation is not essential for anything that follows—in fact, we will later work exclusively with the traditional definition—but we hope that it may provide some intrinsic motivation for considering this kind of structure.

To make our statement more precise, let $\mathbb{W}$ be the ring of symmetric functions. That is, $\mathbb{W}$ consists of all formal power series $\phi(x_1, x_2, \dots)$ in countably many variables $x_1, x_2, \dots$ with coefficients in $\mathbb{Z}$ such that ϕ has bounded degree and such that the image $\phi(x_1, \dots, x_n, 0, 0, \dots)$ of ϕ under the projection to $\mathbb{Z}[x_1, \dots, x_n]$ is a symmetric polynomial for all n. For example, $\mathbb{W}$ contains ...

... the **elementary symmetric functions** $\lambda_k := \sum_{i_1 < \dots < v_k} x_{i_1} \cdot \ldots \cdot x_{i_k}$,
... the **complete symmetric functions** $\sigma_k := \sum_{i_1 \leq \dots \leq i_k} x_{i_1} \cdot \ldots \cdot x_{i_k}$,
... the **Adams symmetric functions** $\psi_k := \sum_i x_i^k$.

The first two families of symmetric functions each define a set of algebraically independent generators of $\mathbb{W}$ over $\mathbb{Z}$, so they can be used to identify $\mathbb{W}$ with a polynomial ring in countably many variables. Another, equivalent set of generators is given by the so-called Witt symmetric functions ([Bor13, 4.5]). The Adams symmetric functions are also algebraically independent, but they only generate $\mathbb{W} \otimes_{\mathbb{Z}} \mathbb{Q}$ over $\mathbb{Q}$. In any case, we have no need to choose any specific set of generators just now.

Given an arbitrary ring A, we write $\mathbb{W}A$ for the **universal λ-ring over A**/ the Big Witt Ring of A. As a set, it consists of all ring homomorphisms from $\mathbb{W}$ to A:

$$\mathbb{W}A = \mathcal{R}ings(\mathbb{W}, A).$$

In particular, for every symmetric function $\phi \in \mathbb{W}$, we have an evaluation map $\mathrm{ev}_\phi \colon \mathbb{W}A \to A$. The universal λ-ring $\mathbb{W}A$ becomes a ring via a coproduct Δ^+ and a comultiplication $\Delta^\times$ on $\mathbb{W}$. (These cooperations are determined by the equations $\Delta^+(\psi_n) = 1 \otimes \psi_n + \psi_n \otimes 1$ and $\Delta^\times(\psi_n) = \psi_n \otimes \psi_n$ for all n [Bor13, Introduction].)

1.1 Definition. A **pre-λ-ring** is a ring A together with a group homomorphism $\theta_A\colon A \to \mathbb{W}A$ such that

$$\begin{array}{ccc} A & \xrightarrow{\theta_A} & \mathbb{W}A \\ & \searrow\!\!\!\!= & \downarrow{\scriptstyle \mathrm{ev}_{\psi_1}} \\ & & A \end{array}$$

commutes. A morphism of pre-λ-rings $(A, \theta_A) \to (B, \theta_B)$ is a ring homomorphism $f\colon A \to B$ such that $\mathbb{W}(f)\theta_A = \theta_B f$. We refer to such a morphism as a **λ-morphism**.

It would, of course, appear more natural to ask for the map θ_A to be a *ring* homomorphism. But this condition is one of two additional requirements reserved for λ-rings. The second additional requirement takes into account that the universal λ-ring $\mathbb{W}A$ can itself be equipped with a canonical pre-λ-structure for any ring A. Thus, it makes sense to define:

1.2 Definition. A **λ-ring** is a pre-λ-ring (A, θ_A) such that θ_A is a λ-morphism.

It turns out that the canonical pre-λ-structure does make the universal λ-ring $\mathbb{W}A$ a λ-ring, so the terminology is sane. The observation alluded to at the beginning of this section is that any symmetric function $\phi \in \mathbb{W}$ defines an "operation" on any (pre-)λ-ring A, i. e. a map $A \to A$: namely, the composition of ev_ϕ with θ_A.

$$\begin{array}{ccc} A & \xrightarrow{\theta_A} & \mathbb{W}A \\ & \searrow & \downarrow{\scriptstyle \mathrm{ev}_{\phi}} \\ & & A \end{array}$$

In particular, we have families of operations λ^k, σ^k and ψ^k corresponding to the symmetric functions specified above. They are referred to as **exterior power operations**, **symmetric power operations** and **Adams operations**, respectively.

The underlying additive group of the universal λ-ring $\mathbb{W}A$ is isomorphic to the multiplicative group $(1 + A[\![t]\!])^\times$ inside the ring of invertible power series over A, and the isomorphism can be chosen such that the projection onto the coefficient of t^i corresponds to ev_{λ_i} (e.g. [Hes04, Prop. 1.14, Rem. 1.21(2)]). Thus, a pre-λ-structure is completely determined by the operations λ^i, and conversely, any family of operations λ^i for which the map

$$\begin{array}{rcl} A & \xrightarrow{\lambda_t} & (1 + A[\![t]\!])^\times \\ a & \mapsto & 1 + \lambda^1(a)t + \lambda^2(a)t^2 + \dots \end{array}$$

is a group homomorphism, and for which $\lambda^1(a) = a$, defines a λ-structure. This recovers the traditional definition of a pre-λ-structure as a family of operations λ^i (with $\lambda^0 = 1$ and $\lambda^1 = \mathrm{id}$) satisfying the relation $\lambda^k(x+y) = \sum_{i+j=k} \lambda^i(x)\lambda^j(y)$ for all $k \geq 0$ and all $x, y \in A$.

The question whether the resulting pre-λ-structure is a λ-structure can similarly be reduced to certain polynomial identities, though these are more difficult to state and often also more difficult to verify in practice. However, for pre-λ-rings with some additional structure, there are certain standard criteria that make life easier.

1.3 Definition. An **augmented (pre-)λ-ring** is a (pre-)λ-ring A together with a λ-morphism

$$d\colon A \to \mathbb{Z},$$

where the (pre-)λ-structure on $\mathbb{Z}$ is defined by $\lambda^i(n) := \binom{n}{i}$.

A **(pre-)λ-ring with positive structure** is an augmented (pre-)λ-ring A together with a specified subset $A_{>0} \subset A$ on which d is positive and which generates A in the strong sense that any element of A can be written as a difference of elements in $A_{>0}$; it is moreover required to satisfy a list of axioms for which we refer to [Zib13, §3].

For example, one of the axioms for a positive structure is that for an element $e \in A_{>0}$, the exterior powers $\lambda^k e$ vanish for all $k > d(e)$. We will refer to elements of $A_{>0}$ as **positive elements**, and to positive elements l of augmentation $d(l) = 1$ as **line elements**. The motivating example, the K-ring $\mathrm{K}(X)$ of a scheme X, is augmented by the rank homomorphism, and a set of positive elements is given by the classes of vector bundles. The situation for the Grothendieck-Witt ring will be analogous.

Here are two simple criteria for showing that a pre-λ-ring with positive structure is a λ-ring:

Splitting Criterion If all positive elements of A decompose into sums of line elements, then A is a λ-ring.

Detection Criterion If for any pair of positive elements $e_1, e_2 \in A_{>0}$ we can find a λ-ring A' and a λ-morphism $A' \to A$ with both e_1 and e_2 in its image, then A is a λ-ring.

We again refer to [Zib13] for details.

1.2 The γ-filtration

The **γ-operations** on a pre-λ-ring A can be defined as $\gamma^n(x) := \lambda^n(x+n-1)$. They again satisfy the identity $\gamma^k(x+y) = \sum_{i+j=k} \gamma^i(x)\gamma^j(y)$.

1.4 Definition. The γ-filtration on an augmented pre-λ-ring A is defined as follows:

$$F^0_\gamma A := A$$
$$F^1_\gamma A := \ker(A \xrightarrow{d} \mathbb{Z})$$
$$F^k_\gamma A := \begin{pmatrix}\text{subgroup generated by all finite products} \\ \prod_j \gamma^{i_j}(a_j) \text{ with } a_j \in F^1_\gamma A \text{ and } \sum_j i_j \geq k\end{pmatrix} \qquad \text{for } i > 1$$

This is in fact a filtration by ideals, multiplicative in the sense that $F^i_\gamma A \cdot F^j_\gamma A \subset F^{i+j}_\gamma A$, hence we have an associated graded ring

$$\operatorname{gr}^*_\gamma A := \bigoplus_i F^i_\gamma A / F^{i+1}_\gamma A.$$

See [AT69, §4] or [FL85, III §1] for details. The following lemma is sometimes useful for concrete computations.

1.5 Lemma. *If A is a pre-λ-ring with positive structure such that every positive element in A can be written as a sum of line elements, then $F^k_\gamma A = (F^1_\gamma A)^k$.*

More generally, suppose that A is an augmented pre-λ-ring, and let $E \subset A$ be some set of additive generators of $F^1_\gamma A$. Then $F^k_\gamma A$ is additively generated by finite products of the form $\prod_j \gamma^{i_j}(e_j)$ with $e_j \in E$ and $\sum_j i_j \geq k$.

Proof. The first assertion may be found in [FL85, III §1]. It also follows from the second, which we now prove. As each $x \in F^1_\gamma A$ can be written as a linear combination of elements of E, we can write any $\gamma^i(x)$ as a linear combination of products of the form $\prod_j \gamma^{i_j}(\pm e_j)$ with $e_j \in E$ and $\sum_j i_j = i$. Thus, $F^k_\gamma A$ can be generated by finite products of the form $\prod_j \gamma^{i_j}(\pm e_j)$, with $e_j \in E$ and $\sum_j i_j \geq k$. Moreover, $\gamma^i(-e)$ is a linear combination of products of the form $\prod_j \gamma^{i_j}(e)$ with $\sum_j i_j = i$: this follows from the above identity for $\gamma^k(x+y)$. Thus, $F^k_\gamma A$ is already generated by products of the form described. □

For λ-rings with positive structure, we also have the following general fact:

1.6 Lemma ([FL85, III, Thm 1.7])**.** *For any λ-ring A with positive structure, the additive group $\operatorname{gr}^1 A = F^1_\gamma A / F^2_\gamma A$ is isomorphic to the multiplicative group of line elements in A.*

2 The λ-structure on the Grothendieck-Witt ring

2.1 The pre-λ-structure

2.1 Proposition. *Let X be a scheme. The exterior power operations $\lambda^k\colon (\mathcal{M}, \mu) \mapsto (\Lambda^k \mathcal{M}, \Lambda^k \mu)$ induce well-defined maps on $\mathrm{GW}(X)$ which provide $\mathrm{GW}(X)$ with the structure of a pre-λ-ring.*

The proof of the existence of a pre-λ-structure is closely analogous to the proof for symmetric representation rings in [Zib13].

Step 1. The assignment $\lambda^i(\mathcal{M},\mu) := (\Lambda^i\mathcal{M},\Lambda^i\mu)$ is well-defined on the set of isometry classes of symmetric vector bundles over X, so that we have an induced map

$$\lambda_t\colon \left\{\begin{matrix}\text{isometry classes of symmetric}\\ \text{vector bundles over } X\end{matrix}\right\} \to (1+\mathrm{GW}(X)[\![t]\!])^\times.$$

Step 2. The map λ_t is additive in the sense that

$$\lambda_t((\mathcal{M},\mu)\perp(\mathcal{N},\nu)) = \lambda_t(\mathcal{M},\mu)\lambda_t(\mathcal{N},\nu).$$

Therefore, λ_t can be extended linearly to a group homomorphism

$$\lambda_t\colon \bigoplus \mathbb{Z}(\mathcal{M},\mu) \to (1+\mathrm{GW}(X)[\![t]\!])^\times,$$

where the sum on the left is over all isometry classes of symmetric vector bundles over X. By the additivity property, this extension factors through the quotient of $\bigoplus\mathbb{Z}(\mathcal{M},\mu)$ by the ideal generated by the relations

$$((\mathcal{M},\mu)\perp(\mathcal{N},\nu)) = (\mathcal{M},\mu)+(\mathcal{N},\nu).$$

Step 3. The homomorphism λ_t respects the relation $(\mathcal{M},\mu) = H(\mathcal{L})$ for every metabolic vector bundle $(\mathcal{M},\mu)$ with Lagrangian $\mathcal{L}$. Thus, we obtain the desired a factorization

$$\lambda_t\colon \mathrm{GW}(X) \to (1+\mathrm{GW}(X)[\![t]\!])^\times.$$

To carry out these steps, we only need to replace all arguments on the level of vector spaces in [Zib13] with local arguments. We formulate the key lemma in detail and then sketch the remaining part of the proof.

2.2 Filtration Lemma (c. f. [SGA6, Exposé V, Lemme 2.2.1])**.** *Let $0\to\mathcal{L}\to\mathcal{M}\to\mathcal{N}\to 0$ be an extension of vector bundles over a scheme X. Then we can find a filtration of $\Lambda^n\mathcal{M}$ by sub-vector bundles $\Lambda^n\mathcal{M} = \mathcal{M}^0\supset\mathcal{M}^1\supset\mathcal{M}^2\supset\cdots$ together with isomorphisms*

$$\mathcal{M}^i\big/\mathcal{M}^{i+1} \cong \Lambda^i\mathcal{L}\otimes\Lambda^{n-i}\mathcal{N}. \tag{1}$$

More precisely, there is a unique choice of such filtrations and isomorphisms subject to the following conditions:

(1) The filtration is natural with respect to isomorphisms of extensions. That is, given another extension $\widetilde{\mathcal{M}}$ of $\mathcal{L}$ by $\mathcal{N}$, any isomorphism $\phi\colon\mathcal{M}\to\widetilde{\mathcal{M}}$ for which

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{L} & \longrightarrow & \mathcal{M} & \longrightarrow & \mathcal{N} & \longrightarrow & 0\\ & & \| & & \cong\downarrow\phi & & \| & & \\ 0 & \longrightarrow & \mathcal{L} & \longrightarrow & \widetilde{\mathcal{M}} & \longrightarrow & \mathcal{N} & \longrightarrow & 0 \end{array}$$

commutes restricts to isomorphisms $\mathcal{M}^i \to \widetilde{\mathcal{M}}^i$ *compatible with* (1) *in the sense that*

$$\begin{array}{ccc} \mathcal{M}^i \big/ \mathcal{M}^{i+1} & \xrightarrow[\cong]{\bar{\phi}} & \widetilde{\mathcal{M}}^i \big/ \widetilde{\mathcal{M}}^{i+1} \\ & {\scriptstyle\cong}\searrow \quad \swarrow{\scriptstyle\cong} & \\ & \Lambda^i\mathcal{L} \otimes \Lambda^{n-i}\mathcal{N} & \end{array}$$

commutes.

(1') The filtration is natural with respect to morphisms of schemes $f\colon Y \to X$*:* $(f^*\mathcal{M})^i = f^*(\mathcal{M}^i)$ *under the identification of* $\Lambda^n(f^*\mathcal{M})$ *with* $f^*(\Lambda^n\mathcal{M})$*.*

(2) For the trivial extension, $(\mathcal{L} \oplus \mathcal{N})^i \subset \Lambda^n(\mathcal{L} \oplus \mathcal{N})$ *corresponds to the submodule*

$$\bigoplus_{j\geq i} \Lambda^j\mathcal{L} \otimes \Lambda^{n-j}\mathcal{N} \quad \subset \quad \bigoplus_j \Lambda^j\mathcal{L} \otimes \Lambda^{n-j}\mathcal{N}$$

under the canonical isomorphism $\Lambda^n(\mathcal{L}\oplus\mathcal{N}) \cong \bigoplus_j \Lambda^j\mathcal{L} \otimes \Lambda^{n-j}\mathcal{N}$*, and the isomorphisms*

$$(\mathcal{L} \oplus \mathcal{N})^i \big/ (\mathcal{L} \oplus \mathcal{N})^{i+1} \xrightarrow{\cong} \Lambda^i\mathcal{L} \otimes \Lambda^{n-i}\mathcal{N}$$

correspond to the canonical projections.

(The numbering *(1), (1'), (2)* is chosen to be as close as possible to the Filtration Lemma in [Zib13]. For a closer analogy, statement *(1)* there should also be split into two parts *(1)* and *(1')*: naturality with respect to isomorphisms of extensions, and naturality with respect to pullback along a group homomorphism. As stated there, only the pullback to the trivial group is covered.)

Proof of the Filtration Lemma 2.2. Uniqueness is clear: if filtrations and isomorphisms satisfying the above conditions exist, they are determined locally by *(2)* and hence on arbitrary extensions by *(1)* and *(1')*.

Existence may be proved via the following direct construction. Let $0 \to \mathcal{L} \xrightarrow{i} \mathcal{M} \xrightarrow{p} \mathcal{N} \to 0$ be an arbitrary short exact sequence of vector bundles over X. Consider the morphism $\Lambda^i\mathcal{L} \otimes \Lambda^{n-i}\mathcal{M} \to \Lambda^n\mathcal{M}$ induced by i. Let $\mathcal{M}_i$ be its kernel and $\mathcal{M}^i$ its image, so that we have a short exact sequences of coherent sheaves

$$0 \longrightarrow \mathcal{M}_i \longrightarrow \Lambda^i\mathcal{L} \otimes \Lambda^{n-i}\mathcal{M} \longrightarrow \mathcal{M}^i \longrightarrow 0$$

We claim *(a)* that the sheaves $\mathcal{M}_i$ and $\mathcal{M}^i$ are again vector bundles, *(b)* that the projection $\Lambda^i\mathcal{L} \otimes \Lambda^{n-i}\mathcal{M} \to \Lambda^i\mathcal{L} \otimes \Lambda^{n-i}\mathcal{N}$ factors through $\mathcal{M}^i$ and induces an isomorphism

$$\mathcal{M}^i \big/ \mathcal{M}^{i+1} \to \Lambda^i\mathcal{L} \otimes \Lambda^{n-i}\mathcal{N},$$

and that the vector bundles $\mathcal{M}^i$ together with these isomorphisms satisfy the properties *(1), (1'), (2)*. This can easily be checked in the following order: First, we check the first half of *(1)*, statement *(1')* and statement *(2)*. Then *(a)* and *(b)* follow because any extension of vector bundles is locally split. Lastly, the commutativity of the triangle in *(1)* can also be checked locally. □

Proof of Proposition 2.1. For Step 1, we note that the exterior power operation $\Lambda^i\colon \mathcal{V}ec(X) \to \mathcal{V}ec(X)$ is a duality functor in the sense that we have an isomorphism $\eta_{\mathcal{M}}\colon \Lambda^i(\mathcal{M}^\vee) \xrightarrow{\cong} (\Lambda^i\mathcal{M})^\vee$ for any vector bundle $\mathcal{M}$. Indeed, we can define $\eta_{\mathcal{M}}$ on sections by

$$\phi_1 \wedge \cdots \wedge \phi_i \mapsto (m_1 \wedge \cdots \wedge m_i \mapsto \det(\phi_\alpha(m_\beta)).$$

The fact that this is an isomorphism can be checked locally and therefore follows from [Bou70, Ch. 3, § 11.5, (30 bis)]. We therefore obtain a well-defined operation on the set of isometry classes of symmetric vector bundles over X by defining

$$\lambda^i(\mathcal{M}, \mu) := (\Lambda^i\mathcal{M}, \eta_{\mathcal{M}} \circ \Lambda^i(\mu)).$$

Step 2 is completely analogous to the argument in [Zib13].

For Step 3, let $(\mathcal{M}, \mu)$ be metabolic with Lagrangian $\mathcal{L}$, so that we have a short exact sequence

$$0 \to \mathcal{L} \xrightarrow{i} \mathcal{M} \xrightarrow{i^\vee\mu} \mathcal{L}^\vee \to 0. \tag{2}$$

When n is odd, say $n = 2k - 1$, we claim that $\mathcal{M}^k$ is a Lagrangian of $\Lambda^n(\mathcal{M}, \mu)$. When n is even, say $n = 2k$, we claim that $\mathcal{M}^{k+1}$ is an admissible sub-Lagrangian of $\Lambda^n(\mathcal{M}, \mu)$ with $(\mathcal{M}^{k+1})^\perp = \mathcal{M}^k$, and that the composition of isomorphisms

$$\mathcal{M}^{k+1}\big/\mathcal{M}^k \cong \Lambda^k\mathcal{L} \otimes \Lambda^k(\mathcal{L}^\vee) \cong H(\mathcal{L})^{k+1}\big/H(\mathcal{L})^k$$

of the Filtration Lemma 2.2 is even an isometry between $(\mathcal{M}^{k+1}/\mathcal{M}^k, \mu)$ and $(H(\mathcal{L})^{k+1}/H(\mathcal{L})^k, \overline{(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix})})$. All of these claims can be checked locally. Both the local arguments and the conclusions are analogous to those of [Zib13]. □

2.3 Remark. In many cases, Step 3 can be simplified. If X is affine, then Step 3 is redundant because any short exact sequence of vector bundles splits. If X is a regular quasiprojective variety over a field, then we can use Jouanolou's trick and homotopy invariance: Jouanolou's trick yields an affine vector bundle torsor $\pi\colon E \to X$ [Wei89, Prop. 4.3]. By what we have just said, the operations λ^i are well-defined on $\mathrm{GW}(E)$. Moreover, as X is regular, π^* is an isomorphism—an isomorphism on $\mathrm{K}(-)$ by [Wei89, below Ex. 4.7], an isomorphism on $\mathrm{W}(-)$ by [Gil03, Cor. 4.2], and hence an isomorphism on $\mathrm{GW}(-)$ by Karoubi induction. So we may deduce that λ^i is also well-defined on $\mathrm{GW}(X)$.

2.2 The pre-λ-structure is a λ-structure

We would now like to show that the pre-λ-structure on $\mathrm{GW}(X)$ discussed above is an actual λ-structure. In some cases, this is easy:

2.4 Proposition. *For any connected semi-local ring R in which two is invertible, the Grothendieck-Witt ring $\mathrm{GW}(R)$ is a λ-ring.*

Proof. Over such a ring, any symmetric space decomposes into a sum of line elements [Bae78, Prop. I.3.4/3.5], so the result follows from the Splitting Criterion. □

The following result is more interesting:

2.5 Theorem. *For any connected scheme X over a field of characteristic not two, the canonical pre-λ-structure on $\mathrm{GW}(X)$ discussed above is a λ-structure.*

Let us recall one of the proofs of the corresponding fact for $\mathrm{K}(X)$ in [SGA6, Exposé VI] (see Theorem 3.3.). Let G be a linear algebraic group scheme over k. The principal ingredients of the proof are:

(K1) The representation ring $\mathrm{K}(\mathcal{R}ep(G))$ is a λ-ring [Ser68].

(K2) For any G-torsor $\mathcal{S}$ over X, the map

$$\mathrm{K}(\mathcal{R}ep(G)) \to \mathrm{K}(X)$$

sending a representation V of G to the vector bundle $\mathcal{S} \times_G V$ is a λ-morphism. Here, the second V is to be interpreted as a trivial vector bundle over X with G acting as on V.

(K3) Any pair of vector bundles over X lies in the image of a common morphism $\mathrm{K}(\mathcal{R}ep(G)) \to \mathrm{K}(X)$ defined by some G-torsor as above. (G can be chosen to be a product of general linear groups.)

The claim that $\mathrm{K}(X)$ is a λ-ring follows from these three points via the detection criterion (Section 1.1). The same argument will clearly work for $\mathrm{GW}(X)$ provided the following three analogous statements hold. (We now assume that $\operatorname{char} k \neq 2$.)

(GW1) The symmetric representation ring $\mathrm{GW}(\mathcal{R}ep(G))$ is a λ-ring.

(GW2) For any G-torsor $\mathcal{S}$ over X, the map

$$\mathrm{GW}(\mathcal{R}ep(G)) \to \mathrm{GW}(X)$$

sending a symmetric representation V of G to the symmetric vector bundle $\mathcal{S} \times_G V$ is a λ-morphism.

(GW3) Any pair of symmetric vector bundles lies in the image of some common morphism $\mathrm{GW}(\mathcal{R}ep(G)) \to \mathrm{GW}(X)$ defined by a G-torsor as above. (G can be chosen to be a product of split orthogonal groups.)

Statement (GW1) is the main result of [Zib13]. The remaining points (GW2) and (GW3) are discussed below: see Corollary 2.13 and Lemma 2.15. Any reader to whom (K2) and (K3) are obvious will most likely consider (GW2) and (GW3) equally obvious—the only point of note is that for (GW3) we have to work in the étale topology rather than in the Zariski topology.

Twisting by torsors

Let X be a scheme with structure sheaf $\mathcal{O}$. We fix some Grothendieck topology on X. (For (K2) and (GW2), the topology is irrelevant. For (K3) we can take any topology at least as fine as the Zariski topology, while for (GW3) we will need the étale topology.)

Given a sheaf of (not necessarily abelian) groups $\mathcal{G}$ over X, recall that a (right) **$\mathcal{G}$-torsor** is a sheaf of sets $\mathcal{S}$ over X with a right $\mathcal{G}$-action such that:

(1) There exists a cover $\{U_i \to X\}_i$ such that $\mathcal{S}(U_i) \neq \emptyset$ for all U_i.
(2) For all open $(U \to X)$, and for one (hence for all) $s \in \mathcal{S}(U)$, the map

$$\begin{aligned} \mathcal{G}_{|U} &\to \mathcal{S}_{|U} \\ g &\mapsto s.g \end{aligned}$$

is an isomorphism.

2.6 Definition. Let $\mathcal{S}$ be a $\mathcal{G}$-torsor as above. For any presheaf $\mathcal{E}$ of $\mathcal{O}$-modules with an $\mathcal{O}$-linear left $\mathcal{G}$-action, we define a new presheaf of $\mathcal{O}$-modules by

$$\begin{aligned} \mathcal{S} \hat{\times}_{\mathcal{G}} \mathcal{E} &:= \operatorname{coeq}\big((\mathcal{S} \times \mathcal{G}) \otimes \mathcal{E} \rightrightarrows \mathcal{S} \otimes \mathcal{E}\big) \\ &= \operatorname{coker}\big(\textstyle\bigoplus_{\mathcal{S},\mathcal{G}} \mathcal{E} \longrightarrow \bigoplus_{\mathcal{S}} \mathcal{E}\big), \end{aligned}$$

where the morphism in the second line has the form ${}_{s,g}v \mapsto {}_{s.g}v - {}_{s}(g.v)$.[2] If $\mathcal{E}$ is a sheaf, we define $\mathcal{S} \times_{\mathcal{G}} \mathcal{E}$ as the sheafification of $\mathcal{S} \hat{\times}_{\mathcal{G}} \mathcal{E}$.

2.7 Remark. The $\mathcal{O}$-module $\mathcal{S} \times_{\mathcal{G}} \mathcal{E}$ can alternatively be described as follows: Fix a cover $\{U_i \to X\}_i$ which splits $\mathcal{S}$, and fix an element $s_i \in S(U_i)$ for each U_i. Let $g_{ij} \in \mathcal{G}(U_i \times_X U_j)$ be the unique element satisfying $s_j = s_i.g_{ij}$. Then $\mathcal{S} \times_{\mathcal{G}} \mathcal{E}$ is isomorphic to the sheaf given on any open $(V \to X)$ by

$$\{(v_i)_i \in \textstyle\prod_i \mathcal{E}(V_i) \mid v_i = g_{ij}.v_j \text{ on } V_{ij}\},$$

where $V_i := V \times_X U_i$, $V_{ij} := V_i \times_X V_j$.

We next recall the basic properties of $\mathcal{S} \times_{\mathcal{G}} \mathcal{E}$. We call two presheaves of $\mathcal{O}$-modules **locally isomorphic** if X has a cover such that the restrictions to each open of the cover are isomorphic. A morphism of presheaves of $\mathcal{O}$-modules is said to be **locally an isomorphism** if X has a cover such that the restriction of the morphism to each open of the cover is an isomorphism.

[2] We frequently use a subscript on the left to indicate in which summand an element lies. A general element in $\bigoplus_{\mathcal{S},\mathcal{G}} \mathcal{E}(V)$ would have the form $\sum_{s,g} {}_{s,g}(v_{s,g})$ with only finitely many non-zero summands in this notation.

2.8 Lemma.

(i) For any presheaf $\mathcal{E}$ as above, $\mathcal{S} \hat{\times}_{\mathcal{G}} \mathcal{E}$ is locally isomorphic to $\mathcal{E}$.

(ii) For any sheaf $\mathcal{E}$ as above, $\mathcal{S} \times_{\mathcal{G}} \mathcal{E}$ is locally isomorphic to $\mathcal{E}$.

(iii) The canonical morphism $\mathcal{S} \hat{\times}_{\mathcal{G}} \mathcal{E} \to \mathcal{S} \times_{\mathcal{G}} \mathcal{E}$ is locally an isomorphism for any sheaf as above.

More precisely, the presheaves in (i) & (ii) are isomorphic over any open $(U \to X)$ such that $\mathcal{S}(U) \neq \emptyset$, and likewise the morphism in (iii) is an isomorphism over any such U.

Proof. For *(i)* of the lemma, let $(U \to X)$ be an open such that $\mathcal{S}(U) \neq \emptyset$. Fix any $s \in \mathcal{S}(U)$. For each $(V \to U)$ and each $t \in \mathcal{S}(V)$, there exists a unique element $g_t \in \mathcal{G}(V)$ such that $t = s.g_t$. Therefore, the morphism $\bigoplus_S \mathcal{E}_{|U} \to \mathcal{E}_{|U}$ sending ${}_t v$ to $g_t.v$ describes the cokernel defining $\mathcal{S} \hat{\times}_{\mathcal{G}} \mathcal{E}$ over U. Statements *(ii) & (iii)* of the lemma follow from *(i)*. □

2.9 Lemma.

(i) The functor $\mathcal{S} \times_{\mathcal{G}} -$ is exact, i. e. it takes exact sequences of $\mathcal{O}$-modules with $\mathcal{O}$-linear $\mathcal{G}$-action to exact sequences of $\mathcal{O}$-modules.

(ii) If $\mathcal{E}$ is a sheaf of $\mathcal{O}$-modules with trivial *$\mathcal{G}$-action, then $\mathcal{S} \times_{\mathcal{G}} \mathcal{E} \cong \mathcal{E}$.*

(iii) Given arbitrary sheaves of $\mathcal{O}$-modules $\mathcal{E}$ and $\mathcal{F}$ with $\mathcal{O}$-linear $\mathcal{G}$-actions, consider $\mathcal{E} \oplus \mathcal{F}$, $\mathcal{E} \otimes \mathcal{F}$, $\Lambda^i \mathcal{E}$ and $\mathcal{E}^\vee$ with the induced $\mathcal{G}$-actions. Then we have the following isomorphisms of $\mathcal{O}$-modules, natural in $\mathcal{E}$ and $\mathcal{F}$:

$$\begin{aligned}
\sigma &: \mathcal{S} \times_{\mathcal{G}} (\mathcal{E} \oplus \mathcal{F}) \cong (\mathcal{S} \times_{\mathcal{G}} \mathcal{E}) \oplus (\mathcal{S} \times_{\mathcal{G}} \mathcal{F}) \\
\theta &: \mathcal{S} \times_{\mathcal{G}} (\mathcal{E} \otimes \mathcal{F}) \cong (\mathcal{S} \times_{\mathcal{G}} \mathcal{E}) \otimes (\mathcal{S} \times_{\mathcal{G}} \mathcal{F}) \\
\lambda &: \quad \mathcal{S} \times_{\mathcal{G}} (\Lambda^k \mathcal{E}) \cong \Lambda^k (\mathcal{S} \times_{\mathcal{G}} \mathcal{E}) \\
\eta &: \quad\quad \mathcal{S} \times_{\mathcal{G}} (\mathcal{E}^\vee) \cong (\mathcal{S} \times_{\mathcal{G}} \mathcal{E})^\vee
\end{aligned}$$

Proof. (i) If we fix s and U as in the proof of Lemma 2.8, then the induced isomorphism $\mathcal{S} \times_{\mathcal{G}} \mathcal{E}_{|U} \to \mathcal{E}_{|U}$ is functorial for morphisms of $\mathcal{O}$-modules with $\mathcal{O}$-linear $\mathcal{G}$-action. The claim follows as exactness of a sequence of sheaves can be checked locally.

(ii) When $\mathcal{G}$ acts trivially on $\mathcal{E}$, the local isomorphisms of Lemma 2.8 do not depend on choices and glue to a global isomorphism.

(iii) It is immediate from Lemma 2.8 that in each case the two sides are locally isomorphic, but we still need to construct global morphisms between them.

For $\oplus$ everything is clear.

For $\otimes$ and Λ^k, we first note that all constructions involved are compatible with sheafification, in the following sense: let $\hat{\otimes}$ and $\hat{\Lambda}$ denote the presheaf tensor product and the presheaf exterior power. Then, for arbitrary presheaves $\mathcal{E}$ and $\mathcal{F}$, the canonical morphisms

$$\begin{gathered}
(\mathcal{E} \mathbin{\hat{\otimes}} \mathcal{F})^+ \to (\mathcal{E}^+) \otimes (\mathcal{F}^+) \\
(\hat{\Lambda}^k \mathcal{E})^+ \to \Lambda^k (\mathcal{E}^+) \\
(\mathcal{S} \hat{\times}_{\mathcal{G}} \mathcal{E})^+ \to \mathcal{S} \times_{\mathcal{G}} (\mathcal{E}^+)
\end{gathered}$$

are isomorphisms. (In the third case, this follows from Lemma 2.8.)

As the arguments for $\otimes$ and Λ^k are very similar, so we only discuss the latter functor. We first check that the morphism

$$\bigoplus_{\mathcal{S}}(\hat{\Lambda}^k\mathcal{E}) \to \hat{\Lambda}^k(\bigoplus_{\mathcal{S}}\mathcal{E})$$

which identifies the summand ${}_s(\hat{\Lambda}^k\mathcal{E})$ on the left with $\hat{\Lambda}^k({}_s\mathcal{E})$ on the right induces a well-defined morphism

$$\mathcal{S}\mathbin{\hat{\times}_{\mathcal{G}}}(\hat{\Lambda}^k\mathcal{E}) \xrightarrow{\hat{\lambda}} \hat{\Lambda}^k(\mathcal{S}\mathbin{\hat{\times}_{\mathcal{G}}}\mathcal{E}).$$

Secondly, we claim that $\hat{\lambda}$ is locally an isomorphism. For this, we only need to observe that over any U such that $\mathcal{S}(U) \neq \emptyset$, we have a commutative triangle

$$\begin{array}{ccc}
\left(\mathcal{S}\mathbin{\hat{\times}_{\mathcal{G}}}(\hat{\Lambda}^k\mathcal{E})\right)(U) & \xrightarrow{\qquad\hat{\lambda}\qquad} & \left(\hat{\Lambda}^k(\mathcal{S}\mathbin{\hat{\times}_{\mathcal{G}}}\mathcal{E})\right)(U) \\
& {\scriptstyle\cong}\searrow \quad \swarrow{\scriptstyle\cong} & \\
& (\hat{\Lambda}^k\mathcal{E})(U) &
\end{array}$$

where the diagonal arrows are induced by the isomorphisms of Lemma 2.8.

For dualization, one of the sheafification morphisms goes in the wrong direction, so the argument is slightly different. Again, we first construct a morphism of presheaves $\hat{\eta}\colon \mathcal{S}\mathbin{\hat{\times}_{\mathcal{G}}}\mathcal{E}^\vee \to (\mathcal{S}\mathbin{\hat{\times}_{\mathcal{G}}}\mathcal{E})^\vee$. Over opens U such that $\mathcal{S}(U) = \emptyset$, the left-hand side is zero, so we take the zero morphism. Over opens U with $\mathcal{S}(U) \neq \emptyset$, we define

$$\begin{aligned}
(\mathcal{S}\mathbin{\hat{\times}_{\mathcal{G}}}\mathcal{E}^\vee)(U) &\xrightarrow{\hat{\eta}} (\mathcal{S}\mathbin{\hat{\times}_{\mathcal{G}}}\mathcal{E})^\vee(U) \\
{}_s\phi \quad &\mapsto ({}_{s.g}(v) \mapsto \phi(g.v))
\end{aligned}$$

Over these U with $\mathcal{S}(U) \neq \emptyset$, the morphism is in fact an isomorphism. To define a local inverse, pick an arbitrary $s \in \mathcal{S}(U)$, and send ψ on the right-hand side to ${}_s(v \mapsto \psi({}_sv))$ on the left.

Finally, given the morphism $\hat{\eta}$, we consider the following square in which α and β are sheafification morphisms:

$$\begin{array}{ccc}
\mathcal{S}\mathbin{\hat{\times}_{\mathcal{G}}}\mathcal{E}^\vee & \xrightarrow{\hat{\eta}} & (\mathcal{S}\mathbin{\hat{\times}_{\mathcal{G}}}\mathcal{E})^\vee \\
{\scriptstyle\alpha}\downarrow & \searrow & \uparrow{\scriptstyle\beta^\vee} \\
\mathcal{S}\times_{\mathcal{G}}\mathcal{E}^\vee & \dashrightarrow_{\eta} & (\mathcal{S}\times_{\mathcal{G}}\mathcal{E})^\vee
\end{array}$$

By Lemma 2.8, both α and β are locally isomorphisms, and it follows that $\beta^\vee$ is likewise locally an isomorphism. The diagonal morphism is defined as follows: over any U with $\mathcal{S}(U) = \emptyset$, it is the zero morphism, and over all other U, it is the composition of $\hat{\eta}$ with the (local) inverse of $\beta^\vee$. Thus, the dotted diagonal is a factorization of $\hat{\eta}$ over $(\mathcal{S}\times_{\mathcal{G}}\mathcal{E})^\vee$. The latter being a sheaf, this factorization must further factor through $\mathcal{S}\times_G\mathcal{E}^\vee$. We thus obtain the horizontal morphism of sheaves η which, being a locally an isomorphism, must be an isomorphism. $\square$

Twisting symmetric bundles

Recall that a duality functor is a functor between categories with dualities

$$F\colon (\mathcal{A}, \vee, \omega) \to (\mathcal{B}, \vee, \omega)$$

together with a natural isomorphism $\eta\colon F(-^{\vee}) \xrightarrow{\cong} F(-)^{\vee}$ such that

$$\begin{array}{ccccc} & & FA & & \\ & \swarrow^{F\omega_A} & & \searrow^{\omega_{FA}} & \\ F(A^{\vee\vee}) & \xrightarrow[\eta_{A^{\vee}}]{} & F(A^{\vee})^{\vee} & \xleftarrow[\eta_A^{\vee}]{} & (FA)^{\vee\vee} \end{array}$$

commutes [Bal05, Def. 1.1.15]. Such a functor induces a functor F_{sym} on the category of symmetric spaces over $\mathcal{A}$: it sends a symmetric space (A, α) over $\mathcal{A}$ to the symmetric space $(FA, \eta_A F\alpha)$ over $\mathcal{B}$. Moreover, we have isometries $H(FA) \cong F_{\mathrm{sym}}(HA)$, where H denotes the hyperbolic functor. It is customary to simply write F for F_{sym}.

Now consider the functor $\mathcal{S} \times_{\mathcal{G}} -$. By Lemma 2.8, we can restricted $\mathcal{S} \times_{\mathcal{G}} -$ to a functor from $\mathcal{G}$-equivariant vector bundles to vector bundles over X. Moreover, by Lemma 2.9, we have a natural isomorphism $\eta\colon \mathcal{S} \times_{\mathcal{G}} (-^{\vee}) \cong (\mathcal{S} \times_{\mathcal{G}} (-))^{\vee}$. The commutativity of the triangle in the definition of a duality functor is easily checked, so we deduce:

2.10 Lemma. *$(\mathcal{S} \times_{\mathcal{G}} -, \eta)$ is a duality functor $\mathcal{G}\mathcal{V}ec \to \mathcal{V}ec$.*

For any symmetric vector bundle $(\mathcal{E}, \varepsilon)$ with an $\mathcal{O}$-linear left $\mathcal{G}$-action, we can now define

$$\mathcal{S} \times_{\mathcal{G}} (\mathcal{E}, \varepsilon) := (\mathcal{S} \times_{\mathcal{G}} -)_{\mathrm{sym}}(\mathcal{E}, \varepsilon).$$

No compatibility of the $\mathcal{G}$-action with the symmetric structure is required for this definition, but we will insist in the following that $\mathcal{G}$ acts via isometries:

2.11 Lemma. *If $\mathcal{G}$ acts on $\mathcal{E}$ via isometries, then $\mathcal{S} \times_{\mathcal{G}} (\mathcal{E}, \varepsilon)$ is locally* isometric *to $(\mathcal{E}, \varepsilon)$.*

More precisely, the local isomorphisms of Lemma 2.8 are isometries in this case. Moreover, natural isomorphisms σ, θ and λ of Lemma 2.9 respect the symmetric structures:

2.12 Lemma. *For symmetric vector bundles $(\mathcal{E}, \varepsilon)$ and $(\mathcal{F}, \phi)$ on which $\mathcal{G}$ acts through isometries, σ, θ and λ respect the induced symmetries.*

Proof. Temporarily writing F for the functor $\mathcal{S} \times_{\mathcal{G}} -$, checking the claim for θ amounts to the following: For symmetric bundles $(\mathcal{E}, \varepsilon)$ and $(\mathcal{F}, \phi)$, the isomorphism $\theta_{\mathcal{E},\mathcal{F}}$ is an isometry from $F(\mathcal{E} \otimes \mathcal{F})$ to $F\mathcal{E} \otimes F\mathcal{F}$ with respect to the induced symmetries if and only if the outer square of the following diagram commutes:

$$\begin{array}{ccc}
F(\mathcal{E}\otimes\mathcal{F}) & \xrightarrow{\theta_{\mathcal{E},\mathcal{F}}} & F\mathcal{E}\otimes F\mathcal{F} \\
\downarrow{\scriptstyle F(\varepsilon\otimes\phi)} & & \downarrow{\scriptstyle F\varepsilon\otimes F\phi} \\
F(\mathcal{E}^\vee\otimes\mathcal{F}^\vee) & \xrightarrow{\theta_{\mathcal{E}^\vee,\mathcal{F}^\vee}} & F(\mathcal{E}^\vee)\otimes F(\mathcal{F}^\vee) \\
\downarrow & & \downarrow{\scriptstyle \eta_{\mathcal{E}}\otimes\eta_{\mathcal{F}}} \\
F((\mathcal{E}\otimes\mathcal{F})^\vee) & & (F\mathcal{E})^\vee\otimes(F\mathcal{F})^\vee \\
\downarrow{\scriptstyle \eta_{\mathcal{E}\otimes\mathcal{F}}} & & \downarrow \\
(F(\mathcal{E}\otimes\mathcal{F}))^\vee & \xleftarrow[\theta^\vee_{\mathcal{E},\mathcal{F}}]{} & (F\mathcal{E}\otimes F\mathcal{F})^\vee
\end{array}$$

Similarly, for a symmetric vector bundle $(\mathcal{E},\varepsilon)$, the isomorphism $\lambda_{\mathcal{E}}$ is an isometry from $F(\Lambda^k\mathcal{E})$ to $\Lambda^k(F\mathcal{E})$ if and only if the outer square of the following diagram commutes:

$$\begin{array}{ccc}
F\Lambda^k\mathcal{E} & \xrightarrow{\lambda_{\mathcal{E}}} & \Lambda^k F\mathcal{E} \\
\downarrow{\scriptstyle F\Lambda^k\varepsilon} & & \downarrow{\scriptstyle \Lambda^k F\varepsilon} \\
F\Lambda^k(\mathcal{E}^\vee) & \xrightarrow{\lambda_{\mathcal{E}^\vee}} & \Lambda^k F(\mathcal{E}^\vee) \\
\downarrow & & \downarrow{\scriptstyle \Lambda^k\eta_{\mathcal{E}}} \\
F((\Lambda^k\mathcal{E})^\vee) & & \Lambda^k((F\mathcal{E})^\vee) \\
\downarrow{\scriptstyle \eta_{\Lambda^k\mathcal{E}}} & & \downarrow \\
(F\Lambda^k\mathcal{E})^\vee & \xleftarrow[\lambda^\vee_{\mathcal{E}}]{} & (\Lambda^k F\mathcal{E})^\vee
\end{array}$$

In both cases, we already know that the the upper square commutes for all $\mathcal{E}$ and $\mathcal{F}$, by naturality of θ and λ. So it suffices to verify that the lower square commutes. This can be checked locally, and follows easily from the descriptions of η, θ and λ given in the proof of Lemma 2.8. □

Proofs of the statements

Statements (K2) and (GW2) are special cases of the following immediate corollary of Lemmas 2.9 and 2.12.

2.13 Corollary. *Let $\pi\colon X\to \operatorname{Spec}(k)$ be a scheme over some field k. For any algebraic group scheme G over k, and for any G-torsor $\mathcal{S}$, the maps*

$$\begin{array}{rlrl}
\mathrm{K}(\mathcal{R}ep_k G) & \to \mathrm{K}(X) & \mathrm{GW}(\mathcal{R}ep_k G) & \to \mathrm{GW}(X) \\
V & \mapsto \mathcal{S}\times_G \pi^* V & (V,\nu) & \mapsto \mathcal{S}\times_G \pi^*(V,\nu)
\end{array}$$

are λ-morphisms.

It remains to prove (K3) and (GW3), which we now state in a more detailed form.

2.14 Lemma. *Let V_n be the standard representation of GL_n.*

(a) Any vector bundle $\mathcal{E}$ is isomorphic to $\mathcal{S} \times_{\mathrm{GL}_n} \pi^* V_n$ for some Zariski GL_n-torsor $\mathcal{S}$.

(b) For any two vector bundles $\mathcal{E}$ and $\mathcal{F}$, there exists a Zariski $\mathrm{GL}_n \times \mathrm{GL}_m$-torsor $\mathcal{S}$ such that

$$\mathcal{E} \cong \mathcal{S} \times_{\mathrm{GL}_n \times \mathrm{GL}_m} V_n$$
$$\mathcal{F} \cong \mathcal{S} \times_{\mathrm{GL}_n \times \mathrm{GL}_m} V_m$$

Here GL_m is supposed to act trivially on V_n, and GL_n is supposed to act trivially on V_m.

2.15 Lemma. *Let X be a scheme over a field k of characteristic not two. Let (V_n, q_n) be the standard representation of O_n over k.*

(a) Any symmetric vector bundle $(\mathcal{E}, \varepsilon)$ is isomorphic to $\mathcal{S} \times_{\mathrm{O}_n} \pi^*(V_n, q_n)$ for some étale O_n-torsor $\mathcal{S}$.

(b) For any two symmetric vector bundles $(\mathcal{E}, \varepsilon)$ and $(\mathcal{F}, \phi)$, there exists an étale $\mathrm{O}_n \times \mathrm{O}_m$-torsor $\mathcal{S}$ such that

$$(\mathcal{E}, \varepsilon) \cong \mathcal{S} \times_{\mathrm{O}_n \times \mathrm{O}_m} \pi^*(V_n, q_n)$$
$$(\mathcal{F}, \phi) \cong \mathcal{S} \times_{\mathrm{O}_n \times \mathrm{O}_m} \pi^*(V_m, q_m)$$

Here, O_m is supposed to act trivially on V_n, and O_n is supposed to act trivially on V_m.

Proof of Lemma 2.14.

(a) Let $\mathcal{S}$ be the sheaf of isomorphisms $\mathcal{S} := \mathcal{I}so(\mathcal{O}^{\oplus n}, \mathcal{E})$ with $\mathrm{GL}_n = \mathcal{A}ut(\mathcal{O}^{\oplus n})$ acting by precomposition. This is a GL_n-torsor as $\mathcal{E}$ is locally isomorphic to $\mathcal{O}^{\oplus n}$. Moreover, we have a well-defined morphism

$$\begin{aligned} \mathrm{ev}\colon \mathcal{S} \hat{\times}_{\mathcal{G}} \mathcal{O}^{\oplus n} &\to \mathcal{E} \\ {}_f v &\mapsto f(v) \end{aligned}$$

which is locally an isomorphism: for any $s \in \mathcal{I}so(\mathcal{O}^{\oplus n}, \mathcal{E})(V)$, the restriction $\mathrm{ev}_{|V}$ factors as

$$(\mathcal{S} \hat{\times}_{\mathcal{G}} \mathcal{O}^{\oplus n})_{|V} \xrightarrow{\cong} \mathcal{O}^{\oplus n}_{|V} \xrightarrow[s]{\cong} \mathcal{E}_{|V},$$

where the first arrow is the isomorphism ${}_f v \mapsto g_f(v)$ of Lemma 2.8 determined by s.

(b) Suppose $\mathcal{S}$ is a $\mathcal{G}$-torsor, $\mathcal{S}'$ is a $\mathcal{G}'$-torsor, and $\mathcal{E}$ is a sheaf of $\mathcal{O}$-modules with $\mathcal{O}$-linear actions by both $\mathcal{G}$ and $\mathcal{G}'$. Then if $\mathcal{G}'$ acts trivially,

$$(\mathcal{S} \times \mathcal{S}') \times_{\mathcal{G} \times \mathcal{G}'} \mathcal{E} \cong \mathcal{S} \times_{\mathcal{G}} \mathcal{E}.$$

We can therefore simply take $\mathcal{S} := \mathcal{I}so(\mathcal{O}^{\oplus n}, \mathcal{E}) \times \mathcal{I}so(\mathcal{O}^{\oplus m}, \mathcal{F})$. □

Proof of Lemma 2.15. We have $\mathrm{O}_n = \mathcal{A}ut(\mathcal{O}^{\oplus n}, q_n)$. So let $\mathcal{S}$ be the sheaf of isometries $\mathcal{S} := \mathcal{I}so((\mathcal{O}^{\oplus n}, q_n), (\mathcal{E}, \varepsilon))$ with O_n acting by precomposition. This is an O_n-torsor in the étale topology since any symmetric vector bundle $(\mathcal{E}, \varepsilon)$ is étale locally isometric to $(\mathcal{O}^{\oplus n}, q_n)$ (e. g. [Hor05, 3.6]). The rest of the proof works exactly as in the non-symmetric case. □

3 (No) splitting principle

The splitting principle in K-theory asserts that any vector bundle behaves like a sum of line bundles. There are two incarnations:

The algebraic splitting principle: For any positive element e of a λ-ring with positive structure A, there exists an extension of λ-rings with positive structure $A \hookrightarrow A_e$ such that e splits as a sum of line elements in A_e.

The geometric splitting principle: For any vector bundle $\mathcal{E}$ over a scheme X, there exists an X-scheme $\pi\colon X_{\mathcal{E}} \to X$ such that the induced morphism $\pi^*\colon \mathrm{K}(X) \hookrightarrow \mathrm{K}(X_{\mathcal{E}})$ is an extension of λ-rings with positive structure, and such that $\pi^*\mathcal{E}$ splits as a sum of line bundles in $\mathrm{K}(X_{\mathcal{E}})$.

Both incarnations are discussed in [FL85, I, §2]. An **extension** of a λ-ring with positive structure A is simply an injective λ-morphism to another λ-ring with positive structure $A \hookrightarrow A'$, compatible with the augmentation and such that $A_{\geq 0}$ maps to $A'_{\geq 0}$.

No splitting principle for GW

For $\mathrm{GW}(X)$, the analogues of the geometric splitting principle fails:

> Over any field of characteristic not two, there exists a (smooth, projective) scheme X and a symmetric vector bundle $(\mathcal{E}, \varepsilon)$ over X such that there exists no X-scheme $\pi\colon X_{(\mathcal{E},\varepsilon)} \to X$ for which the class of $\pi^*(\mathcal{E}, \varepsilon)$ in $\mathrm{GW}(X_{(\mathcal{E},\varepsilon)})$ splits into a sum of symmetric line bundles.

The natural analogue of the algebraic splitting principle could be formulated using the notion of a real λ-ring:

3.1 Definition. A **real λ-ring** is a λ-ring with positive structure A in which any line element squares to one.

This property is clearly satisfied by the Grothendieck-Witt ring $\mathrm{GW}(X)$ of any scheme X. However, an algebraic splitting principle for real λ-rings fails likewise:

> There exist a real λ-ring A and a positive element $e \in A$ that does not split into a sum of line elements in any extension of real λ-rings $A \hookrightarrow A_e$.

The failure of both splitting principles is clear from the following simple counterexample:

3.2 Lemma. *Let $\mathbb{P}^2$ be the projective plane over some field k of characteristic not two. Consider the element $e := H(\mathcal{O}(1)) \in \mathrm{GW}(\mathbb{P}^2)$. There exists no extension of λ-rings $\mathrm{GW}(\mathbb{P}^2) \hookrightarrow A_e$ such that A_e is real and such that e splits as a sum of line elements in A_e.*

Proof. For any element a in a real λ-ring that can be written as a sum of line elements, the Adams operations ψ_n are simply given by

$$\psi_n(a) = \begin{cases} \mathrm{rank}(a) & \text{if } n \text{ is even,} \\ a & \text{if } n \text{ is odd.} \end{cases}$$

However, for $e := H(\mathcal{O}(1)) \in \mathrm{GW}(\mathbb{P}^2)$ we have $\psi_2(e) \neq 2$, so $\psi_2(e)$ cannot be a sum of line bundles, neither in $\mathrm{GW}(\mathbb{P}^2)$ itself nor in any real extension. (Explicitly, $\mathrm{GW}(\mathbb{P}^2) = \pi^*\mathrm{GW}(k) \oplus \mathbb{Z}e$ with $e^2 = -2\pi^*\langle 1, -1\rangle + 4e$, so

$$\psi_2(e) = e^2 - 2\lambda^2(e) = -2\pi^*\langle 1, -1, -1\rangle + 4e \neq 2.$$

Compare Example 5.3 below.) □

A splitting principle for étale cohomology

Despite the negative result above, we do have a splitting principle for Stiefel-Whitney classes of symmetric bundles. Let X be any scheme over $\mathbb{Z}[\frac{1}{2}]$.

3.3 Proposition. *For any symmetric bundle $(\mathcal{E}, \varepsilon)$ over X there exists a morphism $\pi\colon X_{(\mathcal{E},\varepsilon)} \to X$ such that $\pi^*(\mathcal{E}, \varepsilon)$ splits as an orthogonal sum of symmetric line bundles over $X_{(\mathcal{E},\varepsilon)}$ and such that π^* is injective on étale cohomology with $\mathbb{Z}/2$-coefficients.*

Proof. Recall the geometric construction of higher Stiefel-Whitney classes of Delzant and Laborde, as explained for example in [EKV93, §5]: given a symmetric vector bundle $(\mathcal{E}, \varepsilon)$ as above, the key idea is to consider the scheme of non-degenerate one-dimensional subspaces $\pi\colon \mathbb{P}_{nd}(\mathcal{E}, \varepsilon) \to X$, i. e. the complement of the quadric in $\mathbb{P}(\mathcal{E})$ defined by ε. (This is an algebraic version of the projective bundle associated with a real vector bundle in topology; c. f. [Zib11, Lem. 1.7].) Let $\mathcal{O}(-1)$ denote the restriction of the universal line bundle over $\mathbb{P}(\mathcal{E})$ to $\mathbb{P}_{nd}(\mathcal{E}, \varepsilon)$. This is a subbundle of $\pi^*\mathcal{E}$, and by construction the restriction of $\pi^*\varepsilon$ to $\mathcal{O}(-1)$ is non-degenerate. Let w be the first Stiefel-Whitney class of the symmetric line bundle $\mathcal{O}(-1)$. The étale cohomology of $\mathbb{P}_{nd}(\mathcal{E}, \varepsilon)$ decomposes as

$$H^*_{et}(\mathbb{P}_{nd}(\mathcal{E}, \varepsilon), \mathbb{Z}/2) = \bigoplus_{i=0}^{r-1} \pi^* H^*_{et}(X, \mathbb{Z}/2) \cdot w^i,$$

and the higher Stiefel-Whitney classes of $(\mathcal{E}, \varepsilon)$ can be defined as the coefficients of the equation expressing w^r as a linear combination of the smaller powers w^i in $H^*_{et}(\mathbb{P}_{nd}(\mathcal{E}, \varepsilon), \mathbb{Z}/2)$.

We only need to note two facts from this construction: Firstly, over $\mathbb{P}_{nd}(\mathcal{E}, \varepsilon)$ we have an orthogonal decomposition

$$\pi^*(\mathcal{E}, \varepsilon) \cong (\mathcal{O}(-1), \epsilon') \oplus (\mathcal{E}'', \varepsilon''),$$

where $\mathcal{E}'' = \mathcal{O}(-1)^\perp$ and ε' and ε'' are the restrictions of $\pi^*\varepsilon$. Secondly, π induces a monomorphism from the étale cohomology of X to the étale cohomology of $\mathbb{P}_{nd}(\mathcal{E}, \varepsilon)$. So the proposition is proved by iterating this construction. □

4 The γ-filtration on the Grothendieck-Witt ring

From now on, we assume that X is connected. As we have seen, $\mathrm{GW}(X)$ is a pre-λ-ring with positive structure, and we can consider the associated γ-filtration $F^i_\gamma \mathrm{GW}(X)$ of $\mathrm{GW}(X)$. The image of this filtration under the canonical epimorphism $\mathrm{GW}(X) \twoheadrightarrow \mathrm{W}(X)$ will be denoted $F^i_\gamma \mathrm{W}(X)$. In particular, by definition,

$$\begin{aligned} F^1_\gamma \mathrm{GW}(X) &= F^1_{clas}\mathrm{GW}(X) = \ker(\mathrm{rank}\colon \mathrm{GW}(X) \to \mathbb{Z}), \\ F^1_\gamma \mathrm{W}(X) &= F^1_{clas}\mathrm{W}(X) = \ker(\overline{\mathrm{rank}}\colon \mathrm{W}(X) \to \mathbb{Z}/2). \end{aligned}$$

For a field, or more generally for a connected semi-local ring R, we also write $\mathrm{GI}(R)$ and $\mathrm{I}(R)$ instead of $F^1_{clas}\mathrm{GW}(R)$ and $F^1_{clas}\mathrm{W}(R)$, respectively.

4.1 Comparison with the fundamental filtration

4.1 Proposition. *For any connected semi-local commutative ring R in which two is invertible, the γ-filtration on* $\mathrm{GW}(R)$ *is the filtration by powers of the augmentation ideal* $\mathrm{GI}(R)$*, and the induced filtration on* $\mathrm{W}(R)$ *is the filtration by powers of the fundamental ideal $I(R)$.*

Proof. As we have already noted in the proof of Proposition 2.4, all positive elements of the Grothendieck-Witt ring $\mathrm{GW}(R)$ can be written as sums of line elements. Thus, the claim concerning $\mathrm{GW}(R)$ is immediate from Lemma 1.5. Moreover, the fundamental filtration on $\mathrm{W}(R)$ is the image of the fundamental filtration on $\mathrm{GW}(R)$. □

4.2 Remark. In the situation above, the projection $\mathrm{GW}(R) \to \mathrm{W}(R)$ even induces isomorphisms $\mathrm{GI}^n(R) \to \mathrm{I}^n(R)$, so that $\mathrm{gr}^i_\gamma \mathrm{GW}(R) \cong \mathrm{gr}^i_\gamma \mathrm{W}(R)$ in degrees $i > 0$. This fails for general schemes in place of R, of course (see Section 5).

4.3 Remark. It may seem more natural to define a filtration on $\mathrm{GW}(X)$ starting with the kernel not of the rank morphism but of the rank reduced modulo two, as for example in [Aue12]:

$$\mathrm{GI}' := \ker\left(\mathrm{GW}(X) \to H^0(X, \mathbb{Z}/2)\right)$$

For connected X, this is a split extension of the augmentation ideal by a copy of $\mathbb{Z}$ generated by the hyperbolic plane $\mathbb{H}$. In particular, GI and GI' have the same image in $\mathrm{W}(X)$. However, even over a field, the filtration by powers of GI' does *not* yield the same graded ring as the filtration by powers of (GI or) I. For example, for $R = \mathbb{R}$, we find that

$$\begin{aligned} \mathrm{GI}^n(\mathbb{R})\big/\mathrm{GI}^{n+1}(\mathbb{R}) &\cong \mathbb{Z}/2 \qquad\qquad (n > 0) \\ (\mathrm{GI}')^n(\mathbb{R})\big/(\mathrm{GI}')^{n+1}(\mathbb{R}) &\cong \mathbb{Z}/2 \oplus \mathbb{Z}/2 \end{aligned}$$

It is the filtration by powers of GI that yields an associated graded ring isomorphic to $H^*(\mathbb{R}, \mathbb{Z}/2)$ in positive degrees, not the filtration by powers of GI'.

4.2 Comparison with the classical filtration

A common filtration on the Witt ring of a scheme is given by the kernels of the first two étale Stiefel-Whitney classes w_1 and w_2 on the Grothendieck-Witt ring and of the induced classes $\overline{w}_1$ and $\overline{w}_2$ on the Witt ring:

$$\begin{aligned} F^2_{clas}\mathrm{GW}(X) &:= \ker\left(F^1_{clas}\mathrm{GW}(X) \xrightarrow{w_1} H^1_{et}(X, \mathbb{Z}/2)\right) \\ F^2_{clas}\mathrm{W}(X) &:= \ker\left(F^1_{clas}\mathrm{W}(X) \xrightarrow{\overline{w}_1} H^1_{et}(X, \mathbb{Z}/2)\right) \end{aligned}$$

$$\begin{aligned} F^3_{clas}\mathrm{GW}(X) &:= \ker\left(F^2_{clas}\mathrm{GW}(X) \xrightarrow{w_2} H^2_{et}(X, \mathbb{Z}/2)\right) \\ F^3_{clas}\mathrm{W}(X) &:= \ker\left(F^2_{clas}\mathrm{W}(X) \xrightarrow{\overline{w}_2} H^2_{et}(X, \mathbb{Z}/2)/\mathrm{Pic}(X)\right) \end{aligned}$$

4.4 Proposition. *Let X be any connected scheme such that the canonical pre-λ-structure on* $\mathrm{GW}(X)$ *is a λ-structure. Then:*

$$\begin{aligned} F^2_\gamma\mathrm{GW}(X) &= F^2_{clas}\mathrm{GW}(X) \\ F^2_\gamma\mathrm{W}(X) &= F^2_{clas}\mathrm{W}(X) \end{aligned}$$

Proof. The first identity is a consequence of Lemma 1.6: In our case, the group of line elements may be identified with $H^1_{et}(X, \mathbb{Z}/2)$; then the determinant $\mathrm{GW}(X) \to H^1_{et}(X, \mathbb{Z}/2)$ is precisely the first Stiefel-Whitney class w_1. In particular, the kernel of the restriction of w_1 to $F^1_{clas}\mathrm{GW}(X)$ is $F^2_\gamma\mathrm{GW}(X)$, as claimed. For the second identity, it suffices to observe that $F^2_{clas}\mathrm{GW}(X)$ maps surjectively onto $F^2_{clas}\mathrm{W}(X)$. □

In order to analyse the relation of $F^3_\gamma\mathrm{GW}(X)$ to $F^3_{clas}\mathrm{GW}(X)$, we need a few lemmas concerning products of "reduced line elements":

4.5 Lemma. *Let $u_1, \ldots, u_l, v_1, \ldots, v_l$ be line elements in a pre-λ-ring with positive structure A. Then $\gamma^k\left(\sum_i (u_i - v_i)\right)$ can be written as a linear combination of products*

$$(u_{i_1} - 1)\cdots(u_{i_s} - 1)(v_{j_1} - 1)\cdots(v_{j_t} - 1)$$

with $s + t = k$ factors.

Proof. This is easily seen by induction over l. For $l = 1$ and $k = 0$ the statement is trivial, while for $l = 1$ and $k \geq 1$ we have

$$\begin{aligned} \gamma^k(u - v) &= \gamma^k((u-1) + (1-v)) \\ &= \gamma^0(u-1)\gamma^k(1-v) + \gamma^1(u-1)\gamma^{k-1}(1-v) \\ &= \pm(v-1)^k \mp (u-1)(v-1)^{k-1} \end{aligned}$$

For the induction step, we observe that every summand in

$$\gamma^k\left(\sum_{i=1}^{l+1} u_i - v_i\right) = \sum_{i=0}^{k} \gamma^i\left(\sum_{i=1}^{l} u_i - v_i\right)\gamma^{k-i}(u_l - v_l)$$

can be written as a linear combination of the required form. □

4.6 Lemma. *Let X be a scheme over $\mathbb{Z}[\frac{1}{2}]$, and let $u_1, \dots, u_n \in \mathrm{GW}(X)$ be classes of symmetric line bundles with Stiefel-Whitney classes $w_1(u_i) =: \overline{u}_i$. Let ρ denote the product*

$$\rho := (u_1 - 1)\cdots(u_n - 1).$$

Then $w_i(\rho) = 0$ for $0 < i < 2^{n-1}$, and

$$w_{2^{n-1}}(\rho) = \prod_{\substack{1\le i_1<\dots<i_k\le n \\ \text{with } k \text{ odd}}} (\overline{u}_{i_1} + \dots + \overline{u}_{i_k}) = \sum_{\substack{r_1,\dots,r_n: \\ 2^{r_1}+\dots+2^{r_n}=2^{n-1}}} \overline{u}_1^{2^{r_1}} \cdot \dots \cdot \overline{u}_n^{2^{r_n}}.$$

Proof. The lemma generalizes Lemma 3.2/Corollary 3.3 of [Mil69]. The first part of Milnor's proof applies verbatim. Consider the evaluation map

$$\mathbb{Z}/2[\![x_1, \dots, x_n]\!] \xrightarrow{\text{ev}} \prod_i H^i_{et}(X, \mathbb{Z}/2)$$

sending x_i to $\overline{u}_i$. The total Stiefel-Whitney class $w(\rho) = 1 + w_1(\rho) + w_2(\rho) + \dots$ is the evaluation of the power series

$$\omega(x_1, \dots, x_n) := \left(\frac{\prod_{|\boldsymbol{\epsilon}| \text{ even}} (1 + \boldsymbol{\epsilon}\boldsymbol{x})}{\prod_{|\boldsymbol{\epsilon}| \text{ odd}} (1 + \boldsymbol{\epsilon}\boldsymbol{x})}\right)^{(-1)^n},$$

where the products range over all $\boldsymbol{\epsilon} = (\epsilon_1, \dots, \epsilon_n) \in (\mathbb{Z}/2)^n$ with $|\boldsymbol{\epsilon}| := \epsilon_1 + \dots + \epsilon_n$ even or odd, and where $\boldsymbol{\epsilon}\boldsymbol{x}$ denotes the sum $\sum_i \epsilon_i x_i$. As Milnor points out, all factors of ω cancel if we substitute $x_i = 0$. More generally, all factors cancel whenever we replace a given variable x_i by the sum of an even number of variables $x_{i_1} + \dots + x_{i_{2l}}$ all distinct from x_i. Indeed, consider the substitution $x_n = \boldsymbol{\alpha}\boldsymbol{x}$ with $|\boldsymbol{\alpha}|$ even and $\alpha_n = 0$. Write $\boldsymbol{x} = (\boldsymbol{x}', x_n)$, $\boldsymbol{\epsilon} = (\boldsymbol{\epsilon}', \epsilon_n)$ and $\boldsymbol{\alpha} = (\boldsymbol{\alpha}', 0)$, so that the substitution may be rewritten as $x_n = \boldsymbol{\alpha}'\boldsymbol{x}'$. Then

$$(\boldsymbol{\epsilon}', \epsilon_n)(\boldsymbol{x}', \boldsymbol{\alpha}'\boldsymbol{x}') = (\boldsymbol{\epsilon}' + \boldsymbol{\alpha}', \epsilon_n + 1)(\boldsymbol{x}', \boldsymbol{\alpha}'\boldsymbol{x}'),$$

but the parities of $|(\boldsymbol{\epsilon}', \epsilon_n)|$ and $|(\boldsymbol{\epsilon}' + \boldsymbol{\alpha}', \epsilon_n + 1)|$ are different. Thus, the corresponding factors of ω cancel. It follows that $\omega - 1$ is divisible by all sums of an odd number of distinct variables $x_{i_1} + \dots + x_{i_k}$. Therefore,

$$\omega = 1 + (\textstyle\prod_{|\boldsymbol{\epsilon}| \text{ odd}} \boldsymbol{\epsilon}\boldsymbol{x}) \cdot f(\boldsymbol{x}) \tag{3}$$

for some power series f. In particular, ω has no non-zero coefficients in positive total degrees below $\sum_{k \text{ odd}} \binom{n}{k} = 2^{n-1}$, proving the first part of the lemma.

For the second part, we need to show that the constant coefficient of f is 1. This can be seen by considering the substitution $x_1 = x_2 = \cdots = x_n = x$ in (3): we obtain

$$\left(\frac{1}{(1+x)^K}\right)^{\pm 1} = 1 + x^K f(x, \ldots, x)$$

with $K = \sum_{k \text{ odd}} \binom{n}{k} = 2^{n-1}$, and as $(1 + x^K) = 1 + x^K \mod 2$ for K a power of two, this equation can be rewritten as

$$(1 + x^K)^{\mp 1} = 1 + x^K f(x, \ldots, x).$$

The claim follows. Finally, the identification of the product expression for $w_{2^{n-1}}(\rho)$ with a sum is taken from [GM14]. It is verified by showing that all factors of the product divide the sum, using similar substitution arguments as above. □

4.7 Remark. Milnor's proof in the case when X is a field k uses the relation $a^{\cup 2} = [-1] \cup a$ in $H^2(k, \mathbb{Z}/2)$, which does not hold in general.

4.8 Proposition. *Let X be a connected scheme over $\mathbb{Z}[\frac{1}{2}]$. Then $w_i(F^n_\gamma \mathrm{GW}(X)) = 0$ for $0 < i < 2^{n-1}$. In particular:*

$$F^2_\gamma \mathrm{GW}(X) \subset F^2_{clas} \mathrm{GW}(X)$$
$$F^3_\gamma \mathrm{GW}(X) \subset F^3_{clas} \mathrm{GW}(X)$$

Proof. Let $x := \gamma^{k_1}(x_1) \cdots \gamma^{k_l}(x_l)$ be an additive generator of $\mathrm{GW}^n(X)$, i. e. $x_i \in \ker(\mathrm{rank})$ and $\sum k_i \geq n$. By writing each x_i as $[\mathcal{E}_i, \epsilon_i] - [\mathcal{F}_i, \phi_i]$ for certain symmetric vector bundles $(\mathcal{E}_i, \epsilon_i)$ and $(\mathcal{F}_i, \phi_i)$ and successively applying the splitting principal for étale cohomology (Proposition 3.3) to each of these, we can find a morphism

$$X_x \to X$$

which is injective on étale cohomology with $\mathbb{Z}/2$-coefficients, and such that each $\pi^* x_i$ is a sum of differences of line bundles. By Lemma 4.5, each $\gamma^{k_i}(\pi^* x_i)$ can therefore be written as a linear combination of products $(u_1 - 1) \cdots (u_m - 1)$ with $m = k_i$ factors, where each u_i is the class of some line bundle over X_x. Using the naturality of the γ-operations, it follows that $\pi^* x$ can be written as a linear combination of such products with $m \geq n$ factors.

By Lemma 4.6, the classes w_i vanish on every summand of this linear combination for $0 < i < 2^{n-1}$. So $w_i(\pi^* x) = 0$ for all $0 < i < 2^{n-1}$, and by the naturality of Stiefel-Whitney classes and the injectivity of π^* on cohomology we may conclude that $w_i(x)$ vanishes in this range. □

4.3 Comparison with the unramified filtration

Here, we quickly summarize some observations on the relation of the γ-filtration with the "unramified filtration". We start from the general and progress towards the special.

So let X be an integral scheme with function field K, and let $F_K^*\mathrm{GW}(X)$ denote the filtration on $\mathrm{GW}(X)$ given by the preimages of $\mathrm{GI}^i(K)$ under the natural map $\mathrm{GW}(X) \to \mathrm{GW}(K)$. Said map is a morphism of augmented λ-rings, so $F_\gamma^i\mathrm{GW}(X)$ maps to $F_\gamma^i\mathrm{GW}(K) = \mathrm{GI}^i(K)$ and we obtain:

4.9 Proposition. *For any integral scheme X, the unramified fitration on $\mathrm{GW}(X)$ is finer than the γ-filtration, i. e. $F_\gamma^i\mathrm{GW}(X) \subset F_K^i\mathrm{GW}(X)$ for all i.* □

The unramified Grothendieck-Witt group of X is defined as

$$\mathrm{GW}_{ur}(X) := \bigcap_{x\in X^{(1)}} \mathrm{im}\left(\mathrm{GW}(\mathcal{O}_{X,x}) \to \mathrm{GW}(K)\right).$$

Let us consider the functors GW and GW_{ur} as the presheaves on our given integral scheme X that send an open subset $U \subset X$ to $\mathrm{GW}(U)$ or $\mathrm{GW}_{ur}(U)$, respectively. Then GW_{ur} is a sheaf, and we have a sequence of morphisms of presheaves

$$\mathrm{GW} \to \mathrm{GW}^+ \to \mathrm{GW}_{ur} \hookrightarrow \mathrm{GW}(K),$$

where $(-)^+$ denotes sheafification and $\mathrm{GW}(K)$ is to be interpreted as the constant sheaf with value $\mathrm{GW}(K)$. The unramified filtration of GW_{ur} is obtained by intersecting the fundamental filtration on $\mathrm{GW}(K)$ with GW_{ur}:

$$F_K^i\mathrm{GW}_{ur} := \mathrm{GW}_{ur} \cap \mathrm{GI}^i(K).$$

This is a filtration by sheaves, and the unramified filtration $F_K^i\mathrm{GW}$ is given by the preimage of $F_K^i\mathrm{GW}_{ur}$ under the above morphisms.

When X is regular integral of finite type over a field of characteristic not two, the purity results of Ojanguren and Panin [Oja80,OP99] imply that the morphism $\mathrm{GW}^+ \to \mathrm{GW}_{ur}$ is an isomorphism. If we further assume that the field is infinite, a result of Kerz and Müller-Stach yields the following:

4.10 Proposition. *For any regular integral scheme of finite type over an infinite field of characteristic not two, the γ-filtration and the unramified filtration GW^+ have the same sheafifications:*

$$(F_\gamma^i\mathrm{GW})^+ = (F_K^i\mathrm{GW})^+ = F_K^i\mathrm{GW}_{ur}$$

Proof. As already mentioned, the results of Ojanguren and Panin imply that GW^+ injects into $\mathrm{GW}(K)$ in this situation, with image GW_{ur}. In particular, the stalks of GW_{ur} are those of GW: $\mathrm{GW}_x = (\mathrm{GW}_{ur})_x = \mathrm{GW}(\mathcal{O}_{X,x})$. Consequently, the unramified filtration has stalks

$$(F_K^i\mathrm{GW})_x = (F_K^i\mathrm{GW}_{ur})_x = \mathrm{GW}(\mathcal{O}_{X,x}) \cap \mathrm{GI}^i(K).$$

The γ-filtration $F_\gamma^i\mathrm{GW}$ on the other hand, also viewed as a presheaf, has stalks $F_\gamma^i\mathrm{GW}(\mathcal{O}_{X,x})$. By Proposition 4.1 above and Corollary 0.5 of [KMS07], these stalks agree. □

Both propositions apply verbatim to the Witt ring W in place of GW. If, in addition to being regular integral of finite type over a field of characteristic not two, our scheme is separated and of dimension at most three, then by [BW02] the Witt presheaf W is already a sheaf, and hence also $F^i_K\mathrm{W}$ is a filtration by sheaves. This justifies the claim made in the introduction, that the "the unramified filtration of the Witt ring is the sheafification of the γ-filtration" in this situation.

5 Examples

All our examples will be smooth quasiprojective varieties over a field of characteristic different from two. The lower-degree pieces of the filtrations on the K-, Grothendieck-Witt and Witt rings therefore always fit the following pattern:

$$\begin{array}{lll}
F^0_\gamma\mathrm{K} = F^0_{top}\mathrm{K} = \mathrm{K} & F^0_\gamma\mathrm{GW} = \mathrm{GW} & F^0_\gamma\mathrm{W} = \mathrm{W} \\
F^1_\gamma\mathrm{K} = F^1_{top}\mathrm{K} = \ker(\mathrm{rank}) & F^1_\gamma\mathrm{GW} = \ker(\mathrm{rank}) & F^1_\gamma\mathrm{W} = \ker(\overline{\mathrm{rank}}) \\
F^2_\gamma\mathrm{K} = F^2_{top}\mathrm{K} = \ker(c_1) & F^2_\gamma\mathrm{GW} = \ker(w_1) & F^2_\gamma\mathrm{W} = \ker(\overline{w}_1) \\
F^3_\gamma\mathrm{K} \subset F^3_{top}\mathrm{K} = \ker(c_2) & F^3_\gamma\mathrm{GW} \subset \ker(w_2) & F^3_\gamma\mathrm{W} \subset \ker(\overline{w}_2)
\end{array}$$

For the topological filtration F^*_{top} on the K-ring, see [Ful98, Example 15.3.6]. The symbols c_i denote the Chern classes with values in Chow groups.

Some details concerning the computations in each of the examples are provided at the end of this section.

5.1 Example (Curve). Let C be a smooth curve over a field of 2-cohomological dimension at most 1, e. g. over an algebraically closed field or over a finite field. Then

$$\begin{aligned}
\mathrm{gr}^*_\gamma \mathrm{GW}(C) &= \mathrm{gr}^*_{clas} \mathrm{GW}(C) \cong \mathbb{Z} \oplus H^1_{et}(C,\mathbb{Z}/2) \oplus H^2_{et}(C,\mathbb{Z}/2) \\
\mathrm{gr}^*_\gamma \mathrm{W}(C) &= \mathrm{gr}^*_{clas} \mathrm{W}(C) \cong \mathbb{Z}/2 \oplus H^1_{et}(C,\mathbb{Z}/2)
\end{aligned}$$

5.2 Example (Curve $\times\, \mathbb{P}^1$). For a smooth surface X over an algebraically closed field, the filtration $\mathrm{GW} \supset F^1_\gamma\mathrm{GW} \supset F^2_\gamma\mathrm{GW} \supset \ker(w_2)$ yields the graded group

$$\begin{aligned}
\mathrm{gr}^*_{clas} \mathrm{GW}(X) &\cong \mathbb{Z} \oplus H^1_{et}(X,\mathbb{Z}/2) \oplus H^2_{et}(X,\mathbb{Z}/2) \oplus \mathrm{CH}^2(X) \\
\mathrm{gr}^*_\gamma \mathrm{W}(X) = \mathrm{gr}^*_{clas} \mathrm{W}(X) &\cong \mathbb{Z}/2 \oplus H^1_{et}(X,\mathbb{Z}/2) \oplus H^2_{et}(X,\mathbb{Z}/2)/\mathrm{Pic}(X)
\end{aligned}$$

However, in general $F^3_\gamma\mathrm{GW}(X) \subsetneq F^3_{clas}\mathrm{GW}(X) = \mathrm{CH}^2(X)$. For a concrete example, consider the product $X = C \times \mathbb{P}^1$, where C is any smooth projective curve. In this case

$$\begin{aligned}
F^3_{clas}\mathrm{GW}(X) &\cong \mathrm{Pic}(C) \\
F^3_\gamma\mathrm{GW}(X) &\cong \mathrm{Pic}(C)[2] \qquad \text{(kernel of multiplication by 2).}
\end{aligned}$$

5.3 Example ($\mathbb{P}^r$). Let $\mathbb{P}^r$ be the r-dimensional projective space over a field k. We first describe its Grothendieck-Witt ring. Let $a := H_0(\mathcal{O}(1) - 1)$ and $\rho := \lceil \frac{r}{2} \rceil$. Then

$$\mathrm{GW}(\mathbb{P}^r) \cong \begin{cases} \mathrm{GW}(k) \oplus \mathbb{Z}a \oplus \mathbb{Z}a^2 \oplus \cdots \oplus \mathbb{Z}a^\rho & \text{if } r \text{ is even} \\ \big(\mathrm{GW}(k) \oplus \mathbb{Z}a \oplus \mathbb{Z}a^2 \oplus \cdots \oplus \mathbb{Z}a^\rho\big)/(2f_r(a)) & \text{if } r \equiv 1 \mod 4 \\ \big(\mathrm{GW}(k) \oplus \mathbb{Z}a \oplus \mathbb{Z}a^2 \oplus \cdots \oplus \mathbb{Z}a^\rho\big)/(f_r(a)) & \text{if } r \equiv -1 \mod 4 \end{cases}$$

Here, f_r is a polynomial in a of the form

$$f_r(a) = \pm a^\rho + (\text{ intermediate terms }) + b_r \cdot a,$$

with the coefficient of the linear term given by $b_r = \sum_{j=1}^{\rho} (-1)^j \binom{r-1}{\rho-j} j^2$. In particular, in the third case a^ρ is superfluous as an additive generator. The multiplication is determined by the formulas $\phi \cdot a^i = \mathrm{rank}(\phi) a^i$ for $\phi \in \mathrm{GW}(k)$ and $a^k = 0$ for $k > \rho$.

In this description, $F_\gamma^k \mathrm{GW}(\mathbb{P}^r)$ is the ideal generated by $F_\gamma^k \mathrm{GW}(k)$ and $a^{\lfloor \frac{k}{2} \rfloor}$. In particular, $F_\gamma^3 \mathrm{GW}(X)$ is again strictly smaller than $F_{clas}^3 \mathrm{GW}(X)$:

$$\begin{aligned} F_{clas}^3 \mathrm{GW}(\mathbb{P}^r) &= F_\gamma^3 \mathrm{GW}(k) + (a^2, 2a) \\ F_\gamma^3 \mathrm{GW}(\mathbb{P}^r) &= F_\gamma^3 \mathrm{GW}(k) + (a^2) \end{aligned}$$

For the associated graded ring, we obtain:

$$\mathrm{gr}_\gamma^* \mathrm{GW}(\mathbb{P}^r) \cong \begin{cases} \mathrm{gr}_\gamma^* \mathrm{GW}(k) \oplus \mathbb{Z}\, a \oplus \ldots \oplus \mathbb{Z}\, a^\rho & \text{if } r \text{ is even} \\ \mathrm{gr}_\gamma^* \mathrm{GW}(k) \oplus (\mathbb{Z}/2b_r) a \oplus \ldots \oplus (\mathbb{Z}/2b_r) a^\rho & \text{if } r \equiv 1 \mod 4 \\ \mathrm{gr}_\gamma^* \mathrm{GW}(k) \oplus (\mathbb{Z}/b_r)\, a \oplus \ldots \oplus (\mathbb{Z}/b_r)\, a^\rho & \text{if } r \equiv -1 \mod 4 \end{cases}$$

with a of degree 2. In the Witt group, all the hyperbolic elements a^i vanish, so obviously $\mathrm{gr}_\gamma^* \mathrm{W}(\mathbb{P}^r) \cong \mathrm{gr}_\gamma^* \mathrm{W}(k)$.

5.4 Example ($\mathbb{A}^1 - 0$). For the punctured affine line over a field k, we have

$$\begin{aligned} \mathrm{GW}(\mathbb{A}^1 - 0) &\cong \mathrm{GW}(k) \oplus \mathrm{W}(k)\hat{\varepsilon} \\ F_\gamma^i \mathrm{GW}(\mathbb{A}^1 - 0) &\cong \mathrm{GI}^i(k) \oplus \mathrm{I}^{i-1}(k)\hat{\varepsilon} \end{aligned}$$

for some generator $\hat{\varepsilon} \in F_\gamma^1 \mathrm{GW}(\mathbb{A}^1 - 0)$ satisfying $\hat{\varepsilon}^2 = 2\hat{\varepsilon}$. In this example, $F_\gamma^3 \mathrm{GW}(\mathbb{A}^1 - 0) = \ker(w_2)$.

5.5 Example ($\mathbb{A}^{4n+1} - 0$). For punctured affine spaces of dimensions $d \equiv 1 \mod 4$ with $d > 1$, there is a similar result for the Grothendieck-Witt group [BG05]:

$$\mathrm{GW}(\mathbb{A}^{4n+1} - 0) \cong \mathrm{GW}(k) \oplus \mathrm{W}(k)\hat{\varepsilon}$$

for some $\hat{\varepsilon} \in F_\gamma^1 \mathrm{GW}(\mathbb{A}^1 - 0)$. However, in this case $\hat{\varepsilon}^2 = 0$, and the γ-filtration is also different from the γ-filtration in the one-dimensional case. This is already apparent over the complex numbers: $\mathrm{GW}(\mathbb{A}_{\mathbb{C}}^{4n+1} - 0)$ is isomorphic as a λ-ring to $\mathrm{KO}(S^{8n+1})$. We thus find that

$$F_\gamma^i \mathrm{GW}(\mathbb{A}_{\mathbb{C}}^5 - 0) \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z}/2\hat{\varepsilon} & \text{for } i = 0 \\ \mathbb{Z}/2\hat{\varepsilon} & \text{for } i = 1, 2 \\ 0 & \text{for } i \geq 3 \end{cases}$$

This is also a simple example with $F^3_\gamma \mathrm{W}(X) \neq F^3_{clas}\mathrm{W}(X)$, the latter being non-zero since since w_2 and $\overline{w}_2$ are zero.

Calculations for Example 5.1 (curve). Consider the summary at the end of the previous section. In dimension 1, we have $F^2_{top}\mathrm{K} = 0$, so $F^2_\gamma \mathrm{K} = \ker(c_1) = 0$. Moreover, for any curve as above, w_2 is surjective with $\ker(w_2) = H_0(\ker(c_1))$ [Zib14, proof of Cor. 3.7/4.7[3]; H_0 denotes the hyperbolic map]. It follows that $F^3_\gamma\mathrm{GW} = 0$, and that $\mathrm{gr}^*_\gamma \mathrm{GW}$ is as described. The surjectivity of w_2 also implies that $\ker(w_2)$ maps surjectively onto $\ker(\overline{w}_2)$, so that $F^3_\gamma \mathrm{W}$ vanishes and $\mathrm{gr}^*_\gamma \mathrm{W}$ is as displayed. □

Calculations for Example 5.2 (surface). The classical filtration is computed in [Zib14, Cor. 3.7/4.7]. In the case $X = C \times \mathbb{P}^1$, Walter's projective bundle formula [Wal03, Thm 1.5] and the results on $\mathrm{GW}^*(C)$ of [Zib14, Thm 2.1/3.1] yield:

$$\mathrm{GW}(X) \cong \overbrace{\mathbb{Z} \oplus \underbrace{\mathrm{Pic}(C)[2]}_{H^1_{et}(X,\mathbb{Z}/2)} \oplus \mathbb{Z}/2}^{\pi^*\mathrm{GW}(C)} \oplus \overbrace{\underbrace{\mathbb{Z}/2 \oplus}_{H^2_{et}(X,\mathbb{Z}/2)} \underbrace{\mathrm{Pic}(C)}_{\mathrm{CH}^2(X)}}^{\pi^*\mathrm{GW}^{-1}(C)\cdot\Psi}$$

Here, $\pi\colon X \twoheadrightarrow C$ is the projection and $\Psi \in \mathrm{GW}^1(\mathbb{P}^1)$ a generator. Writing $H_i\colon \mathrm{K} \to \mathrm{GW}^i$ for the hyperbolic maps, we can describe the additive generators of $\mathrm{GW}(X)$ explicitly as follows:

- 1 (the trivial symmetric line bundle)
- $a_\mathcal{L} := \pi^*\mathcal{L} - 1$, for each line bundle $\mathcal{L} \in \mathrm{Pic}(C)[2]$
- $b := H_0(\pi^*\mathcal{L}_1 - 1)$, where $\mathcal{L}_1$ is a line bundle of degree 1 on C (hence a generator of the free summand of $\mathrm{Pic}(C)$)
- $c := H_0(\mathcal{O}(-1) - 1) = H_{-1}(1) \cdot \Psi$, where $\mathcal{O}(-1)$ is the pullback of the canonical line bundle on $\mathbb{P}^1$
- $d_\mathcal{N} := H_{-1}(\pi^*\mathcal{N} - 1) \cdot \Psi$, for each $\mathcal{N} \in \mathrm{Pic}(C)$.

The order of the generators in this list is chosen to coincide with the order of the direct summands of $\mathrm{GW}(X)$ which they generate in the formula above. Alternatively, we may replace the generators $d_\mathcal{N}$ by the generators

$$\begin{aligned} d'_\mathcal{N} &:= H_0(\pi^*\mathcal{N} \otimes \mathcal{O}(-1) - 1) \\ &= \begin{cases} d_\mathcal{N} + c & \text{if } \mathcal{N} \text{ is of even degree} \\ d_\mathcal{N} + b + c & \text{if } \mathcal{N} \text{ is of odd degree} \end{cases} \end{aligned}$$

The only non-trivial products of the alternative generators are $a_\mathcal{L} c = a_\mathcal{L} d'_\mathcal{N} = d'_\mathcal{L} + c \; (= d_\mathcal{L})$. Moreover, the effects of the operations γ^i on the alternative generators is immediate from Lemma 5.6 below. So Lemma 1.5 tells us that $F^3_\gamma\mathrm{GW}$ has additive generators

$$\gamma^1(a_\mathcal{L}) \cdot \gamma^2(c) = a_\mathcal{L} \cdot (-c) = d_\mathcal{L}$$

with $\mathcal{L} \in \mathrm{Pic}(C)[2]$. Thus, $F^3_\gamma\mathrm{GW}(X) \cong \mathrm{Pic}(C)[2]$, viewed as subgroup of the last summand in the formula above. We also find that $F^4_\gamma\mathrm{GW}(X) = 0$. □

[3] numberings in preprint/in published article differ

Calculation of the ring structure on GW($\mathbb{P}^r$) *(Example 5.3).*
By [Wal03, Thms 1.1 and 1.5], the Grothendieck-Witt ring of projective space can be additively described as

$$\mathrm{GW}(\mathbb{P}^r) = \begin{cases} \mathrm{GW}(k) \oplus \mathbb{Z}a_1 \oplus \cdots \oplus \mathbb{Z}a_\rho & \text{if } r \text{ is even} \\ \mathrm{GW}(k) \oplus \mathbb{Z}a_1 \oplus \cdots \oplus \mathbb{Z}a_{\rho-1} \oplus (\mathbb{Z}/2)H_0(F\Psi) & \text{if } r \equiv 1 \mod 4 \\ \mathrm{GW}(k) \oplus \mathbb{Z}a_1 \oplus \cdots \oplus \mathbb{Z}a_{\rho-1} & \text{if } r \equiv -1 \mod 4 \end{cases}$$

where $a_i = H_0(\mathcal{O}(i) - 1)$ and Ψ is a certain element in $\mathrm{GW}^r(\mathbb{P}^r)$. Moreover, by tracing through Walter's computations, we find that

$$H_0(F\Psi) = -\sum_{j=1}^{\rho} (-1)^j \binom{r-1}{\rho - j} a_j. \tag{4}$$

Indeed, we see from the proof of [Wal03, Thm 1.5] that $F\Psi = \mathcal{O}^{\oplus N} - \lambda^\rho(\Omega)(\rho)$ in $\mathrm{K}(\mathbb{P}^r)$, where Ω is the cotangent bundle of $\mathbb{P}^r$ and N is such that the virtual rank of this element is zero.
The short exact sequence $0 \to \Omega \to \mathcal{O}^{\oplus(r+1)}(-1) \to \mathcal{O} \to 0$ over $\mathbb{P}^r$ implies that

$$\lambda^\rho(\Omega) = \lambda^\rho(\mathcal{O}^{\oplus(r+1)}(-1) - 1) \text{ in } \mathrm{K}(\mathbb{P}^r),$$

from which (4) follows by a short computation.
An element $\Psi \in \mathrm{GW}^r(\mathbb{P}^r)$ also exists in the case $r \equiv -1 \mod 4$, and (4) is likewise valid in this case. However, in this case, we see from Karoubi's exact sequence

$$\mathrm{GW}^{-1}(\mathbb{P}^r) \xrightarrow{F} \mathrm{K}(\mathbb{P}^r) \xrightarrow{H_0} \mathrm{GW}^0(\mathbb{P}^r)$$

that $H_0(F\Psi) = 0$. We can thus rewrite the above result for the Grothendieck-Witt group as

$$\mathrm{GW}(\mathbb{P}^r) = \begin{cases} \mathrm{GW}(k) \oplus \mathbb{Z}a_1 \oplus \cdots \oplus \mathbb{Z}a_\rho & \text{if } r \text{ is even} \\ \big(\mathrm{GW}(k) \oplus \mathbb{Z}a_1 \oplus \cdots \oplus \mathbb{Z}a_{\rho-1} \oplus \mathbb{Z}a_\rho\big)/2h_r & \text{if } r \equiv 1 \mod 4 \\ \big(\mathrm{GW}(k) \oplus \mathbb{Z}a_1 \oplus \cdots \oplus \mathbb{Z}a_{\rho-1} \oplus \mathbb{Z}a_\rho\big)/h_r & \text{if } r \equiv -1 \mod 4 \end{cases}$$

with $h_r := \sum_{j=1}^{\rho} (-1)^j \binom{r-1}{\rho-j} a_j$.
To see that we can alternatively use powers of $a := a_1$ as generators, it suffices to observe that for all $k \geq 1$,

$$a_k = a^k + \begin{pmatrix} \text{a linear combination of} \\ a, a^2, \ldots, a^{k-1} \end{pmatrix}, \tag{5}$$

which follows inductively from the recursive relation

$$a_k = (a+2)a_{k-1} - a_{k-2} + 2a. \tag{6}$$

for all $k \geq 2$. ($a_0 := 0$.)
Next, we show that $a^k = 0$ for all $k > \rho$. Let $x := \mathcal{O}(1)$, viewed as an element of $\mathrm{K}(\mathbb{P}^r)$. The relation $(x-1)^{r+1} = 0$ in $\mathrm{K}(\mathbb{P}^r)$ implies that

$$(x-1) + (x^{-1} - 1) = \sum_{i=2}^{r} (-1)^i (x-1)^i,$$

so that we can compute:

$$\begin{aligned} a^k = [H(x-1)]^k &= H\left([FH(x-1)]^{k-1}(x-1)\right) \\ &= H\left(\left[(x-1)+(x^{-1}-1)\right]^{k-1}(x-1)\right) \\ &= H\left((x-1)^{2k-1} + \text{higher order terms in } (x-1)\right) \\ &= 0 \quad \text{for } 2k-1>r, \text{ or, equivalently, for } k>\rho. \end{aligned}$$

Equation (5) also allows us to rewrite h_r in terms of the powers of a, yielding some polynomial $f_r(a)$ and the multiplicative description of $\mathrm{GW}(\mathbb{P}^r)$ displayed in Example 5.3. Equation (6) implies that the linear term on the right-hand side of (5) has coefficient k^2, so the linear term of $f_r(a)$ has coefficient $b_r := \sum_{j=1}^{\rho}(-1)^j\binom{r-1}{\rho-j}$. □

Calculation of the γ-filtration on $\mathrm{GW}(\mathbb{P}^r)$ *(Example 5.3, continued).*
We claim above that $F_\gamma^k\mathrm{GW}(\mathbb{P}^r)$ is the ideal generated by $F_\gamma^k\mathrm{GW}(k)$ and $a^{\lfloor\frac{k}{2}\rfloor}$. Equivalently, it is the subgroup generated by $F_\gamma^k\mathrm{GW}(k)$ and by all powers a^i with $i \geq \frac{k}{2}$. To verify the claim, we note that by Lemma 5.6, we have $\gamma_i(a_k) = \pm a_k$ for $i = 1, 2$, while for all $i > 2$ we have $\gamma_i(a_k) = 0$. In particular, $a = a_1 \in F_\gamma^2\mathrm{GW}(\mathbb{P}^r)$, and therefore $a^i \in F_\gamma^{2i}\mathrm{GW}(\mathbb{P}^r)$. This shows that all the above named additive generators indeed lie in $F_\gamma^k\mathrm{GW}(\mathbb{P}^r)$. For the converse inclusion, we note that by Lemma 1.5, $F_\gamma^k\mathrm{GW}(\mathbb{P}^r)$ is additively generated by $F_\gamma^k\mathrm{GW}(k)$ and by all finite products of the form

$$\prod_j \gamma^{i_j}(a_{\alpha_j})$$

with $\sum_j i_j \geq k$. Such a product is non-zero only if $i_j \in \{0,1,2\}$ for all j, in which case it is of the form $\pm\prod_j a_{\alpha_j}$ with at least $\frac{k}{2}$ non-trivial factors. By (5), each non-trivial factor a_{α_j} can be expressed as a non-zero polynomial in a with no constant term. Thus, the product itself can be rewritten as a linear combination of powers a^i with $i \geq \frac{k}{2}$.
Finally, we consider the associated graded ring. When r is even, the description given in Example 5.3 follows immediately. For $r \equiv -1 \mod 4$, we observe that the intersection of $F_\gamma^k\mathrm{GW}(\mathbb{P}^r)$ with the primary ideal $(f_r(a))$ is the primary ideal generated by $a^j f_r(a)$, where j is the minimal exponent such that $j+1 \geq \lfloor\frac{k}{2}\rfloor$. Thus,

$$\begin{aligned} \mathrm{gr}_\gamma^{2j}\mathrm{GW}(\mathbb{P}^r) &= \left(F_\gamma^{2j}\mathrm{GW}(k) + (a^j)\right)/\left(F_\gamma^{2j+1}\mathrm{GW}(k) + (a^{j+1}, a^{j-1}f_r(a))\right) \\ &= \left(F_\gamma^{2j+1}\mathrm{GW}(k) + (a^j)\right)/\left(F_\gamma^{2j+2}\mathrm{GW}(k) + (a^{j+1}, a^j b_r)\right) \\ &= \mathrm{gr}_\gamma^{2j}\mathrm{GW}(k) \oplus (\mathbb{Z}/b_r)a^j. \end{aligned}$$

The case $r \equiv 1 \mod 4$ works analogously, the only difference being that we need to take into account an additional factor of 2. In odd degrees, $\mathrm{gr}_\gamma^*\mathrm{GW}(\mathbb{P}^r)$ of course agrees with $\mathrm{gr}_\gamma^*\mathrm{GW}(k)$ for all values of r. □

Calculations for Example 5.4 ($\mathbb{A}^1 - 0$).
The Witt group of the punctured affine line has the form $\mathrm{W}(\mathbb{A}^1 - 0) \cong \mathrm{W}(k) \oplus \mathrm{W}(k)\varepsilon$, where $\varepsilon = (\mathcal{O}, t)$, the trivial line bundle with the symmetric form given by multiplication with the standard coordinate (e. g. [BG05]). It follows that

$$\mathrm{GW}(\mathbb{A}^1 - 0) \cong \mathrm{GW}(k) \oplus \mathrm{W}(k)\hat{\varepsilon},$$

where $\hat{\varepsilon} := \varepsilon - 1$. As for any symmetric line bundle, $\varepsilon^2 = 1$ in the Grothendieck-Witt ring; equivalently, $\hat{\varepsilon}^2 = -2\hat{\varepsilon}$. To compute the γ-filtration, it need only observe that $\mathrm{GW}(\mathbb{A}^1 - 0)$ is generated by line elements. So

$$\begin{aligned} F^i_\gamma \mathrm{GW}(\mathbb{A}^1 - 0) &= \left(F^1_\gamma \mathrm{GW}(\mathbb{A}^1 - 0)\right)^i \\ &= (\mathrm{GI}(k) \oplus \mathrm{W}(k)\hat{\varepsilon})^i \\ &= \mathrm{GI}^i(k) \oplus \mathrm{I}^{i-1}(k)\hat{\varepsilon} \end{aligned}$$

The étale cohomology of $\mathbb{A}^1 - 0$ has the form

$$H^*_{et}(\mathbb{A}^1 - 0, \mathbb{Z}/2) \cong H^*_{et}(k, \mathbb{Z}/2) \oplus H^*_{et}(k, \mathbb{Z}/2) w_1\varepsilon.$$

Recall that when we write $\ker(w_1)$ and $\ker(w_2)$, we necessarily mean the kernels of the restrictions of these maps to $\ker(\mathrm{rank})$ and $\ker(w_1)$, respectively. An arbitrary element of $\mathrm{GW}(\mathbb{A}^1 - 0)$ can be written as $x + y\hat{\varepsilon}$ with $x, y \in \mathrm{GW}(k)$. For such an element, we have $w_1(x + y\hat{\varepsilon}) = w_1 x + \mathrm{rank}(y) w_1\varepsilon$, so the general fact that $\ker(w_1) = F^1_\gamma \mathrm{GW}$ is consistent with our computation. When $\mathrm{rank}(y) = 0$, we further find that

$$w_2(x + y\hat{\varepsilon}) = w_2 x + w_1 y \cup w_1\varepsilon,$$

proving the claim that $\ker(w_2) = F^2_\gamma \mathrm{GW}$ in this example. □

Calculations for Example 5.5 ($\mathbb{A}^{4n+1} - 0$).
Balmer and Gille show in [BG05] that for $d = 4n + 1$ we have $\mathrm{W}(\mathbb{A}^d) \cong \mathrm{W}(k) \oplus \mathrm{W}(k)\varepsilon$ for some symmetric space ε of even rank r such that $\varepsilon^2 = 0$ in the Witt ring. Let $\hat{\varepsilon} := \varepsilon - \frac{r}{2}\mathbb{H}$. Then

$$\mathrm{GW}(\mathbb{A}^d) \cong \mathrm{GW}(k) \oplus \mathrm{W}(k)\hat{\varepsilon}$$

with $\hat{\varepsilon}^2 = 0$. We now switch to the complex numbers. Equipped with the analytic topology, $\mathbb{A}^{4n+1}_{\mathbb{C}}$ is homotopy equivalent to the sphere S^{8n+1}, so we have a comparison map $\mathrm{KO}(S^{8n+1}) \to \mathrm{GW}(\mathbb{A}^{4n+1}_{\mathbb{C}} - 0)$. As the λ-ring structures on both sides are defined via exterior powers, this is clearly a map of λ-rings. We deduce that it is an isomorphism by comparing the localization sequences for $\mathbb{A}^d_{\mathbb{C}} - 0 \hookrightarrow \mathbb{A}^d_{\mathbb{C}} \hookleftarrow \{0\}$, as in the proof of [Zib11, Thm 2.5]. The λ-ring structure on $\mathrm{KO}(S^{8n+1})$ can be deduced from [Ada62, Thm 7.4]: As a special case, the theorem asserts that the projection $\mathbb{RP}^{8n+1} \twoheadrightarrow \mathbb{RP}^{8n+1}/\mathbb{RP}^{8n} \simeq S^{8n+1}$ induces the following map in KO-theory.

$$\begin{array}{ccccc} \mathrm{KO}(S^{8n+1}) & \cong & \mathbb{Z}[\hat{\varepsilon}]/(2\hat{\varepsilon}, \hat{\varepsilon}^2) & & \hat{\varepsilon} \\ \downarrow & & & & \downarrow \\ \mathrm{KO}(\mathbb{RP}^{8n+1}) & \cong & \mathbb{Z}[\hat{\lambda}]/(\hat{\lambda}^2 - 2\hat{\lambda}, \hat{\lambda}^{f+1}) & & \hat{\lambda}^{f-1} \end{array}$$

Here, λ is the canonical line bundle over the real projective space, $\hat{\lambda} := \lambda - 1$, and f is some integer. Thus, $\gamma_t(\hat{\varepsilon}) = \gamma_t(2^{f-1}\hat{\lambda}) = (1+\hat{\lambda})^{2^{f-1}}$ and we find that $\gamma_i(\hat{\varepsilon}) = c_i\hat{\varepsilon}$ for $c_i := \binom{2^{f-1}}{i}2^{i-f}$. Note that c_i is indeed an integer: by Kummer's theorem, we find that the highest power of two dividing $\binom{2^{f-1}}{i}$ is at least $f-1-k$, where k is the highest power of two such that $2^k \leq i$. In fact, modulo two we have $c_2 \equiv 1$ and $c_i \equiv 0$ for all $i > 2$. So the γ-filtration is as described in the example. □

Finally, here is the lemma referred to multiple times above.

5.6 Lemma. *Let $\mathcal{L}$ be a line bundle over a scheme X over $\mathbb{Z}[\frac{1}{2}]$. Then*

$$\gamma_2(H(\mathcal{L}-1)) = -H(\mathcal{L}-1)$$

and $\gamma_i(H(\mathcal{L}-1)) = 0$ in $\mathrm{GW}(X)$ for all $i > 2$.

Proof. Let us write $\lambda_t(x) = 1 + xt + \lambda^2(x)t^2 + \dots$ for the total λ-operation, and similarly for $\gamma_t(x)$. Then $\lambda_t(x+y) = \lambda_t(x)\lambda_t(y)$, $\gamma_t(x+y) = \gamma_t(x)\gamma_t(y)$, and $\gamma_t(x) = \lambda_{\frac{t}{1-t}}(x)$. Let $a := H(\mathcal{L}-1)$. From

$$\lambda_t(a) = \frac{\lambda_t(H\mathcal{L})}{\lambda_t(H1)} = \frac{1 + (H\mathcal{L})t + \det(H\mathcal{L})t^2}{1 + (H1)t + \det(H1)t^2} = \frac{1 + (H\mathcal{L})t + \langle -1\rangle t^2}{1 + (H1)t + \langle -1\rangle t^2}$$

we deduce that

$$\begin{aligned}\gamma_t(a) &= \frac{1 + (H\mathcal{L}-2)t + (1 + \langle -1\rangle - H\mathcal{L})t^2}{1 + (H1-2)t + (1+\langle -1\rangle - H1)t^2} \\ &= \frac{1 + (H\mathcal{L}-2)t - H(\mathcal{L}-1)t^2}{1 + (H1-2)t} \\ &= [1 + (H\mathcal{L}-2)t - H(\mathcal{L}-1)t^2] \cdot \sum_{i\geq 0}(2 - H1)^i t^i.\end{aligned}$$

Here, the penultimate step uses that $H1 \cong 1 + \langle -1\rangle$ when two is invertible. In order to proceed, we observe that $H1 \cdot Hx = H(FH1 \cdot x) = 2Hx$ for any $x \in \mathrm{GW}(X)$. It follows that $(2-H1)^i = 2^{i-1}(2-H1)$ and hence that

$$[1 + (H\mathcal{L}-2)t - H(\mathcal{L}-1)t^2] \cdot (2-H1)^i t^i = 2^{i-1}(2-H1)(1-2t)t^i$$

for all $i \geq 1$. This implies that the above expression for $\gamma_t(a)$ simplifies to $1 + H(\mathcal{L}-1)t - H(\mathcal{L}-1)t^2$, as claimed. □

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