Witt Groups of Complex Varieties

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Declaration of Originality

This dissertation is the result of my own work and includes nothing which is the outcome of work done in collaboration. No part of this dissertation has been submitted for any other qualification.

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Abstract

The thesis *Witt Groups of Complex Varieties* studies and compares two related cohomology theories that arise in the areas of algebraic geometry and topology: the algebraic theory of Witt groups, and real topological K-theory. Specifically, we introduce comparison maps from the Grothendieck-Witt and Witt groups of a smooth complex variety to the KO-groups of the underlying topological space and analyse their behaviour.

We focus on two particularly favourable situations. Firstly, we explicitly compute the Witt groups of smooth complex curves and surfaces. Using the theory of Stiefel-Whitney classes, we obtain a satisfactory description of the comparison maps in these low-dimensional cases. Secondly, we show that the comparison maps are isomorphisms for smooth cellular varieties. This result applies in particular to projective homogeneous spaces. By extending known computations in topology, we obtain an additive description of the Witt groups of all projective homogeneous varieties that fall within the class of hermitian symmetric spaces.

Contents

Introduction

In this thesis, we study and compare two related cohomology theories that arise in the areas of algebraic geometry and topology: the algebraic theory of Witt groups, and real topological K-theory. Specifically, we introduce comparison maps from the Grothendieck-Witt and Witt groups of a smooth complex variety to the KO-groups of the underlying topological space and analyse their behaviour. Satisfactory results are obtained in low dimensions and for cellular varieties.

The set-up is analogous to the situation one finds in K-theory. Given a smooth complex variety X, we have on the one hand the algebraic K-group $\mathrm{K}_0(X)$ and on the other the complex topological K-group $\mathrm{K}^0(X)$. One is defined in terms of algebraic vector bundles over X, the other in terms of complex continuous vector bundles with respect to the analytic topology on X. Moreover, we have a natural map

$$k\colon \mathrm{K}_0(X) \to \mathrm{K}^0(X)$$

that sends an algebraic vector bundle to the underlying continuous bundle. Of course, a given continuous vector bundle may have several different algebraic structures. For example, the bundles $\mathcal{O}(1) \oplus \mathcal{O}(-1)$ and $\mathcal{O} \oplus \mathcal{O}$ over the projective line $\mathbb{P}^1$ are both topologically trivial but they are non-isomorphic as algebraic vector bundles. In this particular case, they nevertheless represent the same equivalence class in the algebraic K-group $\mathrm{K}_0(X)$, and in fact the comparison map k is an isomorphism for $X = \mathbb{P}^1$. But more serious problems already appear on curves of positive genus. Over any such curve, we have an infinite family of line bundles of degree zero, all of which are topologically trivial, but all of which define distinct elements in $\mathrm{K}_0(X)$. It may also happen that a continuous complex vector bundle has no algebraic structure at all. This occurs, for example, over projective surfaces of positive geometric genus. In general, the comparison map k is neither surjective nor injective.

The idea behind Witt groups is to study not simply vector bundles but vector bundles endowed with the additional structure of a symmetric bilinear form. In this context, the analogues of the K-groups are given by the Grothendieck-Witt group $\mathrm{GW}^0(X)$ and the real topological K-group $\mathrm{KO}^0(X)$. Again, we have a comparison map

$$gw^0\colon\ \mathrm{GW}^0(X) \to \mathrm{KO}^0(X)$$

The Witt group $\mathrm{W}^0(X)$ may be defined as the cokernel of a natural map from $\mathrm{K}_0(X)$ to $\mathrm{GW}^0(X)$. Writing $(\mathrm{KO}^0/\,\mathrm{K})(X)$ for the corresponding cokernel in topology, we obtain an induced comparison map

$$w^0\colon\ \mathrm{W}^0(X) \to (\mathrm{KO}^0/\,\mathrm{K})(X)$$

These two comparison maps and their generalizations will be our main objects of study.

In Chapter I we collect background material. Precise definitions of the objects discussed so far can be found there, along with an introduction to Balmer and Walter's shifted Grothendieck-Witt and Witt groups $\mathrm{GW}^i(X)$ and $\mathrm{W}^i(X)$. In particular, we explain in what sense these groups constitute a cohomology theory.

The aim of Chapter II is to generalize the comparison with topology to these shifted groups, so that we obtain maps

$$gw^i\colon\ \mathrm{GW}^i(X) \to \mathrm{KO}^{2i}(X)$$
$$w^i\colon\ \mathrm{W}^i(X) \to (\mathrm{KO}^{2i}/\,\mathrm{K})(X)$$

Two different approaches are discussed. The first is more elementary but has some shortcomings. The second, more elegant approach requires the context of $\mathbb{A}^1$-homotopy theory. A drawback of our construction is that it relies on some recent results that have not hitherto been properly published. Nonetheless, we believe this is ultimately the more promising route to follow, not least because it places the comparison maps in their natural context. It is this approach that we will focus on.

After these preliminaries, we show in Chapter III that the comparison maps behave well in low dimensions. In particular, the maps w^i on the Witt groups are isomorphisms for all smooth complex curves. For a surface X, they are isomorphisms if and only if every continuous complex line bundle over X is algebraic. For example, this is the case for all projective surfaces of geometric genus zero. As a first step towards the proof, we explicitly compute the Witt groups of arbitrary smooth complex curves and surfaces. The behaviour of w^0 is then analysed using the theory of Stiefel-Whitney classes and the usual comparison theorem for étale cohomology with finite coefficients. For the comparison maps on shifted groups, more work is required, and the advantages of the homotopy-theoretic approach to their construction become essential. To round off this chapter, we briefly discuss how the results are related to more general comparison theorems involving Grothendieck-Witt groups with torsion coefficients.

In Chapter IV, we show that the comparison maps gw^i and w^i are isomorphisms for smooth cellular varieties. This result is illustrated with a series of examples to which it applies. Namely, by extending known computations in topology, we obtain a complete additive description of the Witt groups of all projective homogeneous varieties that fall into the class of hermitian symmetric spaces, such as complex Grassmannians, projective quadrics and spinor varieties.

Chapter I

Two Cohomology Theories

In this chapter, we introduce the two main cohomology theories that we will be dealing with: the theory of Witt groups and KO-theory.

Witt groups originated in the study of quadratic forms over fields, but the concept was later extended to rings, varieties and schemes by Knebusch [Kne77]. They may be thought of as variants of K-groups: in the same sense that the K-group of a variety X classifies vector bundles over it, there is a Grothendieck-Witt group $\mathrm{GW}^0(X)$ classifying vector bundles equipped with symmetric forms, of which the Witt group is a quotient. Thus, we begin our discussion by introducing such symmetric bundles and spelling out the necessary definitions. The construction of cohomology theories based on Witt groups is fairly recent. Roughly ten years ago, Balmer gave a purely algebraic construction of a Witt cohomology theory by introducing Witt groups of triangulated categories [Bal00, Bal01a]. Rather than giving a detailed account of the general theory, we concentrate here on those aspects relevant in the context of regular schemes. More comprehensive introductions may be found in [Bal01b] and [Bal05]. We also briefly touch on a more general cohomology theory known as hermitian K-theory, which parallels algebraic K-theory more closely. This theory is currently being developed mainly by Schlichting [HS04, Sch10a, Sch10b, Sch].

The corresponding cohomology theory in topology, known as real topological K-theory or simply as KO-theory, has been well-known and studied since the early work of Atiyah. Its construction, which parallels that of complex topological K-theory perfectly, is recalled in the second half of this chapter. We close with a description of the Atiyah-Hirzebruch spectral sequence, a standard tool in topology that allows us to relate the KO-groups of a space to its singular cohomology.

1 Witt groups

The algebraic K-group $\mathrm{K}_0(X)$ of a scheme X can be defined as the free abelian group on isomorphism classes of vector bundles over X modulo the following relation: for any short exact sequence of vector bundles

$$0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$$

over X, we have $[\mathcal{F}] = [\mathcal{E}] + [\mathcal{G}]$ in $\mathrm{K}_0(X)$. In particular, as far as $\mathrm{K}_0(X)$ is concerned, we may pretend that all exact sequences of vector bundles over X split. Witt groups can be defined similarly, using the notion of symmetric bundles.

1a Symmetric bundles

Let X be a scheme over $\mathbb{Z}[\frac{1}{2}]$. A symmetric bundle $(\mathcal{E}, \varepsilon)$ over X is a vector bundle[1] $\mathcal{E}$ over X equipped with a non-degenerate symmetric bilinear form $\varepsilon\colon \mathcal{E} \otimes \mathcal{E} \to \mathcal{O}$. Alternatively, we may view ε as an isomorphism from $\mathcal{E}$ to its dual bundle $\mathcal{E}^\vee$. In this case, the symmetry of ε is encoded by the fact that it agrees with its dual $\varepsilon^\vee$ under the canonical identification ω of $\mathcal{E}$ with $(\mathcal{E}^\vee)^\vee$. That is, the following triangle commutes:

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\ \varepsilon\ } & \\ \omega \downarrow \cong & & \mathcal{E}^\vee \\ (\mathcal{E}^\vee)^\vee & \xrightarrow{\ \varepsilon^\vee\ } & \end{array} \tag{1}$$

Two symmetric bundles $(\mathcal{E}, \varepsilon)$ and $(\mathcal{F}, \varphi)$ are isometric if there is an isomorphism of vector bundles $i\colon \mathcal{E} \to \mathcal{F}$ compatible with the symmetries, i. e. such that $i^\vee \varphi i = \varepsilon$. The orthogonal sum of two symmetric bundles has the obvious definition $(\mathcal{E}, \varepsilon) \perp (\mathcal{F}, \varphi) := (\mathcal{E} \oplus \mathcal{F}, \varepsilon \oplus \varphi)$.

1.1 Example (Symmetric line bundles). Let $\mathrm{Pic}(X)[2]$ be the subgroup of line bundles of order ≤ 2 in the Picard group $\mathrm{Pic}(X)$. Any line bundle $\mathcal{L} \in \mathrm{Pic}(X)[2]$ defines a symmetric bundle over X. When X is a projective variety over an algebraically closed field, all symmetric line bundles arise in this way. In general, the set of isometry classes of symmetric line bundles over X is described by $H^1_{\mathrm{et}}(X; \mathbb{Z}/2)$, the first étale cohomology group of X with coefficients in $\mathbb{Z}/2 = \mathrm{O}(1)$. The Kummer sequence exhibits this group as an extension of $\mathrm{Pic}(X)[2]$:

$$0 \to \frac{\mathcal{O}^*(X)}{\mathcal{O}^*(X)^2} \to H^1_{\mathrm{et}}(X; \mathbb{Z}/2) \to \mathrm{Pic}(X)[2] \to 0$$

The additional contribution comes from symmetric line bundles of the form $(\mathcal{O}, \varphi)$, where φ is some invertible regular function which has no globally defined square root. For example, the trivial line bundle over the punctured disk $\mathbb{A}^1 - 0$ carries a non-trivial symmetric form given by multiplication with the standard coordinate function.

1.2 Example (Hyperbolic bundles). Any vector bundle $\mathcal{E}$ gives rise to a symmetric bundle $H(\mathcal{E}) := (\mathcal{E} \oplus \mathcal{E}^\vee, (\begin{smallmatrix} 0 & 1 \\ \omega & 0 \end{smallmatrix}))$ over X, the hyperbolic bundle associated with $\mathcal{E}$.

Hyperbolic bundles are the simplest members of the wider class of metabolic bundles. A metabolic bundle is a symmetric bundle $(\mathcal{M}, \mu)$ which contains a Lagrangian, a subbundle

[1] By convention, we identify a vector bundle with its sheaf of sections. Thus, the common notation $\mathcal{O}$ for the sheaf of regular functions on X will be used to denote the trivial line bundle over X.

$j\colon \mathcal{N} \hookrightarrow \mathcal{M}$ of half the rank of $\mathcal{M}$ on which the symmetric form μ vanishes. In other words, $(\mathcal{M}, \mu)$ is metabolic if it fits into a short exact sequence of the form

$$0 \to \mathcal{N} \xrightarrow{j} \mathcal{M} \xrightarrow{j^\vee \mu} \mathcal{N}^\vee \to 0 \tag{2}$$

The sequence splits if and only if $(\mathcal{M}, \mu)$ is isometric to $H(\mathcal{N})$ [Bal05, Example 1.1.21]. This motivates the definition of the Grothendieck-Witt group.

1.3 Definition. [Kne77, § 4] The Grothendieck-Witt group $\mathrm{GW}^0(X)$ of a scheme X over $\mathbb{Z}[\frac{1}{2}]$ is the free abelian group on isometry classes of symmetric bundles over X modulo the following two relations:

- $[(\mathcal{E}, \varepsilon) \perp (\mathcal{G}, \gamma)] = [(\mathcal{E}, \varepsilon)] + [(\mathcal{G}, \gamma)]$
- $[(M, \mu)] = [H(\mathcal{N})]$ for any metabolic bundle (M, μ) with Lagrangian $\mathcal{N}$

The Witt group $\mathrm{W}^0(X)$ is defined similarly, except that the second relation reads $[(M, \mu)] = 0$. Equivalently, we may define $\mathrm{W}^0(X)$ by the exact sequence

$$\mathrm{K}_0(X) \xrightarrow{H} \mathrm{GW}^0(X) \longrightarrow \mathrm{W}^0(X) \to 0$$

In addition to the hyperbolic map H appearing here, we have a forgetful map F in the opposite direction, sending symmetric bundles to their underlying vector bundles:

$$\mathrm{GW}^0(X) \xrightarrow{F} \mathrm{K}_0(X)$$

The most basic invariant of a vector bundle is its rank. For a connected scheme X, it induces the following well-defined homomorphisms on the above groups:

$$\begin{aligned} \mathrm{K}_0(X) &\xrightarrow{\mathrm{rk}} \mathbb{Z} \\ \mathrm{GW}^0(X) &\xrightarrow{\mathrm{rk}} \mathbb{Z} \\ \mathrm{W}^0(X) &\xrightarrow{\mathrm{rk}} \mathbb{Z}/2 \end{aligned}$$

Of course, on the Witt group the rank is only well-defined modulo two, since arbitrary metabolic bundles are equivalent to zero.

1.4 Example. If k is a field (of characteristic not 2), we simply write $\mathrm{K}_0(k)$, $\mathrm{GW}^0(k)$ and $\mathrm{W}^0(k)$ for the corresponding groups of $\mathrm{Spec}(k)$. Since short exact sequences of vector spaces always split, these groups may be defined more directly: if $\mathrm{Vect}(k)$ and $\mathrm{Bil}(k)$ denote the monoids of vector spaces and of symmetric forms over k, then $\mathrm{K}_0(k)$ and $\mathrm{GW}^0(k)$ are simply the Grothendieck groups of $\mathrm{Vect}(k)$ and $\mathrm{Bil}(k)$, respectively, and $\mathrm{W}^0(k)$ may be identified with the quotient $\mathrm{Bil}(k)/(\mathbb{N} \cdot \mathbb{H})$, where $\mathbb{H}$ is the hyperbolic form $\mathbb{H} = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.

Since vector spaces are uniquely determined by their rank, the rank homomorphism on the K-group yields an isomorphism $\mathrm{K}_0(k) \xrightarrow{\cong} \mathbb{Z}$. In contrast, the rank homomorphisms on the Grothendieck-Witt and Witt groups are not isomorphisms in general unless k is algebraically closed. For example, the Witt group of $\mathbb{R}$ can be identified with $\mathbb{Z}$ via the map that sends a real symmetric form to its signature.

Variants. Returning to the world of schemes, we may more generally consider vector bundles equipped with non-degenerate symmetric bilinear forms with values in any fixed line bundle $\mathcal{L}$ over X. Such vector bundles may be viewed as symmetric bundles with respect to the twisted duality

$$\mathcal{E}^{\vee_{\mathcal{L}}} := \mathcal{H}om(\mathcal{E}, \mathcal{L})$$

Under this interpretation, all notions introduced above immediately generalize, leading to the definition of twisted groups $\mathrm{GW}^0(X; \mathcal{L})$ and $\mathrm{W}^0(X; \mathcal{L})$. If $\mathcal{E}$ is symmetric with respect to the twisted duality $\vee_{\mathcal{L}\otimes\mathcal{L}}$, then $\mathcal{E} \otimes \mathcal{L}^\vee$ is symmetric with respect to the usual duality. Thus, these groups depend only on the class of $\mathcal{L}$ in $\mathrm{Pic}(X)/2$.

We could also work with $-\omega$ in place of the usual double-dual identification, leading to the notion of *anti*-symmetric bundles. The corresponding Grothendieck-Witt and Witt groups are denoted by $\mathrm{GW}^2(X; \mathcal{L})$ and $\mathrm{W}^2(X; \mathcal{L})$. The choice of notation will become clear in the next section, where shifted groups $\mathrm{GW}^i(X; \mathcal{L})$ and $\mathrm{W}^i(X; \mathcal{L})$ are introduced for arbitrary integers i.

Witt groups of exact categories

The situation may be formalized by introducing the notion of an exact category with duality $(\mathcal{A}, \vee, \omega)$. An exact category is essentially an additive category $\mathcal{A}$ equipped with an abstract notion of short exact sequences, and a duality is determined by an endofunctor $\vee\colon \mathcal{A}^{\mathrm{op}} \to \mathcal{A}$ together with a natural isomorphism

$$\omega\colon \mathrm{id}_{\mathcal{A}} \xrightarrow{\cong} \vee \circ \vee$$

which one refers to as double-dual identification. These structures are required to satisfy certain compatibility conditions. Precise definitions may be found in [Sch10a, § 2] or [Bal05, Definitions 1.1.1 and 1.1.13]. The notions of symmetric and metabolic objects over an arbitrary exact category with duality $(\mathcal{A}, \vee, \omega)$ are completely analogous to the notions for bundles given above, and they can be used to define the Grothendieck-Witt and Witt groups $\mathrm{GW}^0(\mathcal{A}, \vee, \omega)$ and $\mathrm{W}^0(\mathcal{A}, \vee, \omega)$.

From this point of view, $\mathrm{GW}^0(X)$ and $\mathrm{W}^0(X)$ are simply the Grothendieck-Witt and Witt groups of the exact category $\mathrm{Vect}(X)$ of vector bundles over X equipped with its usual duality and double-dual identification. For the twisted groups, we use the same category equipped with the twisted duality $\vee_{\mathcal{L}}$ and the canonical identification $\omega_{\mathcal{L}}$ of $\mathcal{E}$

with $(\mathcal{E}^{\vee_{\mathcal{L}}})^{\vee_{\mathcal{L}}}$, while for the groups of anti-symmetric bundles we use $\vee_{\mathcal{L}}$ and $-\omega_{\mathcal{L}}$:

$$\mathrm{GW}^0(X;\mathcal{L}) := \mathrm{GW}^0(\mathrm{Vect}(X), \vee_{\mathcal{L}}, \omega_{\mathcal{L}})$$
$$\mathrm{GW}^2(X;\mathcal{L}) := \mathrm{GW}^0(\mathrm{Vect}(X), \vee_{\mathcal{L}}, -\omega_{\mathcal{L}})$$

1b Symmetric complexes

The notions of duality and symmetry may be generalized to complexes. If we fix an exact category with duality $(\mathcal{A}, \vee, \omega)$, then the usual dual of a complex

$$\mathcal{E}_\bullet\colon \qquad \cdots \to \mathcal{E}_2 \xrightarrow{d_2} \mathcal{E}_1 \xrightarrow{d_1} \mathcal{E}_0 \xrightarrow{d_0} \mathcal{E}_{-1} \xrightarrow{d_{-1}} \mathcal{E}_{-2} \to \cdots$$

over $\mathcal{A}$ is defined term-by-term, i. e. $\mathcal{E}_\bullet^\vee$ is given by

$$\mathcal{E}_\bullet^\vee\colon \qquad \cdots \to \mathcal{E}_{-2}^\vee \xrightarrow{d_{-1}^\vee} \mathcal{E}_{-1}^\vee \xrightarrow{d_0^\vee} \mathcal{E}_0^\vee \xrightarrow{d_1^\vee} \mathcal{E}_1^\vee \xrightarrow{d_2^\vee} \mathcal{E}_2^\vee \to \cdots$$

with $\mathcal{E}_0^\vee$ placed in degree zero. Likewise, the dual of a morphism $\varphi_\bullet\colon \mathcal{E}_\bullet \to \mathcal{F}_\bullet$ is given by $(\varphi^\vee)_l := \varphi_{-l}^\vee$.

Alternative dualities may be defined by composing with the shift functor, as follows. By convention, the shifted complex $\mathcal{E}_\bullet[1]$ is obtained from $\mathcal{E}_\bullet$ by moving all terms by one position to the left and changing the signs of the differentials, while a morphism φ is shifted without any sign changes. Thus, $(\varphi[1])_l = \varphi_{l-1}$, and $\mathcal{E}_\bullet[1]$ takes the form

$$\mathcal{E}_\bullet[1]\colon \qquad \cdots \to \mathcal{E}_1 \xrightarrow{-d_1} \mathcal{E}_0 \xrightarrow{-d_0} \mathcal{E}_{-1} \xrightarrow{-d_{-1}} \mathcal{E}_{-2} \xrightarrow{-d_{-2}} \mathcal{E}_{-3} \to \ldots$$

with $\mathcal{E}_{-1}$ in degree zero [Wei94, 1.2.8]. Shifts $[i]$ for arbitrary integers i are obtained by iterating this construction or its inverse. Thus, for any integer i, we can define a shifted duality $\vee_i$ on complexes by

$$\mathcal{E}_\bullet^{\vee_i} := (\mathcal{E}_\bullet^\vee)[i]$$

The double-dual identification ω on $\mathcal{A}$ induces a canonical identification of $\mathcal{E}_\bullet$ with $(\mathcal{E}_\bullet^{\vee_i})^{\vee_i}$, which we denote by ω_i.

1.5 Definition. An i-symmetric complex $(\mathcal{E}_\bullet, \varepsilon)$ is a bounded complex $\mathcal{E}_\bullet$ together with a quasi-isomorphism

$$\varepsilon\colon \mathcal{E}_\bullet \xrightarrow{\simeq} \mathcal{E}_\bullet^{\vee_i}$$

which is symmetric with respect to $\vee_i$ and the double-dual identification $(-1)^{\frac{i(i+1)}{2}} \cdot \omega_i$. In other words, ε satisfies the condition

$$\varepsilon = (-1)^{\frac{i(i+1)}{2}} \cdot \varepsilon^{\vee_i} \circ \omega_i$$

Because the double-dual identification is modified by a sign, this definition generalizes at the same time the notions of symmetric and anti-symmetric objects: while symmetric objects over $\mathcal{A}$ may be viewed as 0-symmetric complexes concentrated in degree zero, anti-symmetric objects may be viewed as 2-symmetric complexes concentrated in degree one. Given any i-symmetric complex $(\mathcal{E}_\bullet, \varepsilon)$, its two-fold shift $(\mathcal{E}_\bullet[2], \varepsilon[2])$ is $(i+4)$-symmetric. Thus, the essential aspects of i-symmetry depend only on the value of i modulo 4.

Simple examples of i-symmetric complexes for arbitrary i are given by hyperbolic complexes, i. e. complexes of the form $H^i(\mathcal{E}) := (\mathcal{E} \oplus \mathcal{E}^{\vee_i}, \left(\begin{smallmatrix} 0 & 1 \\ (-1)^{\frac{i(i+1)}{2}}\omega_i & 0 \end{smallmatrix}\right))$. A less trivial example is the following.

1.6 Example (A 1-symmetric complex over $\mathbb{P}^1$). Consider the complex of vector bundles $\mathcal{O}(-1) \xrightarrow{\cdot x} \mathcal{O}$ over the projective line $\mathbb{P}^1$ with coordinates $[x : y]$. Place $\mathcal{O}$ in degree zero. Multiplication by y induces a symmetric quasi-isomorphism with the dual complex shifted one to the left, so that we obtain a 1-symmetric complex

$$\Psi_0 := \left(\begin{array}{ccc} \mathcal{O}(-1) & \xrightarrow{\cdot x} & \mathcal{O} \\ \downarrow{\scriptstyle \cdot y} & & \downarrow{\scriptstyle \cdot(-y)} \\ \mathcal{O} & \xrightarrow{\cdot(-x)} & \mathcal{O}(1) \end{array} \right) \tag{3}$$

Analogues of this 1-symmetric complex over projective curves of higher genus are described in Remark III.2.2.

Witt groups of triangulated categories

The category $\mathcal{C}h^b(\mathcal{A})$ of bounded complexes over $\mathcal{A}$ equipped with the exact structure inherited from $\mathcal{A}$ and any of the dualities defined above is again an exact category with duality. However, i-symmetric complexes are not symmetric objects over $(\mathcal{C}h^b(\mathcal{A}), \vee_i, \pm\omega_i)$ in the above sense, since the symmetries are only required to be quasi-isomorphisms. On the other hand, the derived category $\mathcal{D}^b(\mathcal{A})$ obtained from $\mathcal{C}h^b(\mathcal{A})$ by formally inverting all quasi-isomorphisms is no longer exact but rather triangulated.

One is thus led to develop the above theory in the context of triangulated categories. The definition of the K-group of a triangulated category is straightfoward: one simply replaces the short exact sequence in the definition of K_0 by an exact triangle. For any exact category $\mathcal{A}$, we then have $\mathrm{K}_0(\mathcal{D}^b(\mathcal{A})) \cong \mathrm{K}_0(\mathcal{A})$. So this is a perfect generalization.

Witt groups of triangulated categories are developed by Balmer in [Bal00] and [Bal01a]. Naturally enough, the idea is to replace the short exact sequence (2) defining metabolic objects by an appropriate exact triangle. Once the notions of a triangulated category with duality $(\mathcal{D}, \vee, \omega)$ and its Witt group $\mathrm{W}(\mathcal{D}, \vee, \omega)$ are settled, the Witt groups of i-symmetric complexes over an exact category with duality $(\mathcal{A}, \vee, \omega)$ may be defined as

$$\mathrm{W}^i(\mathcal{A}, \vee, \omega) := \mathrm{W}(\mathcal{D}^b(\mathcal{A}), \vee_i, (-1)^{\frac{i(i+1)}{2}}\omega_i)$$

These groups are known as the shifted Witt groups of $(\mathcal{A}, \vee, \omega)$. They are 4-periodic in i, in the sense that we have canonical isomorphisms $\mathrm{W}^i(\mathcal{A}, \vee, \omega) = \mathrm{W}^{i+4}(\mathcal{A}, \vee, \omega)$. For $i = 0$ and $i = 2$, we recover the Witt groups of symmetric and anti-symmetric objects over $\mathcal{A}$ defined above, at least when 2 is invertible in $\mathcal{A}$ (i. e. when all homomorphism groups of $\mathcal{A}$ are uniquely 2-divisible) [Bal01a, Theorem 4.3; BW02, Theorem 1.4].

As Walter notes in [Wal03a], Balmer's approach already works on the level of Grothendieck-Witt groups, so that we also have shifted groups $\mathrm{GW}^i(\mathcal{A}, \vee, \omega)$. Just like the usual Grothendieck-Witt groups, these are equipped with hyperbolic and forgetful maps. They appear in the following exact sequences [Wal03a, Theorem 2.6]:

$$\mathrm{GW}^{i-1}(\mathcal{A}, \vee, \omega) \xrightarrow{F} \mathrm{K}_0(\mathcal{A}) \xrightarrow{H^i} \mathrm{GW}^i(\mathcal{A}, \vee, \omega) \longrightarrow \mathrm{W}^0(\mathcal{A}, \vee, \omega) \to 0$$

Not only do these generalize the exact sequence of Definition 1.3 but they also extend it by one term to the left.

1c Witt groups of schemes

The shifted Witt groups of a scheme X are, of course, defined in terms of the exact category $(\mathrm{Vect}(X), \vee, \omega)$, or more generally in terms of $(\mathrm{Vect}(X), \vee_{\mathcal{L}}, \omega_{\mathcal{L}})$ for any line bundle $\mathcal{L}$ over X. The essence of the previous sections may be summarized as follows.

- For any scheme X over $\mathbb{Z}[\frac{1}{2}]$, any line bundle $\mathcal{L}$ over X and any integer i, we have groups

$$\mathrm{GW}^i(X; \mathcal{L}) := \mathrm{GW}^i(\mathrm{Vect}(X), \vee_{\mathcal{L}}, \omega_{\mathcal{L}})$$
$$\mathrm{W}^i(X; \mathcal{L}) := \mathrm{W}^i(\mathrm{Vect}(X), \vee_{\mathcal{L}}, \omega_{\mathcal{L}})$$

  which generalize the Witt groups of symmetric and anti-symmetric bundles discussed earlier [Bal01a, Theorem 4.7]. They are referred to as shifted (Grothendieck-)Witt groups of X with coefficients in $\mathcal{L}$, or as such groups twisted by $\mathcal{L}$. When $\mathcal{L}$ is trivial, it is frequently dropped from the notation.

- The groups are four-periodic in i and two-periodic in $\mathcal{L}$. That is, for any X, any i and arbitrary line bundles $\mathcal{L}$ and $\mathcal{M}$ over X, we have canonical isomorphisms

$$\mathrm{GW}^i(X; \mathcal{L}) \cong \mathrm{GW}^{i+4}(X; \mathcal{L})$$
$$\mathrm{GW}^i(X; \mathcal{L}) \cong \mathrm{GW}^i(X; \mathcal{L} \otimes \mathcal{M}^{\otimes 2})$$

  and similarly for the Witt groups.

- We have hyperbolic maps $H^i_{\mathcal{L}} \colon \mathrm{K}_0(X) \to \mathrm{GW}^i(X; \mathcal{L})$ and forgetful maps F in the opposite direction. They fit into exact sequences of the form

$$\mathrm{GW}^{i-1}(X;\mathcal{L}) \xrightarrow{F} \mathrm{K}_0(X) \xrightarrow{H^i_{\mathcal{L}}} \mathrm{GW}^i(X;\mathcal{L}) \longrightarrow \mathrm{W}^i(X;\mathcal{L}) \to 0 \tag{4}$$

These sequences are known as Karoubi sequences.

- It follows easily from the definitions that all these constructions are natural. That is, for any morphism of schemes $f\colon X' \to X$ and any line bundle $\mathcal{L}$ over X, we have induced maps

$$\begin{aligned} \mathrm{K}_0(X) &\xrightarrow{f^*} \mathrm{K}_0(X') \\ \mathrm{GW}^i(X;\mathcal{L}) &\xrightarrow{f^*} \mathrm{GW}^i(X';f^*\mathcal{L}) \\ \mathrm{W}^i(X;\mathcal{L}) &\xrightarrow{f^*} \mathrm{W}^i(X';f^*\mathcal{L}) \end{aligned}$$

  which allow us to view $\mathrm{K}_0(-)$, $\mathrm{GW}^i(-;-)$ and $\mathrm{W}^i(-;-)$ as contravariant functors from schemes with line bundles to abelian groups. The maps f^* commute with the maps in the Karoubi sequences.

1.7 Example (Witt groups of a geometric point). Let $p = \mathrm{Spec}(k)$, where k is an algebraically closed field of characteristic not 2. Its Grothendieck-Witt and Witt groups are as follows:

$$\begin{aligned} \mathrm{GW}^0(p) &= \mathbb{Z} & \mathrm{W}^0(p) &= \mathbb{Z}/2 \\ \mathrm{GW}^1(p) &= 0 & \mathrm{W}^1(p) &= 0 \\ \mathrm{GW}^2(p) &= \mathbb{Z} & \mathrm{W}^2(p) &= 0 \\ \mathrm{GW}^3(p) &= \mathbb{Z}/2 & \mathrm{W}^3(p) &= 0 \end{aligned}$$

The classical Grothendieck-Witt group $\mathrm{GW}^0(p)$ is generated by the symmetric bundle $(\mathcal{O}, \mathrm{id})$, while $\mathrm{GW}^2(p)$ is generated by the anti-symmetric bundle $(\mathcal{O}\oplus\mathcal{O}, \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right))$. The group $\mathrm{GW}^3(p)$ is generated by the hyperbolic bundle $H^3(\mathcal{O})$. The hyperbolic bundle $H^1(\mathcal{O})$ is trivial in $\mathrm{GW}^1(p)$ because it shares its Lagrangian $\mathcal{O}$ with the following 1-symmetric metabolic complex, which is exact:

$$\begin{array}{ccc} \mathcal{O} & \xrightarrow{\mathrm{id}} & \mathcal{O} \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle -\mathrm{id}} \\ \mathcal{O} & \xrightarrow{-\mathrm{id}} & \mathcal{O} \end{array}$$

It follows from the naturality property that the groups of an arbitrary scheme X contain the groups of a point as direct summands. Namely, the inclusion of a point $j\colon p \hookrightarrow X$ and the projection of X onto p induce decompositions

$$\begin{aligned} \mathrm{K}_0(X) &= \mathbb{Z} \oplus \widetilde{\mathrm{K}}_0(X) \\ \mathrm{GW}^i(X) &= \mathrm{GW}^i(p) \oplus \widetilde{\mathrm{GW}}^i(X) \\ \mathrm{W}^i(X) &= \mathrm{W}^i(p) \oplus \widetilde{\mathrm{W}}^i(X) \end{aligned}$$

in which $\widetilde{\mathrm{K}}^0(X)$, $\widetilde{\mathrm{GW}}^i(X)$ and $\widetilde{\mathrm{W}}^i(X)$ denote the kernels of j^* on the respective groups. When X is connected, these kernels are independent of the choice of p, and we refer to them as the reduced groups of X. Over an algebraically closed field, the pullback maps j^* on $\mathrm{K}^0(X)$, $\mathrm{GW}^0(X)$ and $\mathrm{W}^0(X)$ can be identified with the rank homomorphisms.

1.8 Example (Witt groups of $\mathbb{P}^1$ [Ara80, Wal03b]). For the projective line $\mathbb{P}^1$ over k, where k is as in the previous example, one finds that

$$\begin{aligned}
\mathrm{GW}^0(\mathbb{P}^1) &= [\mathbb{Z}] \oplus \mathbb{Z}/2 & \qquad \mathrm{W}^0(\mathbb{P}^1) &= [\mathbb{Z}/2]\\
\mathrm{GW}^1(\mathbb{P}^1) &= \mathbb{Z} & \mathrm{W}^1(\mathbb{P}^1) &= \mathbb{Z}/2\\
\mathrm{GW}^2(\mathbb{P}^1) &= [\mathbb{Z}] & \mathrm{W}^2(\mathbb{P}^1) &= 0\\
\mathrm{GW}^3(\mathbb{P}^1) &= [\mathbb{Z}/2] \oplus \mathbb{Z} & \mathrm{W}^3(\mathbb{P}^1) &= 0
\end{aligned}$$

Here, the summands in square brackets are those generated by constant bundles, i. e. those that disappear when passing to reduced groups. The groups $\mathrm{GW}^1(\mathbb{P}^1)$ and $\mathrm{W}^1(\mathbb{P}^1)$ are generated by the 1-symmetric complex Ψ_0 given in Example 1.6. Grothendieck-Witt and Witt groups of projective spaces of arbitrary dimensions are discussed in IV.3b.

In general, when working over an algebraically closed ground field, the Witt groups are always 2-torsion: $2\Psi = 0$ for any $\Psi \in \mathrm{W}^i(X;\mathcal{L})$. This is a consequence of the following well-known lemma.

1.9 Lemma. *Let X be a scheme over an algebraically closed field k of characteristic not 2. Then, for any $\Psi \in \mathrm{GW}^i(X;\mathcal{L})$, we have $H^i_{\mathcal{L}}(F(\Psi)) \cong 2\Psi$.*

Proof. We may assume that Ψ is the class of some i-symmetric complex $(\mathcal{E}_\bullet, \varepsilon)$. Using the assumption that $\mathrm{char}(k) \neq 2$, we define an isometry between the hyperbolic complex $H^i_{\mathcal{L}}(\mathcal{E}_\bullet)$ and the direct sum $(\mathcal{E}_\bullet, \varepsilon) \oplus (\mathcal{E}_\bullet, -\varepsilon)$ by $\frac{1}{2}\begin{pmatrix} 1 & \varepsilon^{-1} \\ 1 & -\varepsilon^{-1}\end{pmatrix}$. Since k contains a square root of -1, the symmetric complex $(\mathcal{E}_\bullet, -\varepsilon)$ is isometric to $(\mathcal{E}_\bullet, \varepsilon)$. □

Witt groups of regular schemes

One of the key features of Balmer's triangulated approach is that it elevates the theory of Witt groups into the realm of cohomology theories. This assertion is filled with meaning by the following paragraphs. By convention, a regular scheme will be a *regular, noetherian, separated scheme over* $\mathbb{Z}[\frac{1}{2}]$. In order to avoid the regularity assumption, we would have to work with Witt groups defined in terms of coherent sheaves rather than vector bundles.

Localization sequences. Let X be a regular scheme, in the above sense. Then, for any open subscheme $U \subset X$, we have a long exact sequence relating the Witt groups of X to those of U. These sequences are known as localization sequences. If we denote the closed complement of U by Z and the open inclusion by $j\colon U \hookrightarrow X$, the assoicated sequence can be written as follows:

$$\cdots \to \mathrm{W}^i_Z(X;\mathcal{L}) \to \mathrm{W}^i(X;\mathcal{L}) \xrightarrow{j^*} \mathrm{W}^i(U;\mathcal{L}|_U) \to \mathrm{W}^{i+1}_Z(X;\mathcal{L}) \to \cdots \tag{5}$$

The groups $\mathrm{W}^i_Z(X;\mathcal{L})$ appearing here are the Witt groups of X with support on Z. The terminology comes from the fact that they are defined in terms of complexes whose cohomology is supported on Z.

The groups with support on $Z = X$ agree with the usual Witt groups of X. When Z is a smooth closed subvariety of a smooth quasi-projective variety X, we have a dévissage or Thom isomorphism relating the Witt groups of X with support on Z to the Witt groups of Z. The precise form of these isomorphisms depends not only on the codimension c, but also on the normal bundle $\mathcal{N}$ of Z in X [Nen07, § 4]:

$$\mathrm{W}^{i-c}(Z;\mathcal{L}|_Z \otimes \det \mathcal{N}) \xrightarrow{\cong} \mathrm{W}^i_Z(X;\mathcal{L}) \tag{6}$$

Thus, in the above situation, the localization sequence may be rewritten purely in terms of the Witt groups of X, U and Z.

By periodicity, the localization sequences may be arranged as exact polygons with twelve vertices. We also have localization sequences involving Grothendieck-Witt groups, of the following form [Wal03a, Theorem 2.4]:

$$\begin{aligned} &\mathrm{GW}^i_Z(X) \to \mathrm{GW}^i(X) \to \mathrm{GW}^i(U) \\ &\qquad \to \mathrm{W}^{i+1}_Z(X) \to \mathrm{W}^{i+1}(X) \to \mathrm{W}^{i+1}(U) \to \mathrm{W}^{i+2}_Z(X) \to \cdots \end{aligned} \tag{7}$$

These sequences agree with the localization sequences for Witt groups from the fourth term onwards, and may of course again be defined for arbitrary twists $\mathcal{L}$. However, if one wishes to continue the sequences to the left, one has to revert to the methods of higher algebraic K-theory. This is discussed very briefly in Section 1d.

Excision. Let $f\colon X' \to X$ be a morphism of regular schemes. Given a closed subscheme $Z \subset X$, write Z' for its preimage $X' \times_X Z$ under f. If f is flat and maps Z' isomorphically to Z, then f induces isomorphisms of Witt groups with support [Bal01b, Corollary 2.3]:

$$f^*\colon\ \mathrm{W}^i_Z(X) \xrightarrow{\cong} \mathrm{W}^i_{Z'}(X') \tag{8}$$

In particular, if Z is contained in an open subscheme U of X, then $\mathrm{W}^i_Z(X) \cong \mathrm{W}^i_Z(U)$.

Mayer-Vietoris sequences. The existence of localization sequences and the excision property imply the exactness of Mayer-Vietoris sequences. That is, given a covering of a regular scheme X by two open subschemes U and V, we have a long exact sequence of the following form [Bal01b, Theorem 2.5]:

$$\cdots \to \mathrm{W}^i(X) \to \mathrm{W}^i(U) \oplus \mathrm{W}^i(V) \to \mathrm{W}^i(U \cap V) \to \mathrm{W}^{i+1}(X) \to \cdots$$

Homotopy invariance. If E is the total space of a vector bundle over a regular scheme X, then the projection $\pi\colon E \to X$ induces isomorphisms of Witt groups:

$$\pi^*\colon \mathrm{W}^i(X) \xrightarrow{\cong} \mathrm{W}^i(E)$$

More generally, this holds for any flat affine morphism of regular schemes $\pi\colon E \to X$ whose fibres are affine spaces [Gil03, Corollary 4.2].

The properties of Witt groups discussed also hold on the level of Grothendieck-Witt groups. This may be deduced in each case from the corresponding properties of K-groups and the Karoubi sequences (4), using Karoubi induction (c. f. the proof of Proposition II.1.3 in the next chapter).

Multiplication. Finally, we mention the multiplicative structure on shifted Witt groups. In the same way that the tensor product of vector bundles or complexes over X induces a ring structure on $\mathrm{K}_0(X)$, we have a ring structure on $\mathrm{GW}^0(X)$ and $\mathrm{W}^0(X)$ induced by the tensor product of symmetric bundles. More generally, in [GN03] Gille and Nenashev develop pairings between the shifted Witt groups of X, of the following form:

$$\star\colon \mathrm{W}^i_Z(X;\mathcal{L}) \otimes \mathrm{W}^j_{Z'}(X;\mathcal{M}) \to \mathrm{W}^{i+j}_{Z\cap Z'}(X;\mathcal{L}\otimes\mathcal{M})$$

Again, this product may be lifted to Grothendieck-Witt groups [Wal03a, end of § 2]. It is graded-commutative in the sense that

$$\Psi_i \star \Psi_j = (\mathcal{O}_X, -\mathrm{id})^{ij} \star \Psi_j \star \Psi_i$$

for $\Psi_i \in \mathrm{W}^i_Z(X;\mathcal{L})$ and $\Psi_j \in \mathrm{W}^j_{Z'}(X;\mathcal{M})$. Over an algebraically closed field, $(\mathcal{O}_X, -\mathrm{id})$ is isometric to the trivial symmetric bundle $(\mathcal{O}_X, \mathrm{id})$, acting as the unit, so the product becomes honestly commutative.

As usual, this "internal" product gives rise to an "external" product between the (Grothendieck-)Witt groups of two different schemes X and Y: if $\mathcal{L}$ and $\mathcal{M}$ are line bundles over X and Y, respectively, we have a cross product

$$\times\colon \mathrm{W}^i_Z(X;\mathcal{L}) \otimes \mathrm{W}^j_W(Y;\mathcal{M}) \to \mathrm{W}^{i+j}_{Z\times W}(X\times Y;\pi_X^*\mathcal{L}\otimes\pi_Y^*\mathcal{M})$$

Here, π_X and π_Y denote the respective projections from $X \times Y$ to X and Y. For $\Psi \in \mathrm{W}^i_Z(X;\mathcal{L})$ and $\Phi \in \mathrm{W}^j_W(Y;\mathcal{M})$, the product is defined as $\Psi \times \Phi := \pi_X^*(\Psi) \star \pi_Y^*(\Phi)$.

1d Hermitian K-theory

The algebraic K-group of a scheme fits into a family of higher algebraic K-groups $\mathrm{K}_n(X)$. In general, however, there seems to be no purely algebraic description of the higher groups. Rather, following Quillen [Qui73], one defines the higher K-groups of a scheme X as the homotopy groups of a topological space $\mathrm{K}(X)$ associated with X. By working with an appropriate spectrum in place of $\mathrm{K}(X)$, one may further define groups $\mathrm{K}_n(X)$ in all degrees $n \in \mathbb{Z}$. For a regular scheme, however, the groups in negative degrees vanish [TT90, Proposition 6.8].

An analogous construction for Grothendieck-Witt groups, usually referred to as (higher) hermitian K-theory, is developed in [Sch10b, Section 10]. Given a scheme X and a line bundle $\mathcal{L}$ over it, Schlichting constructs a family of spectra $\mathbb{G}\mathrm{W}^i(X;\mathcal{L})$ from which hermitian K-groups can be defined as

$$\mathrm{GW}^i_n(X;\mathcal{L}) := \pi_n(\mathbb{G}\mathrm{W}^i(X;\mathcal{L}))$$

These groups satisfy many properties analogous to those discussed in the case of Witt groups above. The localization sequences now have the following form [Sch10b, Theorem 14]:

$$\cdots \to \mathrm{GW}^i_{n,Z}(X) \to \mathrm{GW}^i_n(X) \to \mathrm{GW}^i_n(U) \to \mathrm{GW}^i_{n-1,Z}(X) \to \cdots \tag{9}$$

In degree $n = 0$, one recovers Walter's Grothendieck-Witt groups, while Balmer's Witt groups appear as hermitian K-groups in negative degrees. More precisely, for any regular scheme X, we have natural identifications

$$\begin{aligned} \mathrm{GW}^i_0(X;\mathcal{L}) &\cong \mathrm{GW}^i(X;\mathcal{L}) \\ \mathrm{GW}^i_n(X;\mathcal{L}) &\cong \mathrm{W}^{i-n}(X;\mathcal{L}) \quad \text{for } n < 0 \end{aligned} \tag{10}$$

These identifications are due to appear in full generality in [Sch]. For affine varieties, the identifications of Witt groups may be found in [Hor05]: see Proposition A.4 and Corollary A.5. For a general regular scheme X, we can pass to a vector bundle torsor T over X such that T is affine [Jou73, Lemma 1.5; Hor05, Lemma 2.1].[1] By homotopy invariance, the Witt groups of T may be identified with those of X, and the same is true for the hermitian K-groups: in this case, homotopy invariance follows from homotopy invariance in the affine case, as shown in [Hor05, Corollary 1.12], and the Mayer-Vietoris sequences established in [Sch10b, Theorem 1].

[1] This step is known as Jouanolou's trick.

2 KO-theory

We now turn to the corresponding cohomology theories in topology. These are known as complex and real topological K-theory, or simply as K- and KO-theory. To ensure that the definitions given here are consistent with the literature, we restrict our attention to finite-dimensional CW complexes.[1] Since we are ultimately only interested in topological spaces that arise as complex varieties, this is not a problem.

So let X be such a CW complex. If we imitate the definitions of $\mathrm{K}_0(X)$ and $\mathrm{GW}^0(X)$ given in Section 1a, using continuous complex vector bundles over X in place of algebraic vector bundles, we obtain the complex and real topological K-groups $\mathrm{K}^0(X)$ and $\mathrm{KO}^0(X)$. However, since short exact sequences of vector bundles over CW complexes always split, the definitions may be simplified:

2.1 Definition. For a finite-dimensional CW complex X, the complex topological K-group $\mathrm{K}^0(X)$ is the free abelian group on isomorphism classes of continuous complex vector bundles over X modulo the relation $[\mathcal{E} \oplus \mathcal{G}] = [\mathcal{E}] + [\mathcal{G}]$. Similarly, the KO-group $\mathrm{KO}^0(X)$ is the free abelian group on isometry classes of continuous complex vector bundles equipped with non-degenerate symmetric bilinear forms over X, modulo the relation $[(\mathcal{E}, \varepsilon) \perp (\mathcal{G}, \gamma)] = [(\mathcal{E}, \varepsilon)] + [(\mathcal{G}, \gamma)]$.

In the following, continuous vector bundles will be referred to simply as vector bundles, and a vector bundle equipped with a non-degenerate symmetric bilinear form will be referred to as a symmetric bundle.

As in the algebraic case, we may view K^0 and KO^0 as contravariant functors, defined in this case on the category of finite-dimensional CW complexes and continuous maps between them. Also, we again have natural transformations from K^0 to KO^0 and vice versa, corresponding to the hyperbolic and the forgetful map. These are traditionally referred to as "realification" and "complexification", and written as

$$r\colon\ \mathrm{K}^0(X) \to \mathrm{KO}^0(X)$$
$$c\colon\ \mathrm{KO}^0(X) \to \mathrm{K}^0(X)$$

The terminology is explained in the next section.

2a Real versus symmetric bundles

There is a more common description of $\mathrm{KO}^0(X)$ as the K-group of continuous real vector bundles over X. The equivalence with the definition given here can be traced back to the fact that the orthogonal group $\mathrm{O}(n)$ is a maximal compact subgroup of both $\mathrm{GL}_n(\mathbb{R})$ and $\mathrm{O}_n(\mathbb{C})$. Alternatively, this equivalence may be seen concretely along the following lines.

[1] The key property we need is that any vector bundle over a finite-dimensional CW complex has a stable inverse. See the proof of Theorem 2.7.

We say that a complex bilinear form ε on a (real or complex) vector bundle $\mathcal{F}$ is real if $\varepsilon\colon \mathcal{F}\otimes\mathcal{F}\to\mathbb{C}$ factors through $\mathbb{R}$.

2.2 Lemma. *Any complex symmetric bundle $(\mathcal{E},\varepsilon)$ has a unique real subbundle $\Re(\mathcal{E},\varepsilon)\subset\mathcal{E}$ such that $\Re(\mathcal{E},\varepsilon)\otimes_{\mathbb{R}}\mathbb{C}=\mathcal{E}$ and such that the restriction of ε to $\Re(\mathcal{E},\varepsilon)$ is real and positive definite. Concretely, a fibre of $\Re(\mathcal{E},\varepsilon)$ is given by the real span of any orthonormal basis of the corresponding fibre of $\mathcal{E}$.*

To prove this lemma, we need the following fact:

2.3 Lemma. *Any complex symmetric bundle $(\mathcal{F},\varphi)$ over a topological space X is locally trivial. That is, any point of X has an open neighbourhood over which $(\mathcal{F},\varphi)$ is isometric to the trivial symmetric bundle $(\mathbb{C}^{\mathrm{rk}\mathcal{F}},\mathrm{id})$.*

Proof. Let p be an arbitrary point of X. To prove the claim, we may assume without loss of generality that $\mathcal{F}$ is trivial as a complex vector bundle, and that the fibre of $(\mathcal{F},\varphi)$ at p is the trivial symmetric space

$$\left(\mathbb{C}^r,\begin{pmatrix}1 & & 0\\ & \ddots & \\ 0 & & 1\end{pmatrix}\right)$$

Let $e_1,\ldots,e_r$ be the sections of the trivial bundle $\mathbb{C}^r$ that provide the standard basis at each fibre. Since φ is orthonormal at p, we may find an open neighbourhood of p on which the deviation of φ from orthonormality is arbitrarily small. That is, for any positive $\epsilon\in\mathbb{R}$, we may find an open neighbourhood of p over which

$$\|\varphi(e_i,e_j)-\delta_{ij}\|<\epsilon \tag{$*$}$$

for all i and j, where $\|\cdot\|$ denotes the standard hermitian metric and δ_{ij} is the Kronecker delta. The lemma follows from the existence of a Gram-Schmidt process for such "almost orthonormal" symmetric bilinear forms. More precisely, we have the following statement:

> Let φ be a non-degenerate symmetric bilinear form on an r-dimensional complex vector space with basis $e_1,\ldots,e_r$. Suppose that $(*)$ holds for some sufficiently small ϵ. Then there exists another basis $\widetilde{e_1},\ldots,\widetilde{e_r}$ which is orthonormal with respect to φ, such that each $\widetilde{e_i}$ may be expressed by a formula depending continuously on the coefficients $\varphi(e_i,e_j)$ of φ.

For lack of reference, we include the following proof of this statement. We take "sufficiently small" to mean that $\epsilon\le\frac{1}{2^{r+1}r!}$. Let $\sqrt{\cdot}$ denote the continuous endofunction of $\{z\in\mathbb{C}\mid\Re(z)>0\}$ that assigns to a complex number with positive real part that square root whose real part is also positive. The orthonormal basis $\widetilde{e}_1,\ldots,\widetilde{e}_r$ is constructed inductively, starting with

$$\widetilde{e_1}:=\frac{e_1}{\sqrt{\varphi(e_1,e_1)}}$$

By definition, $\varphi(\widetilde{e_1},\widetilde{e_1})=1$. For $l>1$, we have $\|\varphi(\widetilde{e}_1,e_l)\|<2\epsilon$.

Now, suppose we have already constructed $\widetilde{e_1}, \ldots, \widetilde{e_i}$ satisfying the following two conditions:

$$\varphi(\widetilde{e}_j, \widetilde{e}_k) \;= \delta_{jk} \qquad \text{for all } j,k \leq i \tag{C1}$$

$$\|\varphi(\widetilde{e}_j, e_l)\| < 2^i i! \cdot \epsilon \qquad \text{for all } j \leq i \text{ and } l > i \tag{C2}$$

Set $e'_{i+1} := e_{i+1} - \sum_{k=1}^{i} \varphi(e_{i+1}, \widetilde{e}_k)\widetilde{e}_k$. Then $\varphi(e'_{i+1}, \widetilde{e}_k) = 0$ for all $k \leq i$, and

$$\varphi(e'_{i+1}, e'_{i+1}) = \varphi(e_{i+1}, e_{i+1}) - \sum_{k=1}^{i} \varphi(e_{i+1}, \widetilde{e}_k)^2$$

A crude estimation shows that

$$\|\varphi(e'_{i+1}, e'_{i+1}) - 1\| < \|\varphi(e_{i+1}, e_{i+1}) - 1\| + \sum_{k=1}^{i} \|\varphi(e_{i+1}, \widetilde{e}_k)\|^2 < \epsilon + i(2^i i!\epsilon)^2 < 1/2$$

In particular, $\varphi(e'_{i+1}, e'_{i+1})$ has positive real part, so that we may normalize e'_{i+1} by setting

$$\widetilde{e}_{i+1} := \frac{e'_{i+1}}{\sqrt{\varphi(e'_{i+1}, e'_{i+1})}}$$

Then $\varphi(\widetilde{e}_j, \widetilde{e}_k) = \delta_{ij}$ for all $k, j \leq i+1$, as desired. In other words, (C1) holds with i replaced by $i+1$. The same is true of condition (C2). Indeed, if we fix $l > i+1$, then for any $j \leq i$ we have $\|\varphi(\widetilde{e}_j, e_l)\| < 2^i i!\epsilon < 2^{i+1}(i+1)!\epsilon$, and for $j = i+1$ we find that

$$\|\varphi(\widetilde{e}_{i+1}, e_l)\| \leq \frac{1}{\|\sqrt{\varphi(e'_{i+1}, e'_{i+1})}\|} \cdot \left(\|\varphi(e_{i+1}, e_l)\| + \sum_{k=1}^{i} \|\varphi(e_{i+1}, \widetilde{e}_k)\| \cdot \|\varphi(\widetilde{e}_k, e_l)\| \right)$$

$$< 2 \cdot \left(\epsilon + i(2^i i!\epsilon)^2\right) < 2 \cdot \left(2^i i!\epsilon + i(2^i i!\epsilon)\right) = 2^{i+1}(i+1)!\epsilon \qquad \square$$

Proof of Lemma 2.2. In the case of a vector bundle over a point we may assume as above that $(\mathcal{E}, \varepsilon)$ is isometric to $(\mathbb{C}^r, \mathrm{id})$. Clearly, the subspace $\mathbb{R}^r \subset \mathbb{C}^r$ has the required properties. To see uniqueness, suppose W is another r-dimensional real subspace of $\mathbb{C}^r$ such that $\varepsilon|_W$ is real and positive definite. Pick an orthonormal basis of W with respect to $\varepsilon|_W$,

$$e_1 + if_1,\; e_2 + if_2,\; \ldots,\; e_r + if_r$$

where e_j and f_j are vectors in $\mathbb{R}^r$ and i is the imaginary unit in $\mathbb{C}$. Then we see that

$$\varepsilon(e_j, f_k) = 0 \qquad \text{for all } j,\, k \tag{D1}$$

$$\varepsilon(e_j, e_k) = \varepsilon(f_j, f_k) \qquad \text{for all } j \neq k \tag{D2}$$

$$\varepsilon(e_k, e_k) = \varepsilon(f_k, f_k) + 1 \qquad \text{for all } k \tag{D3}$$

It follows from (D2) and (D3) that the vectors $e_1, \dots, e_r$ are linearly independent. Indeed, if $\sum_j \lambda_j e_j = 0$ for certain $\lambda_j \in \mathbb{R}$, then

$$0 = \varepsilon\left(\sum \lambda_j e_j, \sum \lambda_j e_j\right) = \varepsilon\left(\sum \lambda_j f_j, \sum \lambda_j f_j\right) + \sum \lambda_j^2$$

Since ε is positive definite on $\mathbb{R}^r$, all λ_j must be zero. On the other hand, (D1) implies that the real spans of $e_1, \dots, e_r$ and $f_1, \dots, f_r$ in $\mathbb{R}^r$ are orthogonal. Thus, $f_1 = \dots = f_r = 0$, and $W = \mathbb{R}^r$.

If $(\mathcal{E}, \varepsilon)$ is an arbitrary complex symmetric bundle over a space X, then by Lemma 2.3 any point of X has some neighbourhood over which $(\mathcal{E}, \varepsilon)$ can be trivialized in the form above. We know how to define $\Re(\mathcal{E}, \varepsilon)$ over each such neighbourhood, and by uniqueness these local bundles can be glued together. □

2.4 Corollary. *For any CW complex X, the mapping $(\mathcal{E}, \varepsilon) \mapsto \Re(\mathcal{E}, \varepsilon)$ defines an isomorphism between the monoid of isometry classes of complex symmetric bundles over X and the monoid of isomorphism classes of real vector bundles over X.*

Proof. Given a real vector bundle $\mathcal{E}$ over X, we choose an inner product σ on $\mathcal{E}$ and consider the complex symmetric bundle $(\mathcal{E} \otimes_{\mathbb{R}} \mathbb{C}, \sigma_{\mathbb{C}})$, where $\sigma_{\mathbb{C}}$ denotes the $\mathbb{C}$-linear extension of σ to $\mathcal{E} \otimes_{\mathbb{R}} \mathbb{C}$. Since σ is defined uniquely up to isometry, so is $(\mathcal{E} \otimes_{\mathbb{R}} \mathbb{C}, \sigma_{\mathbb{C}})$, and we obtain a well-defined inverse to the map above. An alternative proof that avoids the uniqueness part of the preceding lemma is given in [MH73, Chapter V, § 2]. □

It follows that the definitions of $\mathrm{KO}^0(X)$ in terms of complex symmetric bundles and in terms of real bundles agree. Moreover, we see from the concrete description of the correspondence between these types of bundles given in the corollary and its proof that the hyperbolic map $\mathrm{K}^0(X) \to \mathrm{KO}^0(X)$ corresponds to the map sending a complex vector bundle to its underlying real bundle, whereas the forgetful map from $\mathrm{KO}^0(X)$ to $\mathrm{K}^0(X)$ corresponds to the map that sends a real vector bundle $\mathcal{F}$ to its complexification $\mathcal{F} \otimes_{\mathbb{R}} \mathbb{C}$. This explains the terminology and the notation used for these maps in topology.

Real versus non-degenerate Grassmannians

The correspondence between complex symmetric and real bundles is reflected by the fact that their projectivizations, or more generally their Grassmannian bundles, are homotopy equivalent. This will be used in the next section to describe a representing space for KO-theory, and it will also be relevant in the discussion of Stiefel-Whitney classes in Section III.1.

For any real or complex vector bundle $\mathcal{E}$ over a topological space X, we have Grassmannian bundles $\mathbb{R}\mathrm{Gr}(k, \mathcal{E})$ or $\mathrm{Gr}(k, \mathcal{E})$ over X whose fibres are given by the Grassmannians of real or complex k-planes in the corresponding fibres of $\mathcal{E}$. There are universal k-bundles over these spaces, which we denote by $\mathcal{U}_{\mathcal{E}}$. For complex symmetric bundles, Grassmannians may be defined as follows.

2.5 Definition. The non-degenerate Grassmannian $\mathrm{Gr}^{\mathrm{nd}}(k,(\mathcal{E},\varepsilon))$ associated with a complex symmetric bundle $(\mathcal{E},\varepsilon)$ is the open subbundle of the complex Grassmannian bundle $\mathrm{Gr}(k,\mathcal{E})$ given in each fibre by those k-planes T for which the restriction $\varepsilon|_T$ is non-degenerate. The universal symmetric bundle $\mathcal{U}_{(\mathcal{E},\varepsilon)}$ is defined as the restriction of the universal bundle over $\mathrm{Gr}^{\mathrm{nd}}(k,\mathcal{E})$ endowed with the symmetric form induced by ε.

The non-degenerate Grassmannian $\mathrm{Gr}^{\mathrm{nd}}(k,(\mathcal{E},\varepsilon))$ contains the Grassmannian of k-planes of the real bundle $\Re(\mathcal{E},\varepsilon)$. An inclusion

$$j\colon \mathbb{R}\mathrm{Gr}(k,\Re(\mathcal{E},\varepsilon)) \hookrightarrow \mathrm{Gr}(k,(\mathcal{E},\varepsilon)) \tag{11}$$

is given by sending k-dimensional subspaces T in the fibres of $\Re(\mathcal{E},\varepsilon)$ to their complexifications $T\otimes_{\mathbb{R}}\mathbb{C}$. In fact, we will see in the next lemma that this inclusion is a homotopy equivalence. Moreover, the restriction of the universal bundle $\mathcal{U}_{(\mathcal{E},\varepsilon)}$ to $\mathbb{R}\mathrm{Gr}(k,\Re(\mathcal{E},\varepsilon))$ corresponds to the universal real bundle over this space in the sense that

$$\Re(j^*\mathcal{U}_{(\mathcal{E},\varepsilon)}) = \mathcal{U}_{\Re(\mathcal{E},\varepsilon)} \tag{12}$$

2.6 Lemma. *For any complex symmetric bundle $(\mathcal{E},\varepsilon)$, the inclusion j defined in* (11) *is a homotopy equivalence. A homotopy inverse is provided by a retract of* $\mathrm{Gr}(k,(\mathcal{E},\varepsilon))$ *onto* $\mathbb{R}\mathrm{Gr}(k,\Re(\mathcal{E},\varepsilon))$.

Proof. Consider the projection $\pi\colon \mathcal{E}=\Re(\mathcal{E},\varepsilon)\oplus i\Re(\mathcal{E},\varepsilon)\twoheadrightarrow\Re(\mathcal{E},\varepsilon)$.

We define a retract r of j by sending a complex k-plane T in a fibre of $\mathcal{E}$ to the subspace $\pi(\Re(T,\varepsilon|_T))$ in the corresponding fibre of $\Re(\mathcal{E},\varepsilon)$. This is indeed a linear subspace of real dimension k: since ε is positive definite on $\Re(T,\varepsilon|_T)$ but negative definite on $i\Re(\mathcal{E},\varepsilon)$, the intersection $\Re(T,\varepsilon|_T)\cap i\Re(\mathcal{E},\varepsilon)$ is trivial.

More generally, we define a family of endomorphisms of $\mathcal{E}$ parametrized by $t\in[0,1]$ by

$$\begin{aligned}\pi_t\colon \Re(\mathcal{E},\varepsilon)\oplus i\Re(\mathcal{E},\varepsilon) &\twoheadrightarrow \Re(\mathcal{E},\varepsilon)\oplus i\Re(\mathcal{E},\varepsilon)\\ (x,y)&\mapsto(x,ty)\end{aligned}$$

This family interpolates between the identity π_1 and the projection π_0, which we can identify with π. We claim that for any k-plane T in a fibre of $\mathcal{E}$ the image

$$\pi_t(\Re(T,\varepsilon|_T))\subset\mathcal{E}$$

is a real linear subspace of dimension k on which ε is real and positive definite. The claim concerning the dimension has already been verified in the case $t=0$ and follows for non-zero t from the fact that π_t is an isomorphism. Now take a non-zero vector $v \in \pi_t(\Re(T, \varepsilon|_T))$ and write it as $v = x + tiy$, where $x, y \in \Re(\mathcal{E}, \varepsilon)$ and $x + iy \in \Re(T, \varepsilon|_T)$. Since $\varepsilon(x, x)$, $\varepsilon(y, y)$ and $\varepsilon(x + iy, x + iy)$ are all real we deduce that $\varepsilon(x, y) = 0$; it follows that $\varepsilon(v, v)$ is real as well. Moreover, since $\varepsilon(x + iy, x + iy)$ is positive we have $\varepsilon(x, x) > \varepsilon(y, y)$, so that $\varepsilon(v, v) > (1 - t^2)\varepsilon(y, y)$. In particular, $\varepsilon(v, v) > 0$ for all $t \in [0, 1]$, as claimed.

It follows that $T \mapsto \pi_t(\Re(T, \varepsilon|_T)) \otimes_{\mathbb{R}} \mathbb{C}$ defines a homotopy from $j \circ r$ to the identity on $\mathrm{Gr}^{\mathrm{nd}}(k, (\mathcal{E}, \varepsilon))$. □

2b Representability

The homotopy classification of vector bundles implies the representability of the functors K^0 and KO^0 in the homotopy category $\mathcal{H}$ of topological spaces. More precisely, there are representable functors on $\mathcal{H}$ which agree with K^0 and KO^0 on all finite-dimensional CW complexes. We first recall how this works in the complex case. For KO^0, we have two equivalent representing spaces, corresponding to the characterizations in terms of real vector bundles and complex symmetric bundles, respectively.

Let us write $\mathrm{Gr}_{r,n}$ for the Grassmannian $\mathrm{Gr}(r, \mathbb{C}^{r+n})$ of complex r-bundles in $\mathbb{C}^{r+n}$, and let Gr_r be the union of $\mathrm{Gr}_{r,n} \subset \mathrm{Gr}_{r,n+1} \subset \cdots$ under the obvious inclusions. Denote the universal r-bundles over these spaces by $\mathcal{U}_{r,n}$ and $\mathcal{U}_r$. For any connected paracompact Hausdorff space X, we have a one-to-one correspondence between the set $\mathrm{Vect}_r(X)$ of isomorphism classes of rank r complex vector bundles over X and homotopy classes of maps from X to Gr_r: a homotopy class $[f]$ in $\mathcal{H}(X, \mathrm{Gr}_r)$ corresponds to the pullback of $\mathcal{U}_r$ along f [Hus94, Chapter 3, Theorem 7.2].

To describe $\mathrm{K}^0(X)$, we need to pass to Gr, the union of the Gr_r under the embeddings $\mathrm{Gr}_r \hookrightarrow \mathrm{Gr}_{r+1}$ that send a complex r-plane W to $\mathbb{C} \oplus W$.

2.7 Theorem. *For finite-dimensional CW complexes X, we have natural identifications*

$$\mathrm{K}^0(X) \cong \mathcal{H}(X, \mathbb{Z} \times \mathrm{Gr}) \tag{13}$$

such that, for $X = \mathrm{Gr}_{r,n}$, the class $[\mathcal{U}_{r,n}] + (d - r)[\mathbb{C}]$ in $\mathrm{K}^0(\mathrm{Gr}_{r,n})$ corresponds to the inclusion $\mathrm{Gr}_{r,n} \hookrightarrow \{d\} \times \mathrm{Gr}_{r,n} \hookrightarrow \mathbb{Z} \times \mathrm{Gr}$.

Proof. The theorem is of course well-known, see for example [Ada95, page 204]. To deduce it from the homotopy classification of vector bundles, we note first that any CW complex is paracompact and Hausdorff [Hat09, Proposition 1.20]. Moreover, we may assume that X is connected. The product $\mathbb{Z} \times \mathrm{Gr}$ can be viewed as the colimit of the inductive system

$$\coprod_{d \geq 0} \{d\} \times \mathrm{Gr}_d \hookrightarrow \coprod_{d \geq -1} \{d\} \times \mathrm{Gr}_{d+1} \hookrightarrow \coprod_{d \geq -2} \{d\} \times \mathrm{Gr}_{d+2} \hookrightarrow \cdots \subset \mathbb{Z} \times \mathrm{Gr}$$

Any continuous map from X to $\mathbb{Z} \times \mathrm{Gr}$ must factor through one of the components $\mathrm{colim}_n(\{d\} \times \mathrm{Gr}_n)$. By cellular approximation, it is in fact homotopic to a map that factors through $\{d\} \times \mathrm{Gr}_n$ for some n. Thus,

$$\mathcal{H}(X, \mathbb{Z} \times \mathrm{Gr}) \cong \coprod_{d \in \mathbb{Z}} \mathrm{colim}_n \, \mathrm{Vect}_n(X)$$

where the colimit is taken over the maps $\mathrm{Vect}_n(X) \to \mathrm{Vect}_{n+1}(X)$ sending a vector bundle $\mathcal{E}$ to $\mathbb{C} \oplus \mathcal{E}$. We define a map from the coproduct to $\mathrm{K}^0(X)$ by sending a vector bundle $\mathcal{E}$ in the d^{th} component to the class $[\mathcal{E}] + (d - \mathrm{rk}\mathcal{E})[\mathbb{C}]$ in $\mathrm{K}^0(X)$. To see that this is a bijection, we use the fact that every vector bundle $\mathcal{E}$ over a finite-dimensional CW complex has a stable inverse: a vector bundle $\mathcal{E}^\perp$ over X such that $\mathcal{E} \oplus \mathcal{E}^\perp$ is a trivial bundle [Hus94, Chapter 3, Proposition 5.8]. □

If we replace the complex Grassmannians by real Grassmannians $\mathbb{R}\mathrm{Gr}_{r,n}$, we obtain the analogous statement that KO^0 can be represented by $\mathbb{Z} \times \mathbb{R}\mathrm{Gr}$. Equivalently, but more in the spirit of Definition 2.1, we could work with the non-degenerate Grassmannians defined in the previous section.

So let $\mathrm{Gr}^{\mathrm{nd}}_{r,n}$ abbreviate $\mathrm{Gr}^{\mathrm{nd}}(r, \mathbb{H}^{r+n})$, where $\mathbb{H}$ is the hyperbolic plane $(\mathbb{C}^2, \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right))$, and let $\mathcal{U}^{\mathrm{nd}}_{r,n}$ denote the restriction of the universal bundle over $\mathrm{Gr}(r, \mathbb{C}^{2r+2n})$ to $\mathrm{Gr}^{\mathrm{nd}}_{r,n}$. Then colimits $\mathrm{Gr}^{\mathrm{nd}}_r$ and $\mathrm{Gr}^{\mathrm{nd}}$ can be defined in the same way as for the usual Grassmannians. By (12) and Lemma 2.6 the homotopy classification of real vector bundles is equivalent to the homotopy classification of complex symmetric bundles in the sense that we have commutative diagrams

$$\begin{array}{ccc}
\left(\begin{array}{c}\text{isomorphism classes of real vector} \\ \text{bundles of rank } r \text{ over } X\end{array}\right) & \xleftarrow{\;\cong\;} & \mathcal{H}(X, \mathbb{R}\mathrm{Gr}_r) \\
\Re \uparrow \cong & & j^* \uparrow \cong \\
\left(\begin{array}{c}\text{isometry classes of complex symmetric} \\ \text{bundles of rank } r \text{ over } X\end{array}\right) & \xleftarrow{\;\cong\;} & \mathcal{H}(X, \mathrm{Gr}^{\mathrm{nd}}_r)
\end{array}$$

For finite-dimensional CW complexes X, we obtain natural identifications

$$\mathrm{KO}^0(X) \cong \mathcal{H}(X, \mathbb{Z} \times \mathrm{Gr}^{\mathrm{nd}}) \tag{14}$$

Here, for even $(d-r)$, the inclusion $\mathrm{Gr}^{\mathrm{nd}}_{r,n} \hookrightarrow \{d\} \times \mathrm{Gr}^{\mathrm{nd}}_{r,n} \hookrightarrow \mathbb{Z} \times \mathrm{Gr}^{\mathrm{nd}}$ corresponds to the class of $[\mathcal{U}^{\mathrm{nd}}_{r,n}] + \frac{d-r}{2}[\mathbb{H}]$ in $\mathrm{GW}^0(\mathrm{Gr}^{\mathrm{nd}}_{r,n})$.

2c Generalized cohomology theories

The representable functors K^0 and KO^0 may be embedded into sequences of functors K^i and KO^i defined for all integers i. These constituted the first examples of generalized cohomology theories in topology. By definition, a generalized cohomology theory is a sequence of functors E^i from topological spaces to abelian groups satisfying all the axioms of a cohomology theory formulated by Eilenberg and Steenrod except for one: the dimension axiom. That is, it is not required that $E^i(\text{point})$ vanishes for $i \neq 0$. Indeed, as we will see, the values of $\mathrm{K}^i(X)$ and $\mathrm{KO}^i(X)$ are periodic in i, and in particular, there are non-zero groups $\mathrm{K}^i(\text{point})$ and $\mathrm{KO}^i(\text{point})$ in arbitrarily high degrees.

Here, we describe the general method of defining generalized cohomology theories via spectra. The construction of spectra $\mathbb{K}^{\mathrm{top}}$ and $\mathbb{K}\mathbf{O}^{\mathrm{top}}$ representing topological K- and KO-theory is sketched at the beginning of the next section. The classical book [Ada95] may be consulted both for general background on spectra and for details concerning the particular case of K-theory; see in particular Example 2.2.

Generalized cohomology theories arise as representable functors on the stable homotopy category $\mathcal{SH}$, the homotopy category of spectra. More precisely, a generalized cohomology theory E^* may be obtained from a spectrum $\mathbb{E}$ as follows. Recall that $\mathcal{SH}$ is a triangulated category in which the shift functor is given by suspension. We will denote the suspension by $S^1 \wedge -$. Given any spectrum $\mathbb{X}$, one defines the cohomology groups of $\mathbb{X}$ with respect to $\mathbb{E}$ by

$$\widetilde{E}^i(\mathbb{X}) := \mathcal{SH}(\mathbb{X}, S^i \wedge \mathbb{E})$$

Since suspension is invertible, these groups are defined for all integers i. Using the functor $\Sigma^\infty \colon \mathcal{H}_\bullet \to \mathcal{SH}$ from the pointed to the stable homotopy category that assigns to a pointed space (X, x) its suspension spectrum, the definition may be specialized to pointed spaces. That is, we define the (reduced) generalized cohomology theory on $\mathcal{H}_\bullet$ corresponding to $\mathbb{E}$ by

$$\widetilde{E}^i(X, x) := E^i(\Sigma^\infty(X, x))$$

If X is connected, we often simply write $\widetilde{E}^i(X)$ for $\widetilde{E}^i(X, x)$, where x is an arbitrary point of X. Finally, the cohomology groups $E^i(X)$ of an unpointed space are defined as the reduced groups of X_+, the union of X with a disjoint base point:

$$E^i(X) := \widetilde{E}^i(X_+)$$

For connected X, we have a canonical decomposition $E^i(X) = E^i(\text{point}) \oplus \widetilde{E}^i(X)$.

The fact that the functors E^* define a generalized cohomology theory follows directly from the construction. We emphasize two properties:

- For the reduced theory $\widetilde{E}^*$, we have suspension isomorphisms

$$\sigma^n \colon \widetilde{E}^{i-n}(X, x) \cong \widetilde{E}^i(S^n \wedge (X, x)) \tag{15}$$

- Exact triangles in $\mathcal{SH}$ give rise to long exact sequences of cohomology groups. For example, any continuous map $f\colon X' \to X$ fits into an exact triangle involving its mapping cone $C(f)$:

$$\begin{tikzcd} \Sigma^\infty(X'_+) \arrow[rr, "\Sigma^\infty(f_+)"] & & \Sigma^\infty(X_+) \arrow[dl] \\ & \Sigma^\infty(C(f)) \arrow[ul, dashed] & \end{tikzcd}$$

This triangle induces the long exact sequence

$$\cdots \to \widetilde{E}^i(C(f)) \to E^i(X) \xrightarrow{f^*} E^i(X') \to \widetilde{E}^{i+1}(C(f)) \to \cdots$$

2d K- and KO-theory

We now specialize the preceding discussion to the cases of K- and KO-theory. After sketching the constructions of the corresponding spectra, we discuss some general consequences along with more specific aspects of these two theories.

The key ingredient in the construction of a K-theory spectrum $\mathbb{K}^{\mathrm{top}}$ is Bott periodicity. One observes that the infinite Grassmannian Gr can be identified with the classifying space BU of the infinite unitary group, so that K^0 is represented by $\mathbb{Z}\times\mathrm{BU}$. By Bott periodicity, this space is equivalent to its own two-fold loop space $\Omega^2(\mathbb{Z}\times\mathrm{BU})$. Thus, one may define a 2-periodic Ω-spectrum $\mathbb{K}^{\mathrm{top}}$ whose even terms are all given by $\mathbb{Z}\times\mathrm{BU}$. Similarly, $\mathbb{R}\mathrm{Gr}$ is equivalent to the classifying space BO of the infinite orthogonal group. In this case, Bott periodicity says that $\mathbb{Z}\times\mathrm{BO}$ is equivalent to $\Omega^8(\mathbb{Z}\times\mathrm{BO})$, so one obtains a spectrum $\mathbb{K}\mathbf{O}^{\mathrm{top}}$ which is 8-periodic. Following the definition in the previous section, we obtain 2- and 8-periodic cohomology theories on reduced spaces:

$$\widetilde{\mathrm{K}}^i(X,x) := \mathcal{SH}(\Sigma^\infty(X,x), S^i \wedge \mathbb{K}^{\mathrm{top}})$$
$$\widetilde{\mathrm{KO}}^i(X,x) := \mathcal{SH}(\Sigma^\infty(X,x), S^i \wedge \mathbb{K}\mathbf{O}^{\mathrm{top}})$$

The groups $\widetilde{\mathrm{K}}^i(X)$ and $\mathrm{K}^i(X)$ of unpointed spaces and the corresponding KO-groups are defined exactly as above.

It follows from the construction that the functors K^{2i} and KO^{8i} agree with the functors K^0 and KO^0 that we started with. If X is a connected finite-dimensional CW complex, then $\widetilde{\mathrm{K}}^0(X)$ may be identified with the subgroup of $\mathrm{K}^0(X)$ given by virtual bundles of rank zero, i. e. by those elements $[\mathcal{E}]-[\mathcal{F}]$ of $\mathrm{K}^0(X)$ for which $\mathrm{rk}\mathcal{E}-\mathrm{rk}\mathcal{F}=0$. Moreover, the suspension isomorphisms allow us to express all the lower K-groups $\mathrm{K}^{-i}(X)$ and $\mathrm{KO}^{-i}(X)$ in terms of $\widetilde{\mathrm{K}}^0$ and $\widetilde{\mathrm{KO}}^0$. That is, for all $i \geq 0$ we have the following isomorphisms:

$$\mathrm{K}^{-i}(X) \cong \widetilde{\mathrm{K}}^0(S^i \wedge (X_+))$$
$$\mathrm{KO}^{-i}(X) \cong \widetilde{\mathrm{KO}}^0(S^i \wedge (X_+))$$

Thus, when X is a finite-dimensional CW complex, its K- and KO-groups may be described in terms of vector bundles over suspensions of X.

2.8 Example (K-groups of a point). The topological K-group of a point is the free abelian group generated by the trivial complex line bundle. Thus, $\mathrm{K}^{2i}(\text{point}) \cong \mathbb{Z}$ for all integers i. On the other hand, the preceding isomorphisms imply that

$$\mathrm{K}^{-2}(\text{point}) \cong \widetilde{\mathrm{K}}^0(S^2)$$

An explicit generator of $\widetilde{\mathrm{K}}^0(S^2)$ is given by $[\tau]-[\mathbb{C}]$, where τ denotes the Hopf bundle over S^2. If we identify S^2 with the complex projective line $\mathbb{CP}^1$, the Hopf bundle corresponds to the tautological or universal bundle $\mathcal{O}(-1)$.

Long exact cohomology sequences. As we have seen, any continuous map $f\colon X' \to X$ gives rise to long exact sequences of K- and KO-groups. By periodicity, these may be arranged as exact polygons with 6 and 24 vertices, respectively. We examine two particular cases.

- If $f\colon A \to X$ is a cofibration, for example an inclusion of a closed subcomplex A into a CW complex X, the mapping cone $C(f)$ is homotopy equivalent to the quotient space X/A. The long exact sequences then take the form in which they appear most often in topology. The groups $\widetilde{\mathrm{KO}}^i(X/A)$ are usually denoted $\mathrm{KO}^i(X,A)$, and similarly for the K-groups.

- Suppose X is a smooth manifold with a smooth submanifold Z. Let $f\colon U \hookrightarrow X$ denote the inclusion of the open complement of Z into X. Then the mapping cone $C(f)$ is the homotopy quotient $X/\!_h U$, which may be realized as the quotient of X by the closed complement of a tubular neighbourhood of Z in X. It follows that $C(f)$ is homotopy equivalent to the Thom space of the normal bundle $\mathcal{N}$ of Z in X. Thus, the long exact sequences have the form

$$\cdots \to \widetilde{\mathrm{KO}}^i(\mathrm{Thom}_Z\,\mathcal{N}) \to \mathrm{KO}^i(X) \to \mathrm{KO}^i(U) \to \widetilde{\mathrm{KO}}^{i+1}(\mathrm{Thom}_Z\,\mathcal{N}) \to \cdots \tag{16}$$

  Alternatively, we will sometimes denote the groups $\widetilde{\mathrm{K}}^i(X/\!_h U)$ and $\widetilde{\mathrm{KO}}^i(X/\!_h U)$ by $\mathrm{K}^i_Z(X)$ and $\mathrm{KO}^i_Z(X)$, in analogy with the notation used in algebraic geometry.

Multiplication. The spectra $\mathbb{K}^{\mathrm{top}}$ and $\mathbb{K}\mathbf{O}^{\mathrm{top}}$ are ring spectra. For the associated cohomology theories, this means that we have multiplication maps

$$\begin{aligned}\widetilde{\mathrm{KO}}^i(X,x) \otimes \widetilde{\mathrm{KO}}^j(Y,y) &\xrightarrow{\cdot} \widetilde{\mathrm{KO}}^{i+j}((X,x)\wedge(Y,y))\\ \mathrm{KO}^i(X)\otimes \mathrm{KO}^j(Y) &\xrightarrow{\times} \mathrm{KO}^{i+j}(X\times Y)\end{aligned}$$

on KO-groups and similarly for K-groups. These products generalize the products that may be defined in terms of vector bundles for finite-dimensional CW complexes [Gra75, Theorem 29.14]. They are natural and respect the suspension isomorphisms (15) in the sense that $\sigma^n(x \cdot y) = \sigma^n(x) \cdot y$. In particular, σ^n itself may be expressed as multiplication with $\sigma^n(1)$, where 1 denotes the unit in $\mathrm{K}^0(\text{point})$ or $\mathrm{KO}^0(\text{point})$, respectively. The periodicity isomorphisms $\mathrm{K}^i(X) \cong \mathrm{K}^{i-2}(X)$ and $\mathrm{KO}^i(X) \cong \mathrm{KO}^{i-8}(X)$ are induced by multiplication with generators $g \in \mathrm{K}^{-2}(\text{point})$ and $\lambda \in \mathrm{KO}^{-8}(\text{point})$. The generator g is described in Example 2.8.

Coefficient rings. The groups $E^i(\text{point})$ of a generalized cohomology theory E^* are known as its coefficient groups. For K-theory, we simply have

$$\begin{aligned} \mathrm{K}^0(\text{point}) &\cong \mathbb{Z} \\ \mathrm{K}^1(\text{point}) &= 0 \end{aligned}$$

The coefficient groups of KO-theory are more complicated [Bot69, page 66]:

$$\begin{aligned} \mathrm{KO}^0(\text{point}) &= \mathbb{Z} & \mathrm{KO}^7(\text{point}) &= \mathbb{Z}/2 \\ \mathrm{KO}^2(\text{point}) &= 0 & \mathrm{KO}^1(\text{point}) &= 0 \\ \mathrm{KO}^4(\text{point}) &= \mathbb{Z} & \mathrm{KO}^3(\text{point}) &= 0 \\ \mathrm{KO}^6(\text{point}) &= \mathbb{Z}/2 & \mathrm{KO}^5(\text{point}) &= 0 \end{aligned} \tag{17}$$

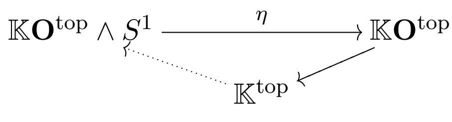
We have arranged them in a slightly unusual fashion so as to facilitate the comparison with Example 1.7. Since our theories are multiplicative, their coefficient groups may moreover be assembled to coefficient rings $E^*(\text{point}) = \oplus E^i(\text{point})$. These can be written as follows [Bot69, page 74[1]]:

$$\mathrm{K}^*(\text{point}) = \mathbb{Z}\big[g, g^{-1}\big] \tag{18}$$

$$\mathrm{KO}^*(\text{point}) = \mathbb{Z}\big[\eta, \alpha, \lambda, \lambda^{-1}\big] \Big/ \left(2\eta,\ \eta^3,\ \eta\alpha,\ \alpha^2 - 4\lambda\right) \tag{19}$$

Here, g and λ are the generators of $\mathrm{K}^{-2}(\text{point})$ and $\mathrm{KO}^{-8}(\text{point})$ inducing the periodicity isomorphisms, as mentioned above. The generators η and α are of degrees -1 and -4, respectively.

The Bott sequence. The spectra $\mathbb{K}^{\text{top}}$ and $\mathbb{K}\mathbf{O}^{\text{top}}$ fit into an exact triangle of the form



This triangle induces long exact sequences known as Bott sequences [Bot69, pages 75 and 112[2]; BG10, 4.I.B]. Concretely, the Bott sequence of a topological space X looks as follows:

[1] Unfortunately, the relation $\eta\alpha = 0$ is missing here, and this omission seems to have pervaded much of the literature. Of course, the relation follows from the fact that $\mathrm{KO}^{-5}(\text{point}) = 0$.

[2] There are misprints on both pages. In particular, the central group in the diagram on page 112 should be K^0.

$$\begin{aligned} \ldots \to \mathrm{KO}^{2i-1}X \to \mathrm{KO}^{2i-2}X \to \mathrm{K}^{2i-2}X \to \mathrm{KO}^{2i}X \to \mathrm{KO}^{2i-1}X \to \mathrm{K}^{2i-1}X \\ \to \mathrm{KO}^{2i+1}X \to \mathrm{KO}^{2i}X \to \mathrm{K}^{2i}X \to \mathrm{KO}^{2i+2}X \to \mathrm{KO}^{2i+1}X \to \ldots \end{aligned} \tag{20}$$

The maps appearing in this sequence may be described explicitly:

- The maps $\mathrm{KO}^j X \to \mathrm{K}^j X$ are the complexification maps c.
- The maps $\mathrm{K}^{j-2}X \to \mathrm{KO}^j X$ are the composites of the periodicity isomorphisms $\mathrm{K}^{j-2}X \cong \mathrm{K}^j X$ and the realification maps r.
- The maps between KO-groups are given by multiplication with $\eta \in \mathrm{KO}^{-1}(\mathrm{point})$.

Twisted KO-groups

It is possible to define twisted KO-groups $\mathrm{KO}^0(X;\mathcal{L})$ for complex line bundles $\mathcal{L}$ over X similarly to the way this is done for Grothendieck-Witt groups. It turns out, however, that these groups may alternatively be expressed as the usual (reduced) KO-groups $\widetilde{\mathrm{KO}}^2(\mathrm{Thom}\,\mathcal{L})$ of the Thom space of $\mathcal{L}$ [AR76, 3.8]. Here, we take this identification as our definition. More generally, we define KO-groups twisted by arbitrary complex bundles $\mathcal{E}$ over X as follows.

2.9 Definition. For any complex vector bundle $\mathcal{E}$ of constant rank r over a topological space X, we define

$$\mathrm{KO}^p(X;\mathcal{E}) := \widetilde{\mathrm{KO}}^{p+2r}(\mathrm{Thom}\,\mathcal{E})$$

When $\mathcal{E}$ is a trivial bundle, its Thom space is just a suspension of X, so that $\mathrm{KO}^p(X;\mathcal{E})$ agrees with the usual KO-group $\mathrm{KO}^p(X)$. Moreover, the Thom isomorphisms for KO-theory of [ABS64] show that the groups $\mathrm{KO}(X;\mathcal{E})$ only depend on the determinant line bundle of $\mathcal{E}$:

2.10 Lemma. *For complex vector bundles $\mathcal{E}$ and $\mathcal{F}$ over a topological space X with identical first Chern class modulo two, we have*

$$\mathrm{KO}^p(X;\mathcal{E}) \cong \mathrm{KO}^p(X;\mathcal{F})$$

Proof. A complex vector bundle $\mathcal{E}$ whose first Chern class vanishes modulo two has a spin structure and is therefore oriented with respect to KO-theory [ABS64, § 12]. That is, we have a Thom isomorphism

$$\mathrm{KO}^p X \xrightarrow{\cong} \widetilde{\mathrm{KO}}^{p+2r}(\mathrm{Thom}\,\mathcal{E})$$

Now suppose $c_1(\mathcal{E}) \equiv c_1(\mathcal{F}) \bmod 2$. We may view $\mathcal{E} \oplus \mathcal{E} \oplus \mathcal{F}$ both as a vector bundle over $\mathcal{E}$ and as a vector bundle over $\mathcal{F}$, and by assumption it is oriented with respect to KO-theory in both cases. Thus, both groups in the lemma can be identified with $\mathrm{KO}^p(X; \mathcal{E} \oplus \mathcal{E} \oplus \mathcal{F})$. □

Remark. In general, the identifications of Lemma 2.10 are non-canonical. Given a spin structure on a real vector bundle, the constructions in [ABS64] do yield a canonical Thom class, but there may be several different spin structures on the same bundle. Still, canonical identifications exist in many cases. For example, there is a canonical spin structure on the square of any complex line bundle, yielding canonical identifications

$$\mathrm{KO}^p(X; \mathcal{L}) \cong \mathrm{KO}^p(X; \mathcal{L} \otimes \mathcal{M}^{\otimes 2})$$

for any two complex line bundles $\mathcal{L}$ and $\mathcal{M}$ over X. In general, different spin structures on a spin bundle over X are classified by $H^1(X; \mathbb{Z}/2)$.

2e The Atiyah-Hirzebruch spectral sequence

In topology, there is a standard computational tool for generalized cohomology theories known as the Atiyah-Hirzebruch spectral sequence (AHSS). Given a generalized cohomology theory E^* and a finite-dimensional CW complex X, this spectral sequence takes the form [Ada95, III.7; Koc96, Theorem 4.2.7]

$$E_2^{p,q} = H^p(X; E^q(\text{point})) \Rightarrow E^{p+q}(X)$$

It is concentrated in the half-plane $p \geq 0$ and has differentials d_r of bidegree $(r, -r+1)$. In good cases, it enables us to compute the cohomology $E^*(X)$ from the coefficient groups of E^* and the singular cohomology of X.

The spectral sequence is natural with respect to continuous maps of finite-dimensional CW complexes. Moreover, if E^* is a multiplicative cohomology theory represented by a ring spectrum, then the spectral sequence is also multiplicative [Koc96, Proposition 4.2.9]. That is, the multiplication on the E_2-page induced by the cup product on singular cohomology and the ring structure of $E^*(\text{point})$ descends to a multiplication on all subsequent pages, such that the multiplication on the E_∞-page is compatible with the multiplication on $E^*(X)$. In particular, each page is a module over $E^*(\text{point})$. In this case, the differentials of the spectral sequence are derivations, i. e. they satisfy a Leibniz rule.

In some situations, it is more natural to consider the spectral sequence for the reduced cohomology theory $\widetilde{E}^*$ associated with E^*. Then the spectral sequence is written as

$$\widetilde{E}_2^{p,q} = \widetilde{H}^p(X; E^q(\text{point})) \Rightarrow \widetilde{E}^{p+q}(X)$$

The AHSS for K- and KO-theory

Specializing to the cases of K- and KO-theory, we obtain the following two multiplicative spectral sequences.

$$E_2^{p,q} = H^p(X; \mathrm{K}^q(\mathrm{point})) \Rightarrow \mathrm{K}^{p+q}(X)$$
$$E_2^{p,q} = H^p(X; \mathrm{KO}^q(\mathrm{point})) \Rightarrow \mathrm{KO}^{p+q}(X)$$

For K-theory, the spectral sequence has the singular cohomology of X with integral coefficients in all even rows, while the odd rows vanish. For KO-theory, the spectral sequence is 8-periodic in q. We have the integral cohomology of X in rows $q \equiv 0$ and $q \equiv -4 \bmod 8$, its cohomology with $\mathbb{Z}/2$-coefficients in rows $q \equiv -1$ and $q \equiv -2$, and all other rows are zero.

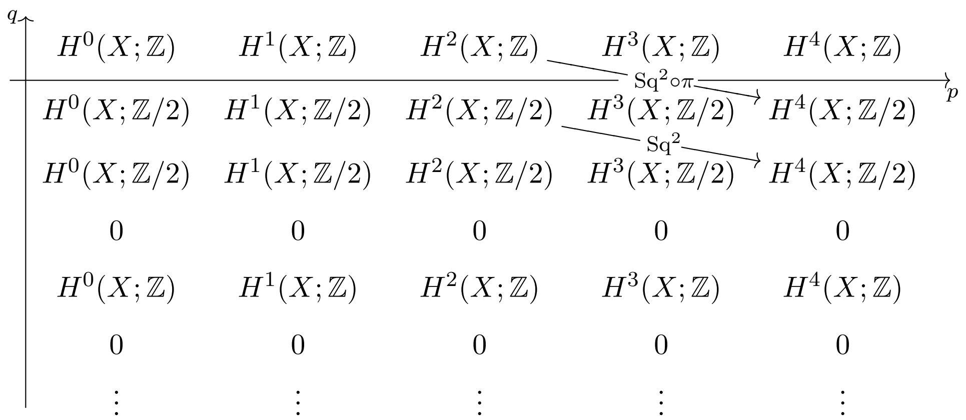


Figure 1: The E_2-page of the Atiyah-Hirzebruch spectral sequence computing $\mathrm{KO}^*(X)$

The low-order differentials of these spectral sequences may be described as follows. Let

$$\pi\colon H^*(X;\mathbb{Z}) \to H^*(X;\mathbb{Z}/2)$$
$$\mathrm{Sq}^2\colon H^*(X;\mathbb{Z}/2) \to H^{*+2}(X;\mathbb{Z}/2)$$
$$\beta\colon H^*(X;\mathbb{Z}/2) \to H^{*+1}(X;\mathbb{Z})$$

denote reduction modulo two, the second Steenrod square, and the Bockstein homomorphism, respectively. For K-theory, the differentials on the E_3-page of the Atiyah-Hirzebruch spectral sequence are given by the composition $\beta \circ \mathrm{Sq}^2 \circ \pi$ [HJJS08, Chapter 21, § 5]. For KO-theory, the differentials $d_2^{*,0}$ and $d_2^{*,-1}$ on the E_2-page are given by $\mathrm{Sq}^2 \circ \pi$ and Sq^2, respectively. The differential $d_3^{*,-2}$ on the E_3-page can be identified with $\beta \circ \mathrm{Sq}^2$ [Fuj67, 1.3].

The AHSS for Thom spaces

In order to compute twisted KO-groups as discussed in Section 2d, we need to apply the Atiyah-Hirzebruch spectral sequence of KO-theory to Thom spaces. So let X be a finite-dimensional CW complex, and let $\pi\colon \mathcal{E} \to X$ be a vector bundle of constant rank over X. Though we will be mainly interested in the case when $\mathcal{E}$ is complex, we may more generally assume here that $\mathcal{E}$ is any real vector bundle which is oriented. Then the Thom isomorphism for singular cohomology tells us that the reduced cohomology of the Thom space $\operatorname{Thom}\mathcal{E}$ is additively isomorphic to the cohomology of X itself, apart from a shift in degrees by $r := \operatorname{rk}_{\mathbb{R}}\mathcal{E}$. The isomorphism is given by multiplication with a Thom class θ in $\widetilde{H}^r(\operatorname{Thom}\mathcal{E};\mathbb{Z})$:

$$\begin{array}{ccc} H^*(X;\mathbb{Z}) & \xrightarrow{\cong} & \widetilde{H}^{*+r}(\operatorname{Thom}\mathcal{E};\mathbb{Z}) \\ x & \mapsto & \pi^*(x)\cdot\theta \end{array}$$

Similarly, the reduction of θ modulo two induces an isomorphism of the respective singular cohomology groups with $\mathbb{Z}/2$-coefficients. Thus, apart from a shift of columns, the entries on the E_2-page of the spectral sequence for $\widetilde{\mathrm{KO}}^*(\operatorname{Thom}\mathcal{E})$ are identical to those on the E_2-page for $\mathrm{KO}^*(X)$. However, the differentials may differ.

2.11 Lemma. *Let $\mathcal{E} \xrightarrow{\pi} X$ be a complex vector bundle of constant rank over a topological space X, with Thom class θ as above. The second Steenrod square on $\widetilde{H}^*(\operatorname{Thom}\mathcal{E};\mathbb{Z}/2)$ is given by "$\mathrm{Sq}^2 + c_1(\mathcal{E})$", where $c_1(\mathcal{E})$ is the first Chern class of $\mathcal{E}$ modulo two. That is,*

$$\mathrm{Sq}^2(\pi^*x\cdot\theta) = \pi^*\left(\mathrm{Sq}^2(x) + c_1(\mathcal{E})x\right)\cdot\theta$$

for any $x \in H^*(X;\mathbb{Z}/2)$. More generally, if $\mathcal{E}$ is a real oriented vector bundle, the second Steenrod square on the cohomology of its Thom space is given by "$\mathrm{Sq}^2 + w_2(\mathcal{E})$", where w_2 is the second Stiefel-Whitney class of $\mathcal{E}$.

Proof. This is a special case of the following well-known identity of Thom [MS74, page 91]:

$$\mathrm{Sq}^i(\pi^*x\cdot\theta) = \pi^*\left(\mathrm{Sq}^i(x) + w_i(\mathcal{E})x\right)\cdot\theta \qquad \square$$

The higher differentials in the spectral sequence for $\widetilde{\mathrm{KO}}^*(\operatorname{Thom}\mathcal{E})$ also depend only on the second Stiefel-Whitney class of $\mathcal{E}$. This follows from the observation that the Atiyah-Hirzebruch spectral sequence is compatible with Thom isomorphisms, which is made more precise by the next lemma.

2.12 Lemma. *Let $\mathcal{E}$ be a real oriented vector bundle over a finite-dimensional CW complex X, of constant rank r. Suppose $\mathcal{E}$ is oriented, and let θ be a Thom class as above. If $\mathcal{E}$ is moreover oriented with respect to* KO*-theory, then θ survives to the E_∞-page of the Atiyah-Hirzebruch spectral sequence computing $\widetilde{\mathrm{KO}}^*(\operatorname{Thom}\mathcal{E})$, and the Thom isomorphism for H^* extends to an isomorphism of spectral sequences. That is, for each* *page right multiplication with the class of* θ *in* $\widetilde{E}_s^{r,0}(\mathrm{Thom}\,\mathcal{E})$ *gives an isomorphism of* $E_s^{*,*}(X)$*-modules*

$$E_s^{*,*}(X) \xrightarrow[\cong]{\cdot\theta} \widetilde{E}_s^{*+r,*}(\mathrm{Thom}\,\mathcal{E})$$

Moreover, any lift of $\theta \in \widetilde{E}_\infty^{r,0}(\mathrm{Thom}\,\mathcal{E})$ *to* $\widetilde{\mathrm{KO}}^r(\mathrm{Thom}\,\mathcal{E})$ *defines a Thom class of* $\mathcal{E}$ *with respect to* KO*-theory. The isomorphism of the* E_∞*-pages of the spectral sequences is induced by the Thom isomorphism given by multiplication with any such class.*

Proof. We may assume without loss of generality that X is connected. Fix a point x on X. The inclusion of the fibre over x into $\mathcal{E}$ induces a map $i_x\colon S^r \hookrightarrow \mathrm{Thom}\,\mathcal{E}$. By assumption, the pullback i_x^* on ordinary cohomology maps θ to a generator of $\widetilde{H}^r(S^r)$, and the pullback on $\widetilde{\mathrm{KO}}^*$ gives a surjection

$$\widetilde{\mathrm{KO}}^*(\mathrm{Thom}\,\mathcal{E}) \overset{i_x^*}{\twoheadrightarrow} \widetilde{\mathrm{KO}}^r(S^r)$$

Consider the pullback along i_x on the E_∞-pages of the spectral sequences for S^r and $\mathrm{Thom}\,\mathcal{E}$. Since we can identify $\widetilde{E}_\infty^{r,0}(\mathrm{Thom}\,\mathcal{E})$ with a quotient of $\widetilde{\mathrm{KO}}^r(\mathrm{Thom}\,\mathcal{E})$ and $\widetilde{E}_\infty^{r,0}(S^r)$ with $\widetilde{\mathrm{KO}}^r(S^r)$, we must have a surjection

$$i_x^*\colon \widetilde{E}_\infty^{r,0}(\mathrm{Thom}\,\mathcal{E}) \twoheadrightarrow \widetilde{E}_\infty^{r,0}(S^r)$$

On the other hand, the behaviour of i_x^* on $\widetilde{E}_\infty^{r,0}$ is determined by its behaviour on $\widetilde{H}^r$, whence we can only have such a surjection if θ survives to the $\widetilde{E}_\infty$-page of $\mathrm{Thom}\,\mathcal{E}$. Thus, all differentials vanish on θ, and if multiplication by θ induces an isomorphism from $E_s^{*,*}(X)$ to $\widetilde{E}_s^{*+r,*}$ on page s, it also induces an isomorphism on the next page. Lastly, consider any lift of θ to an element Θ of $\widetilde{\mathrm{KO}}^r(\mathrm{Thom}\,\mathcal{E})$. It is clear by construction that right multiplication with Θ gives an isomorphism from $E_\infty(X)$ to $\widetilde{E}_\infty(\mathrm{Thom}\,E)$, and thus it also gives an isomorphism from $\mathrm{KO}^*(X)$ to $\widetilde{\mathrm{KO}}^*(\mathrm{Thom}\,\mathcal{E})$. Thus, Θ is a Thom class for $\mathcal{E}$ with respect to KO-theory. □

Lemma 2.12 allows the following strengthening of Lemma 2.10:

2.13 Corollary. *For complex vector bundles* $\mathcal{E}$ *and* $\mathcal{F}$ *over* X *with identical first Chern class modulo two, the spectral sequences computing* $\widetilde{\mathrm{KO}}^*(\mathrm{Thom}\,\mathcal{E})$ *and* $\widetilde{\mathrm{KO}}^*(\mathrm{Thom}\,\mathcal{F})$ *can be identified up to a possible shift of columns when* $\mathcal{E}$ *and* $\mathcal{F}$ *have different ranks.*

Chapter II

Comparison Maps

If X is a smooth complex variety, then we can consider both the algebraic and the topological cohomology theories discussed in the previous chapter. That is, on the one hand we can study the algebraic K-group and the (Grothendieck-)Witt groups of X viewed as a variety, and on the other hand we can study the topological complex and real K-groups of X viewed as a topological space, i. e. the K-groups of the set $X(\mathbb{C})$ of complex points of X equipped with the analytic topology. It follows directly from the definitions of these groups that we have natural maps

$$k\colon \quad \mathrm{K}_0(X) \to \mathrm{K}^0(X(\mathbb{C})) \tag{1}$$
$$gw^0\colon \mathrm{GW}^0(X) \to \mathrm{KO}^0(X(\mathbb{C})) \tag{2}$$

If we write $(\mathrm{KO}^0/\,\mathrm{K})(X(\mathbb{C}))$ for the cokernel of the realification map from $\mathrm{K}^0(X(\mathbb{C}))$ to $\mathrm{KO}^0(X(\mathbb{C}))$, we moreover have an induced map

$$w^0\colon \quad \mathrm{W}^0(X) \to (\mathrm{KO}^0/\,\mathrm{K})(X(\mathbb{C})) \tag{3}$$

Our aim in this chapter is to extend these maps to be defined on shifted groups $\mathrm{GW}^i(X)$ and $\mathrm{W}^i(X)$ for arbitrary integers i in such a way that they are compatible with as much structure of the respective cohomology theories as possible. Two different approaches will be discussed.

The first approach is the more elementary one. The idea is to use the multiplicative structure of the theories to define the maps on shifted Grothendieck-Witt groups in such a way that many of their properties can be checked by explicit calculations. This approach has previously been detailed in [Zib09]. We refrain from reproducing the full details here, but only sketch the key steps in the argument. The results obtained in [Zib09] only just suffice to prove the comparison theorem for cellular varieties included in Chapter IV of this thesis. However, as we will explain, the fundamental question of whether the comparison maps are in general compatible with the boundary maps appearing in long exact localization sequences remains open.

The idea of the second approach is to use a construction of Grothendieck-Witt groups, or more generally of hermitian K-theory, that resembles the homotopy-theoretic definition of KO-theory so closely that all compatibility issues disappear. More precisely, our second definition of the comparison maps will rely on the following theorem from $\mathbb{A}^1$-homotopy theory:

> Hermitian K-theory is representable in the stable $\mathbb{A}^1$-homotopy category by a spectrum whose complex realization in the usual stable homotopy category represents KO-theory.

Unfortunately, although this result seems to be well-known among experts, it has not yet been properly published. We therefore highlight both this and a related statement that we will be using in the form of Assumption 2.2. Proofs of both parts of the assumption are announced by Morel in [Mor06]. To add further credibility to the first statement, we explain in Section 2b how the analogous claim in the unstable homotopy category may be deduced from recent results of Schlichting and Tripathi. A brief summary of those superficial aspects of $\mathbb{A}^1$-homotopy theory that we will constantly be referring to is included in Section 2a.

1 An elementary approach

Recall from I.2d that the suspension isomorphisms in KO-theory are given by multiplication with generators of the KO-groups of spheres. That is, if $\widetilde{e}_n$ denotes the generator of $\widetilde{\mathrm{KO}}^n(S^n)$ corresponding to the unit in $\mathrm{KO}^0(\text{point})$ under suspension, then the suspension isomorphism for an arbitrary pointed space (X,x) is given by multiplication with $\widetilde{e}_n$:

$$\widetilde{\mathrm{KO}}^{i-n}(X,x) \xrightarrow{\cong} \widetilde{\mathrm{KO}}^i(S^n \wedge (X,x))$$

Similarly, for an unpointed space X, we have isomorphisms

$$\begin{aligned} \mathrm{KO}^{i-n}(X) &\xrightarrow{\cong} \widetilde{\mathrm{KO}}^i(S^n \wedge (X_+)) \\ \mathrm{KO}^i(X) \oplus \mathrm{KO}^{i-n}(X) &\xrightarrow{\cong} \mathrm{KO}^i(S^n \times X) \end{aligned}$$

Explicitly, if we write e_n for the element of $\mathrm{KO}^n(S^n)$ corresponding to $\widetilde{e}_n$ and π for the projection $S^n \times X \twoheadrightarrow X$, then the lower map is given by $(x,y) \mapsto \pi^*x + e_n \times y$. Analogous isomorphisms exist for Grothendieck-Witt groups:

1.1 Lemma. *Let X be a smooth variety with some closed subset Z. Let Ψ_0 be the generator of $\mathrm{GW}^1(\mathbb{P}^1)$ described in Example I.1.6. We have isomorphisms*

$$\begin{aligned} \mathrm{GW}^{i-1}_Z(X) &\xrightarrow{\cong} \mathrm{GW}^i_Z(X \times \mathbb{P}^1) \\ \mathrm{GW}^{i-1}_Z(X) &\xrightarrow{\cong} \mathrm{GW}^i_Z(X \times \mathbb{A}^1) \end{aligned}$$

given by multiplication with the restriction of Ψ_0 to $\mathrm{GW}^1_{\{0\}}(\mathbb{P}^1)$ and its further restriction *to* $\mathrm{GW}^1_{\{0\}}(\mathbb{A}^1)$*, respectively. Moreover, we have an isomorphism*

$$\mathrm{GW}^i_Z(X) \oplus \mathrm{GW}^{i-1}_Z(X) \xrightarrow{\cong} \mathrm{GW}^i_{Z\times\mathbb{P}^1}(X \times \mathbb{P}^1)$$

given by $(x,y) \mapsto \pi^* x + y \times \Psi_0$*, where* π *denotes the projection* $X \times \mathbb{P}^1 \twoheadrightarrow X$*. All of these isomorphisms also hold with Witt groups in place of Grothendieck-Witt groups.*

Proof. For Witt groups, the first isomorphism is a special case of Theorem 2.5 in [Nen07], the case when $Z = X$ being Theorem 8.2 in [BG05]. The second isomorphism follows via excision. The decomposition of $\mathrm{W}^i(X \times \mathbb{P}^1)$ into $\mathrm{W}^i(X) \oplus \mathrm{W}^{i-1}(X)$ is a special case of Theorem 1.5 in [Wal03b]. It may be deduced from the second isomorphism by considering the localization sequences associated with the inclusion of $X \times \mathbb{A}^1$ into $X \times \mathbb{P}^1$, since these sequences split. A decomposition of $\mathrm{W}^i_Z(X \times \mathbb{P}^1)$ can be obtained analogously, using the more general localization sequences of triples discussed in Theorem 1.6 of [Bal01b]. Lastly, since the corresponding isomorphisms also hold for K_0, with $F(\Psi_0) \in \mathrm{K}_0(\mathbb{P}^1)$ in place of Ψ_0, the corresponding isomorphisms of Grothendieck-Witt groups may be deduced via Karoubi induction (c. f. the proof of Lemma 1.3 below and Lemma 2.3 in [Zib09]). □

Now let X be a smooth complex variety. Then the two isomorphisms

$$\mathrm{GW}^i(X) \oplus \mathrm{GW}^{i-1}(X) \xrightarrow{\cong} \mathrm{GW}^i(X \times \mathbb{P}^1)$$
$$\mathrm{KO}^{2i}(X(\mathbb{C})) \oplus \mathrm{KO}^{2i-2}(X(\mathbb{C})) \xrightarrow{\cong} \mathrm{KO}^{2i}(X(\mathbb{C}) \times S^2)$$

can be used to define comparison maps

$$gw^i \colon \mathrm{GW}^i(X) \to \mathrm{KO}^{2i}(X(\mathbb{C}))$$

inductively for all non-positive i. Explicitly, if gw^i is already defined for some $i \leq 0$, then for $x \in \mathrm{GW}^{i-1}(X)$ we define $gw^{i-1}(x)$ to be the unique element in $\mathrm{KO}^{2i-2}(X(\mathbb{C}))$ satisfying[1]

$$gw^{i-1}(x) \times e_2 = gw^i(x \times (-\Psi_0))$$

The following properties of these maps can be checked via explicit calculations. Firstly, it is clear from the construction that they are natural with respect to morphisms of smooth varieties, and that they respect the multiplicative structures of Grothendieck-Witt groups and KO-theory. Moreover, they are compatible with the Karoubi and Bott sequences (see

[1] We use $-\Psi_0$ instead of Ψ_0 because the image of e_2 under the complexification map $\mathrm{KO}^2(S^2) \xrightarrow{c} \mathrm{K}^2(S^2)$ and the periodicity isomorphism $\mathrm{K}^2(S^2) \xrightarrow{\cong} \mathrm{K}^0(S^2)$ agrees with the image of $-\Psi_0$ under the forgetful map $\mathrm{GW}^1(\mathbb{P}^1) \xrightarrow{F} \mathrm{K}_0(X)$: we have $k(F(-\Psi)) = k(\mathcal{O}(-1) - \mathcal{O}) = [\tau] - [\mathbb{C}] = \sigma^{-2}(g) = c(e_2) \times g$. See Examples I.1.6 and I.2.8.

(I.4) and (I.20)) in the sense that we have induced maps

$$w^i\colon\ \mathrm{W}^i(X) \to (\mathrm{KO}^{2i}/\,\mathrm{K})(X(\mathbb{C}))$$

and commutative diagrams of the following form:

$$\begin{array}{ccccccccc}
\mathrm{GW}^{i-1}(X) & \xrightarrow{F} & \mathrm{K}_0(X) & \xrightarrow{H^i} & \mathrm{GW}^i(X) & \longrightarrow & \mathrm{W}^i(X) & \longrightarrow & 0 \\
\downarrow{\scriptstyle gw^{i-1}} & & \downarrow{\scriptstyle k} & & \downarrow{\scriptstyle gw^i} & & \downarrow{\scriptstyle w^i} & & \\
\mathrm{KO}^{2i-2}(X(\mathbb{C})) & \xrightarrow{g^{i-1}\circ c} & \mathrm{K}^0(X(\mathbb{C})) & \xrightarrow{r\circ g^{-i}} & \mathrm{KO}^{2i}(X(\mathbb{C})) & \longrightarrow & (\mathrm{KO}^{2i}/\,\mathrm{K})(X(\mathbb{C})) & \longrightarrow & 0
\end{array}$$

To obtain comparison maps in positive degrees, one checks next that the maps already defined respect the periodicity isomorphisms. This can again be done by expressing these isomorphisms in a multiplicative way. We include a short proof to convey the flavour of these arguments.

1.2 Lemma. *The maps* $gw^i\colon\ \mathrm{GW}^i(X) \to \mathrm{KO}^{2i}(X(\mathbb{C}))$ *defined above respect the periodicity isomorphisms* $\mathrm{GW}^i(X) \cong \mathrm{GW}^{i-4}(X)$ *and* $\mathrm{KO}^{2i}(X(\mathbb{C})) \cong \mathrm{KO}^{2i-8}(X(\mathbb{C}))$.

Proof. Recall from Section I.2d that the periodicity isomorphism in KO-theory is given by multiplication with a generator $\lambda \in \mathrm{KO}^{-8}(\text{point})$ whose sign is fixed by the relation $\alpha^2 = 4\lambda$ for a generator $\alpha \in \mathrm{KO}^{-4}(\text{point})$. The periodicity isomorphism for Grothendieck-Witt groups is induced by shifting complexes two positions to the right. This isomorphism may be interpreted as cross product with the element $\Lambda := [\mathcal{O}[-2], \mathrm{id}]$ of $\mathrm{GW}^{-4}(\text{point})$, where $\mathcal{O}[-2]$ denotes the complex consisting of the trivial line bundle in degree -2. To show that gw^{-4} maps Λ to λ, we use the following square from the diagram above, evaluated on a point:

$$\begin{array}{ccc}
\mathrm{K}_0(\text{point}) & \xrightarrow{H^{-4}} & \mathrm{GW}^{-4}(\text{point}) \\
\downarrow{\scriptstyle k} & & \downarrow{\scriptstyle gw^{-4}} \\
\mathrm{K}^0(\text{point}) & \xrightarrow{r\circ g^4} & \mathrm{KO}^{-8}(\text{point})
\end{array}$$

By Lemma I.1.9, we have $H^{-4}(\mathcal{O}) = H^{-4}(F(\Lambda)) = 2\Lambda$, and similarly $r(g^4) = 2\lambda$. Thus, gw^{-4} must map Λ to λ. □

We see similarly that the comparison maps gw^i and w^i, now defined for all shifts $i \in \mathbb{Z}$, induce isomorphisms between the Grothendieck-Witt and KO-groups of a point.

1.3 Proposition. *The comparison maps are isomorphism on a point:*

$$gw^i\colon\ \mathrm{GW}^i(\text{point}) \xrightarrow{\cong} \mathrm{KO}^{2i}(\text{point})$$
$$w^i\colon\ \mathrm{W}^i(\text{point}) \xrightarrow{\cong} (\mathrm{KO}^{2i}/\,\mathrm{K})(\text{point})$$

Proof. We can easily see that the corresponding groups of a point p are isomorphic by direct comparison: since $\mathrm{K}^1(p)$ vanishes, the Bott sequence (I.20) implies that we can identify the quotients $(\mathrm{KO}^{2i}/\,\mathrm{K})(p)$ with the odd KO-groups $\mathrm{KO}^{2i-1}(p)$, so we can read off the values of all groups from Example I.1.7 and (I.17). Moreover, it is clear that the maps k, gw^0 and w^0 are isomorphisms. It follows by periodicity that gw^i and w^i are isomorphisms for all i divisible by four. For all other values of i, the groups $\mathrm{W}^i(p)$ are trivial, so w^i is an isomorphism on a point in general. To see that the maps gw^i are also isomorphisms for arbitrary i, we can use the comparison of the Karoubi and Bott sequences:

$$\begin{array}{ccccccccccccc}
\dots & \longrightarrow & \mathrm{GW}^{i-1}(p) & \longrightarrow & \mathrm{K}_0(p) & \longrightarrow & \mathrm{GW}^i(p) & \longrightarrow & \mathrm{W}^i(p) & \longrightarrow & 0 & \longrightarrow & \dots \\
 & & \downarrow{\scriptstyle gw^{i-1}} & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle gw^i} & & \downarrow{\scriptstyle \cong} & & \downarrow & & \\
\dots & \longrightarrow & \mathrm{KO}^{2i-2}(p) & \longrightarrow & \mathrm{K}^0(p) & \longrightarrow & \mathrm{KO}^{2i}(p) & \longrightarrow & \mathrm{KO}^{2i-1}(p) & \longrightarrow & 0 & \longrightarrow & \dots
\end{array}$$

Given the periodicity of the Grothendieck-Witt groups, the claim concerning the maps gw^i follows from repeated applications of the Five Lemma. This strategy of proof is known as “Karoubi induction”. □

As a next step, we would like to compare the localization sequence (I.7) of a smooth variety X with a smooth closed subvariety Z to the corresponding localization sequence (I.16) in KO-theory. Actually, this only makes sense for small parts of these sequences, and we have to compose the maps w^i defined on Witt groups with multiplication by $\eta \in \mathrm{KO}^{-1}(\text{point})$:

$$\begin{array}{ccccccccccc}
\mathrm{GW}^i_Z(X) & \longrightarrow & \mathrm{GW}^i(X) & \longrightarrow & \mathrm{GW}^i(U) & \xrightarrow{\partial} & \mathrm{W}^{i+1}_Z(X) & \longrightarrow & \mathrm{W}^{i+1}(X) & \longrightarrow & \mathrm{W}^{i+1}(U) \\
\vdots{\scriptstyle ?} & & \downarrow{\scriptstyle gw^i} & \square & \downarrow{\scriptstyle gw^i} & & \vdots{\scriptstyle ?} & & \downarrow{\scriptstyle \eta\circ w^{i+1}} & \square & \downarrow{\scriptstyle \eta\circ w^{i+1}} \\
\mathrm{KO}^{2i}_Z(X) & \longrightarrow & \mathrm{KO}^{2i}(X) & \longrightarrow & \mathrm{KO}^{2i}(U) & \xrightarrow{\partial} & \mathrm{KO}^{2i+1}_Z(X) & \longrightarrow & \mathrm{KO}^{2i+1}(X) & \longrightarrow & \mathrm{KO}^{2i+1}(U)
\end{array}$$

Here, we have written X for $X(\mathbb{C})$ in the second row. The symbol “□” marks the two squares that commute by naturality, and the question is how to fill in the dotted arrows such that all squares commute. Using Thom isomorphisms, we could replace the groups with support on Z by the actual cohomology groups of Z, and then we could use the comparison maps already defined for the dotted arrows. But we do not know a way of proving that the resulting squares are commutative without using comparison maps on groups with restricted support at some point.

Unfortunately, we are similarly unaware of any straight-forward way of defining comparison maps on groups $\mathrm{GW}^0_Z(X)$ with support $Z \neq X$. The solution in [Zib09] is to adapt a known construction of classes in relative K-groups detailed in [Seg68], in terms of complexes of vector bundles whose cohomology is supported on an open subspace. To construct a comparison map on $\mathrm{GW}^0_Z(X)$, we first choose a suitable open neighbourhood of

Z in X and define a map to a KO-group of complexes with support on this neighbourhood. In a second step, we construct maps from such KO-groups defined in terms of complexes to the usual relative KO-groups. We do not know whether these maps are isomorphisms but, in any case, we obtain a comparison map $gw^0\colon \mathrm{GW}^0_Z(X) \to \mathrm{KO}^0_Z(X(\mathbb{C}))$ by composition. Finally, we need to check that this map is independent of the choices made.

Once the construction of comparison maps on $\mathrm{GW}^0_Z(X)$ is settled, maps on shifted groups $\mathrm{GW}^i_Z(X)$ can be defined in exactly the same way as before. Moreover, these constructions also work for groups twisted by line bundles. This allows us to check, again by explicit calculations, that the comparison maps are compatible with Nenashev's Thom isomorphisms, and to conclude that two further squares in the above diagram commute.

Lastly, we are left with the square involving the boundary maps. This square remains problematic. What is needed is either an algebraic description of the boundary morphism in KO-theory, or a homotopy-theoretic description of the boundary morphism for (Grothendieck-)Witt groups. The latter idea led us to consider a totally different approach to the comparison problem altogether, which we describe in the next section.

2 A homotopy-theoretic approach

A homotopy theory of schemes emulating the situation for topological spaces has been developed over the past 20 years mainly by Morel and Voevodsky. Today, it is known as either $\mathbb{A}^1$-homotopy theory or motivic homotopy theory. The authoritative reference is [MV99]. Closely related texts by the same authors are [Voe98], [Mor99] and [Mor04]. A textbook introduction is provided by [DLØ+07]. Below, we summarize the main points relevant for us in just a few sentences.

2a $\mathbb{A}^1$-homotopy theory

In short, the $\mathbb{A}^1$-homotopy category is constructed as follows. Consider the category Sm_k of smooth separated schemes of finite type over a field k. One of the problems with this category that arises if one tries to imitate topological arguments is that it does not have colimits: given two schemes X and Y with a common subscheme A, there is in general no smooth candidate for the union of X and Y "glued along A". However, the category Sm_k can be embedded into some larger category Spc_k of "spaces over k" which is closed under small limits and colimits, and which can even be equipped with a model structure. The $\mathbb{A}^1$-homotopy category $\mathcal{H}(k)$ over k is the homotopy category associated with this model category.

In fact, there are several possible choices for Spc_k and many possible model structures yielding the same homotopy category $\mathcal{H}(k)$. One possibility is to consider the category of simplicial presheaves over Sm_k, or the category of simplicial sheaves with respect to the Nisnevich topology. Both categories contain Sm_k as a full subcategory via the Yoneda embedding, and they also contain simplicial sets viewed as constant (pre)sheaves. One may then apply a general recipe for equipping the category of simplicial (pre)sheaves over a site with a model structure (see [Jar87]). In a crucial last step, one forces the affine line $\mathbb{A}^1$ to become contractible by localizing with respect to the set of all projections $\mathbb{A}^1 \times X \twoheadrightarrow X$.

As in topology, we also have a pointed version $\mathcal{H}_\bullet(k)$ of $\mathcal{H}(k)$. Remarkably, these two categories contain several distinct "circles": the simplicial circle S^1, the "Tate circle" $\mathbb{G}_m = \mathbb{A}^1 - 0$ (pointed at 1) and the projective line $\mathbb{P}^1$ (pointed at ∞). They are related by the intriguing formula $\mathbb{P}^1 = S^1 \wedge \mathbb{G}_m$. A common notational convention which we will follow is to define

$$S^{p,q} := (S^1)^{\wedge(p-q)} \wedge \mathbb{G}_m^{\wedge q}$$

for any $p \geq q$. In particular, we then have $S^1 = S^{1,0}$, $\mathbb{G}_m = S^{1,1}$ and $\mathbb{P}^1 = S^{2,1}$.

One can take the theory one step further by passing to the stable homotopy category $\mathcal{SH}(k)$, a triangulated category in which the suspension functors $S^{p,q} \wedge -$ become invertible. This category is usually constructed using $\mathbb{P}^1$-spectra. The triangulated shift functor is given by suspension with the simplicial sphere $S^{1,0}$.

Finally and crucially, the analogy with topology can be made precise: when we take our ground field k to be the complex numbers, or more general any subfield of $\mathbb{C}$, we have a complex realization functor

$$\mathcal{H}(k) \to \mathcal{H} \tag{4}$$

that sends a smooth scheme X to its set of complex points $X(\mathbb{C})$ equipped with the analytic topology. There is also a pointed realization functor and, moreover, a triangulated functor of the stable homotopy categories

$$\mathcal{SH}(k) \to \mathcal{SH} \tag{5}$$

which takes $\Sigma^\infty(X_+)$ to $\Sigma^\infty(X(\mathbb{C})_+)$ for any smooth scheme X [Rio06, Théorème I.123; Rio07a, Théorème 5.26].

2b Representing algebraic and hermitian K-theory

In the context of $\mathbb{A}^1$-homotopy theory, Theorem I.2.7 describing K^0 in terms of homotopy classes of maps to Grassmannians has an algebraic analogue. Grassmannians of r-planes in k^{n+r} can be constructed as smooth projective varieties over any field k. Viewing them as objects in Spc_k, we can form their colimits Gr_r and Gr in the same way as in topology. The following theorem is established in [MV99, § 4] and spelt out explicitly in [Rio06, Théorème III.3 and Assertion III.4].

2.1 Theorem. *For smooth schemes X over k we have natural identifications*

$$\mathrm{K}_0(X) \cong \mathcal{H}(k)(X, \mathbb{Z} \times \mathrm{Gr}) \tag{6}$$

such that the inclusion $\mathrm{Gr}_{r,n} \hookrightarrow \{d\} \times \mathrm{Gr}_{r,n} \hookrightarrow \mathbb{Z} \times \mathrm{Gr}$ *corresponds to the class* $[\mathcal{U}_{r,n}] + (d-r)[\mathcal{O}]$ *in* $\mathrm{K}_0(\mathrm{Gr}_{r,n})$.

An analogous result for hermitian K-theory has recently been obtained by Schlichting and Tripathi[1]: Let $\mathrm{Gr}^{\mathrm{nd}}_{r,n}$ denote the "non-degenerate Grassmannians" defined as open subvarieties of $\mathrm{Gr}_{r,r+2n}$ as above, and let $\mathrm{Gr}^{\mathrm{nd}}_r$ and $\mathrm{Gr}^{\mathrm{nd}}$ be the respective colimits. Then for smooth schemes over k we have natural identifications

$$\mathrm{GW}^0(X) \cong \mathcal{H}(k)(X, \mathbb{Z} \times \mathrm{Gr}^{\mathrm{nd}}) \tag{7}$$

It follows from the construction that, when $(d-r)$ is even, the inclusion of $\mathrm{Gr}^{\mathrm{nd}}_{r,n} \hookrightarrow \{d\} \times \mathrm{Gr}^{\mathrm{nd}}_{r,n} \hookrightarrow \mathbb{Z} \times \mathrm{Gr}^{\mathrm{nd}}$ corresponds to the class of $[\mathcal{U}^{\mathrm{nd}}_{r,n}] + \frac{d-r}{2}[\mathbb{H}]$ in $\mathrm{GW}^0(\mathrm{Gr}^{\mathrm{nd}}_{r,n})$, where $\mathcal{U}^{\mathrm{nd}}_{r,n}$ is the universal symmetric bundle over $\mathrm{Gr}^{\mathrm{nd}}_{r,n}$.

The fact that hermitian K-theory is representable in $\mathcal{H}(k)$ has been known for longer, see [Hor05]. One of the advantages of having a geometric description of a representing space, however, is that one can easily see what its complex realization is. In particular, this gives us an alternative way to define the comparison maps. For any smooth complex scheme X we have the following commutative squares, in which the left vertical arrows are the comparison maps (1) and (2), and the right vertical arrows are induced by the complex realization functor (4).

$$\begin{array}{ccc} \mathrm{K}_0(X) & \cong & \mathcal{H}(\mathbb{C})(X, \mathbb{Z} \times \mathrm{Gr}) \\ \downarrow & & \downarrow \\ \mathrm{K}^0(X(\mathbb{C})) & \cong & \mathcal{H}(X(\mathbb{C}), \mathbb{Z} \times \mathrm{Gr}) \end{array} \qquad \begin{array}{ccc} \mathrm{GW}^0(X) & \cong & \mathcal{H}(\mathbb{C})(X, \mathbb{Z} \times \mathrm{Gr}^{\mathrm{nd}}) \\ \downarrow & & \downarrow \\ \mathrm{KO}^0(X(\mathbb{C})) & \cong & \mathcal{H}(X(\mathbb{C}), \mathbb{Z} \times \mathrm{Gr}^{\mathrm{nd}}) \end{array}$$

Some of the results quoted here are in fact known in a much greater generality. Firstly, higher algebraic and hermitian K-groups of X are obtained by passing to suspensions of X in (6) and (7). Even better, algebraic and hermitian K-theory are representable in the stable $\mathbb{A}^1$-homotopy category $\mathcal{SH}(k)$. Let us make the statement a little more precise by fixing some notation. With any spectrum $\mathbb{E}$ in $\mathcal{SH}(k)$ one may associate a cohomology theory in exactly the same way as explained in Section I.2c, the only difference being that the theory obtained is now bigraded. Explicitly, a reduced cohomology theory $\widetilde{E}^{*,*}$ on

[1] Talk "Geometric representation of hermitian K-theory in $\mathbb{A}^1$-homotopy theory" at the Workshop "Geometric Aspects of Motivic Homotopy Theory", 6.–10. September 2010 at the Hausdorff Center for Mathematics, Bonn

$\mathcal{H}_\bullet(k)$ and a corresponding unreduced theory $E^{*,*}$ on $\mathcal{H}(k)$ are defined by setting

$$\widetilde{E}^{p,q}(\mathcal{X}) := \mathcal{SH}(k)(\Sigma^\infty\mathcal{X}, S^{p,q}\wedge\mathbb{E}) \qquad \text{for } \mathcal{X}\in\mathcal{H}_\bullet(k)$$
$$E^{p,q}(X) := \widetilde{E}^{p,q}(X_+) \qquad \text{for } X\in\mathcal{H}(k)$$

A spectrum $\mathbb{K}$ representing algebraic K-theory was first constructed in [Voe98, § 6.2]; see [Rio06] or [Rio07b] for some further discussion. It is (2,1)-periodic, meaning that in $\mathcal{SH}(k)$ we have an isomorphism

$$S^{2,1}\wedge\mathbb{K}\xrightarrow{\cong}\mathbb{K}$$

Thus, the bigrading of the corresponding cohomology theory $\mathrm{K}^{p,q}$ is slightly artificial. The identification with the usual notation for algebraic K-theory is given by

$$\mathrm{K}^{p,q}(X) = \mathrm{K}_{2q-p}(X) \tag{8}$$

For hermitian K-theory we have an (8,4)-periodic spectrum $\mathbb{K}\mathbf{O}$, and the corresponding cohomology groups $\mathrm{KO}^{p,q}$ are honestly bigraded. The translation into the notation used for hermitian K-groups in Section I.1d is given by

$$\mathrm{KO}^{p,q}(X) = \mathrm{GW}^q_{2q-p}(X) \tag{9}$$

We will refer to the number $2q-p$ as the degree of the group $\mathrm{KO}^{p,q}(X)$. The relation with Balmer's Witt groups obtained by combining (9) and (I.10) is illustrated by the following table:

$\mathbf{KO}^{p,q}$	$p=0$	1	2	3	4	5	6	7
$q=0$	$\mathbf{GW^0}$	$\mathbf{W^1}$	$\mathbf{W^2}$	$\mathbf{W^3}$	$\mathbf{W^0}$	$\mathbf{W^1}$	$\mathbf{W^2}$	$\mathbf{W^3}$
$q=1$	GW^1_2	GW^1_1	$\mathbf{GW^1}$	$\mathbf{W^2}$	$\mathbf{W^3}$	$\mathbf{W^0}$	$\mathbf{W^1}$	$\mathbf{W^2}$
$q=2$	GW^2_4	GW^2_3	GW^2_2	GW^2_1	$\mathbf{GW^2}$	$\mathbf{W^3}$	$\mathbf{W^0}$	$\mathbf{W^1}$
$q=3$	GW^3_6	GW^3_5	GW^3_4	GW^3_3	GW^3_2	GW^3_1	$\mathbf{GW^3}$	$\mathbf{W^0}$

As for the representing spaces in the unstable homotopy category, it is known that the complex realizations of $\mathbb{K}\mathbf{O}$ and $\mathbb{K}$ represent real and complex topological K-theory. This is well-documented in the latter case, see for example [Rio06, Proposition VI.12]. However, for $\mathbb{K}\mathbf{O}$ our references are slightly thin. Rather than shedding any further light on this problem, we will at this point succumb to an "axiomatic approach". For clarity, we make a record of the precise properties of the spectra $\mathbb{K}\mathbf{O}$ and $\mathbb{K}$ that we will be using.

2.2 Standing assumptions. *There exist spectra $\mathbb{K}$ and $\mathbb{K}\mathbf{O}$ in $\mathcal{SH}(\mathbb{C})$ representing algebraic K-theory and hermitian K-theory in the sense described above, such that:*

(a) The complex realization functor (5) *takes $\mathbb{K}$ to $\mathbb{K}^{top}$ and $\mathbb{K}\mathbf{O}$ to $\mathbb{K}\mathbf{O}^{top}$.*

(b) We have an exact triangle in $\mathcal{SH}(\mathbb{C})$ of the form

$$\mathbb{K}\mathbf{O} \wedge S^{1,1} \xrightarrow{\eta} \mathbb{K}\mathbf{O} \to \mathbb{K} \dashrightarrow \mathbb{K}\mathbf{O} \wedge S^{1,1} \tag{10}$$

which corresponds to the usual triangle in $\mathcal{SH}$.

These results are announced in [Mor06]. Independent constructions of spectra representing hermitian K-theory can be found in [Hor05] and in a recent preprint of Panin and Walter [PW10].

2c The comparison maps

Given the assumptions above, the existence of comparison maps with good properties is immediate. In fact, by assumption (a), complex realization induces comparison maps in arbitrary bidegrees. Thus, we have comparison maps

$$\begin{aligned} \widetilde{k}^{p,q}\colon \quad & \widetilde{\mathrm{K}}^{p,q}(\mathcal{X}) \to \widetilde{\mathrm{K}}^{p}(\mathcal{X}(\mathbb{C})) \\ \widetilde{k}_h^{p,q}\colon \quad & \widetilde{\mathrm{KO}}^{p,q}(\mathcal{X}) \to \widetilde{\mathrm{KO}}^{p}(\mathcal{X}(\mathbb{C})) \end{aligned}$$

for any pointed space $\mathcal{X}$, and similarly comparison maps $k^{p,q}$ and $k_h^{p,q}$ for any unpointed space, including in particular the case of a smooth complex scheme X. The maps we originally intended to study appear as special cases in degrees 0 and -1. They will be denoted by the same letters as before:

$$\begin{aligned} k := k^{0,0}\colon \quad & \mathrm{K}_0(X) \to \mathrm{K}^0(X(\mathbb{C})) \\ gw^q := k_h^{2q,q}\colon \quad & \mathrm{GW}^q(X) \to \mathrm{KO}^{2q}(X(\mathbb{C})) \\ w^q := k_h^{2q-1,q-1}\colon \quad & \mathrm{W}^q(X) \to \mathrm{KO}^{2q-1}(X(\mathbb{C})) \end{aligned}$$

In the following, we will usually write X for $X(\mathbb{C})$ when this is unambiguous. The following properties of the comparison maps follow directly from the construction:

- They commute with pullbacks along morphisms of smooth schemes.
- They are compatible with suspension isomorphisms.
- They are compatible with the periodicity isomorphisms, so that we may identify $k_h^{p,q}$ with $k_h^{p+8,q+4}$ (and hence w^q with w^{q+4} and gw^q with gw^{q+4}).
- They are compatible with long exact sequences arising from exact triangles in $\mathcal{SH}(\mathbb{C})$.

The crucial advantage of the homotopy-theoretic approach is, of course, the last point. It applies in particular to the following two kinds of sequences.

Localization sequences. As in topology, any morphism of smooth schemes $f\colon X' \to X$ gives rise to long exact sequences relating the cohomology groups of X' to those of X. In particular, if X has a smooth closed subscheme Z with open complement U, we have an exact triangle of the form

$$\begin{array}{ccc} \Sigma^\infty(X-Z)_+ & \longrightarrow & \Sigma^\infty(X_+) \\ & \nwarrow \cdots \quad \swarrow & \\ & \Sigma^\infty(X/U) & \end{array}$$

in $\mathcal{SH}(\mathbb{C})$, where X/U is obtained as the homotopy quotient of X by U in the category of spaces[1]. The induced long exact sequences of cohomology groups may be referred to as localization sequences. To clarify how the bigrading works, we write out one sequence explicitly in the example of hermitian K-theory:

$$\cdots \to \widetilde{\mathrm{KO}}^{p,q}(X/U) \to \mathrm{KO}^{p,q}(X) \to \mathrm{KO}^{p,q}(U) \to \widetilde{\mathrm{KO}}^{p+1,q}(X/U) \to \dots \tag{11}$$

As a further analogy with topology, Morel and Voevodsky show that the space X/U only depends on the normal bundle $\mathcal{N}$ of Z in X. To make this more precise, we introduce the Thom space of a vector bundle $\mathcal{E}$ over an arbitrary smooth scheme Z, defined as the homotopy quotient of $\mathcal{E}$ by the complement of the zero section:

$$\mathrm{Thom}_Z(\mathcal{E}) := {}^{\mathcal{E}}\!\big/_{(\mathcal{E}-Z)}$$

Using a geometric construction known as deformation to the normal bundle, Morel and Voevodsky show in Theorem 2.23 of [MV99, Chapter 3] that X/U is canonically isomorphic to $\mathrm{Thom}_Z(\mathcal{N})$ in the unstable pointed $\mathbb{A}^1$-homotopy category.

The comparison maps induce commutative ladder diagrams of the following form:

$$\begin{array}{ccccccccc} \dots \longrightarrow & \widetilde{\mathrm{KO}}^{p,q}(\mathrm{Thom}_Z\,\mathcal{N}) & \longrightarrow & \mathrm{KO}^{p,q}(X) & \longrightarrow & \mathrm{KO}^{p,q}(U) & \longrightarrow & \widetilde{\mathrm{KO}}^{p+1,q}(\mathrm{Thom}_Z\,\mathcal{N}) & \longrightarrow \dots \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ \dots \longrightarrow & \widetilde{\mathrm{KO}}^{p}(\mathrm{Thom}_Z\,\mathcal{N}) & \longrightarrow & \mathrm{KO}^{p}(X) & \longrightarrow & \mathrm{KO}^{p}(U) & \longrightarrow & \widetilde{\mathrm{KO}}^{p+1}(\mathrm{Thom}_Z\,\mathcal{N}) & \longrightarrow \dots \end{array}$$

Of course, part of the statement here is that the complex realization of the Thom space of $\mathcal{N}$ over Z can be identified with the corresponding Thom space in topology, but since we are working in the stable category, there is no need to check this explicitly: the two spaces are isomorphic in $\mathcal{SH}$ because they fit into isomorphic exact triangles. To see that they are already isomorphic in the unstable homotopy category $\mathcal{H}_\bullet$, we would need to be more careful and use a realization functor defined on the level of model categories. Such a functor is constructed in [Dug01].

[1] In the model category of spaces used in [MV99], inclusions are cofibrations, so the homotopy quotient of X by U coincides with the actual quotient.

Karoubi/Bott sequences. The Karoubi sequence arising from triangle (10) may be written explicitly as follows:

$$\begin{aligned}\dots \to \mathrm{KO}^{p-1,q}(X) \to \mathrm{KO}^{p-2,q-1}(X) \to \mathrm{K}_{2q-p}(X) \to \\ \mathrm{KO}^{p,q}(X) \to \mathrm{KO}^{p-1,q-1}(X) \to \mathrm{K}_{2q-p-1}(X) \to \dots\end{aligned}$$

By Assumption 2.2(b), the comparison maps induce a commutative ladder diagram with this sequence in the first row and the Bott sequence (I.20) in the second. Near degree zero, it takes the following familiar form:

$$\begin{array}{ccccccccccccc}\dots \to & \mathrm{KO}^{2i-1,i}X & \to & \mathrm{GW}^{i-1}X & \to & \mathrm{K}_0X & \to & \mathrm{GW}^iX & \to & \mathrm{W}^iX & \to & 0 & \to \dots \\ & \downarrow{\scriptstyle k_h^{2i-1,i}} & & \downarrow{\scriptstyle gw^{i-1}} & & \downarrow{\scriptstyle k} & & \downarrow{\scriptstyle gw^i} & & \downarrow{\scriptstyle w^i} & & \downarrow & \\ \dots \to & \mathrm{KO}^{2i-1}X & \to & \mathrm{KO}^{2i-2}X & \to & \mathrm{K}^0X & \to & \mathrm{KO}^{2i}X & \to & \mathrm{KO}^{2i-1}X & \to & \mathrm{K}^1X & \to \dots\end{array} \quad (12)$$

As a consequence, the comparison map w^i factors through $(\mathrm{KO}^{2i}/\,\mathrm{K})(X) \xrightarrow{\eta} \mathrm{KO}^{2i-1}(X)$, so that we have an induced map

$$w^i\colon\ \mathrm{W}^i(X) \to (\mathrm{KO}^{2i}/\,\mathrm{K})(X)$$

We will often switch between these two versions of w^i without further comment.

Groups with restricted support. Comparing the localization sequences (I.7) and (I.9) discussed in Chapter I to the localization sequence (11) above, we see that the groups $\widetilde{\mathrm{KO}}^{p,q}(X/U)$ play the role of hermitian K-groups of X supported on Z. This should be viewed as part of any representability statement, see for example [PW10, Theorem 6.5]. Alternatively, a formal identification of the groups in degrees zero and below using only the minimal assumptions we have stated could be achieved as follows:

2.3 Lemma. *Let Z be a smooth closed subvariety of a smooth variety X, with open complement U. We have the following isomorphisms:*

$$\begin{aligned}\widetilde{\mathrm{KO}}^{2q,q}(X/U) &\cong \mathrm{GW}^q_Z(X) \\ \widetilde{\mathrm{KO}}^{p,q}(X/U) &\cong \mathrm{W}^{p-q}_Z(X) \quad \textit{for} \quad 2q-p<0\end{aligned}$$

Proof. Consider $Z = Z\times\{0\}$ as a subvariety of $X\times\mathbb{A}^1$. Its open complement $(X\times\mathbb{A}^1)-Z$ contains $X = X\times\{1\}$ as a retract. Thus, the projection from $X\times\mathbb{A}^1$ onto X induces a splitting of the localization sequences associated with $(X\times\mathbb{A}^1 - Z) \hookrightarrow X\times\mathbb{A}^1$, and we have

$$\begin{aligned}\mathrm{GW}^{i+1}_Z(X\times\mathbb{A}^1) &\cong \operatorname{coker}\left(\mathrm{GW}^{i+1}_1(X\times\mathbb{A}^1) \hookrightarrow \mathrm{GW}^{i+1}_1(X\times\mathbb{A}^1-Z)\right) \\ \widetilde{\mathrm{KO}}^{2i+2,i+1}\left(\tfrac{X\times\mathbb{A}^1}{X\times\mathbb{A}^1-Z}\right) &\cong \operatorname{coker}\left(\mathrm{KO}^{2i+1,i+1}(X\times\mathbb{A}^1) \hookrightarrow \mathrm{KO}^{2i+1,i+1}(X\times\mathbb{A}^1-Z)\right)\end{aligned}$$

By (9), we can identify the groups appearing on the right, so we obtain an induced isomorphism of the cokernels. The quotient $X \times \mathbb{A}^1/(X \times \mathbb{A}^1 - Z)$ can be identified with the suspension of X/U by $S^{2,1}$, so we have an isomorphism

$$\widetilde{\mathrm{KO}}^{2i+2,i+1}(\tfrac{X\times\mathbb{A}^1}{X\times\mathbb{A}^1-Z}) \cong \widetilde{\mathrm{KO}}^{2i,i}(X/U)$$

On the other hand, by Lemma 1.1, we have an isomorphism

$$\mathrm{GW}^{i+1}_Z(X \times \mathbb{A}^1) \cong \mathrm{GW}^i_Z(X)$$

The proof in lower degrees is analogous. □

Twisting by line bundles. In the homotopy-theoretic approach, we can define hermitian K-groups with twists in a line bundle, or more generally in any vector bundle, in the same way as we did for KO-theory in Section I.2d.

2.4 Definition. For a vector bundle $\mathcal{E}$ of constant rank r over a smooth scheme X, we define the hermitian K-groups of X with coefficients in $\mathcal{E}$ by

$$\mathrm{KO}^{p,q}(X;\mathcal{E}) := \widetilde{\mathrm{KO}}^{p+2r,q+r}(\mathrm{Thom}\,\mathcal{E})$$

The twisted groups in degrees zero and below agree with the usual twisted Grothendieck-Witt and Witt groups:

2.5 Lemma. *For a vector bundle $\mathcal{E}$ over a quasi-projective variety X, we have isomorphisms*

$$\begin{aligned}\mathrm{KO}^{2q,q}(X;\mathcal{E}) &\cong \mathrm{GW}^q(X;\det\mathcal{E})\\ \mathrm{KO}^{p,q}(X;\mathcal{E}) &\cong \mathrm{W}^{p-q}(X;\det\mathcal{E}) \quad \textit{for} \quad 2q-p<0\end{aligned}$$

Proof. This follows from Lemma 2.3 and Nenashev's Thom isomorphisms for Witt groups: for any vector bundle $\mathcal{E}$ of rank r there is a canonical Thom class in $\mathrm{W}^r_X(\mathcal{E})$ which induces an isomorphism $\mathrm{W}^i(X;\det\mathcal{E}) \cong \mathrm{W}^{i+r}_X(\mathcal{E})$ by multiplication [Nen07, Theorem 2.5]. This Thom class actually comes from a class in $\mathrm{GW}^r_X(\mathcal{E})$. Using the Karoubi sequence (I.4), one sees that multiplication with this class also induces an isomorphism of the corresponding Grothendieck-Witt groups. □

Remark. The isomorphisms of Lemmas 2.3 and 2.5 are constructed here in a rather ad hoc fashion, and we have taken little care in recording their precise form. Whenever we give an argument concerning the comparison maps on "twisted groups" in the following, we do all constructions on the level of representable groups of Thom spaces. The identifications with the usual twisted groups are only needed to identify the final output of concrete calculations as in Section IV.3.

2d Comparison of the coefficient groups

As before, the comparison maps gw^i and w^i in degrees 0 and -1 are isomorphisms on a point. Given that we now also have maps in arbitrary degrees at our disposal, it is natural to ask whether they, too, are isomorphisms. However, a comparison with the situation in K-theory shows that this is not what we should expect. Indeed, it is easy to find counterexamples:

- In degrees -3 or less, the comparison map is necessarily zero. The problem is that while $\eta\colon \mathrm{W}^{p-q}(X) \to \mathrm{W}^{p-q}(X)$ is an isomorphism in all negative degrees, the topological η is nilpotent ($\eta^3 = 0$).

- In degree -2, we have $\mathrm{KO}^{0,-1}(\text{point}) \cong \mathrm{W}^1(\text{point})$ mapping to $\mathrm{KO}^0(\text{point})$. The first group is zero while the second is isomorphic to $\mathbb{Z}$.

- In degree 1, it is known that $\mathrm{KO}^{-1,0}(\text{point}) = \mathbb{Z}/2$ [Kar05, Example 18], from which we may deduce via the Karoubi sequence that $\mathrm{KO}^{1,1}(\text{point}) \cong \mathbb{C}^*$. In particular, $\mathrm{KO}^{1,1}(\text{point})$ cannot map isomorphically to $\mathrm{KO}^1(\text{point}) = 0$.

We now show that the map in degree 1 is always surjective, while the map in degree -2 is always injective.

2.6 Proposition. *For a complex point, the comparison maps in degrees* 1, 0, -1 *and* -2 *have the properties indicated by the following arrows.*

$$\begin{aligned}
\mathrm{KO}^{2q-1,q}(\text{point}) &\twoheadrightarrow \mathrm{KO}^{2q-1}(\text{point})\\
\mathrm{KO}^{2q,q}(\text{point}) &\xrightarrow{\cong} \mathrm{KO}^{2q}(\text{point})\\
\mathrm{KO}^{2q+1,q}(\text{point}) &\xrightarrow{\cong} \mathrm{KO}^{2q+1}(\text{point})\\
\mathrm{KO}^{2q+2,q}(\text{point}) &\rightarrowtail \mathrm{KO}^{2q+2}(\text{point})
\end{aligned}$$

Proof. The isomorphisms in degree zero can be deduced exactly as in Proposition 1.3. To deal with the map in degree 1, we note that the odd KO-groups of a complex point p are all trivial except for $\mathrm{KO}^{-1}(p)$, so $k_h^{2q-1,q}$ is trivially a surjection unless $q \equiv 0 \mod 4$. In that case, we claim that the surjectivity of $k_h^{-1,0}$ is clear from the following diagram:

$$\begin{array}{ccccccccc}
\ldots \longrightarrow & \mathrm{KO}^{-1,0}(p) & \longrightarrow & \mathrm{GW}^{-1}(p) & \longrightarrow & \mathrm{K}_0(p) & \longrightarrow \ldots\\
 & \big\downarrow k_h^{-1,0} & & \big\downarrow \cong & & \big\downarrow \cong & \\
\ldots \longrightarrow & \mathrm{KO}^{-1}(p) & \longrightarrow & \mathrm{KO}^{-2}(p) & \longrightarrow & \mathrm{K}^0(p) & \longrightarrow \ldots
\end{array}$$

Indeed, if we compute the groups appearing here, we obtain:

$$\begin{array}{ccccccccc}
\dots & \longrightarrow & \mathrm{KO}^{-1,0}(p) & \twoheadrightarrow & \mathbb{Z}/2 & \xrightarrow{0} & \mathbb{Z} & \longrightarrow & \dots \\
 & & \downarrow{\scriptstyle k_h^{-1,0}} & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} & & \\
\dots & \longrightarrow & \mathbb{Z}/2 & \xrightarrow{\cong} & \mathbb{Z}/2 & \xrightarrow{0} & \mathbb{Z} & \longrightarrow & \dots
\end{array}$$

In degree -2, three out of four cases are again trivial as $\mathrm{KO}^{2q+2,q}(p) = \mathrm{W}^{q+2}(p)$ is zero unless $q \equiv 2 \mod 4$. For the non-trivial case, consider the map η appearing in triangle (10). As the negative algebraic K-groups of p are zero, η yields automorphisms of $\mathrm{W}^{p-q}(p)$ in negative degrees. In topology, the corresponding maps are given by multiplication with a generator η of $\mathrm{KO}^{-1}(p)$, and η^2 generates $\mathrm{KO}^{-2}(p)$. So the commutative square

$$\begin{array}{ccc}
\mathrm{W}^0(p) & \xrightarrow{\cong} & \mathrm{W}^0(p) \\
\downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle k_h^{0,-2}} \\
\mathrm{KO}^{-1}(p) & \underset{\cong}{\xrightarrow{\eta}} & \mathrm{KO}^{-2}(p)
\end{array}$$

shows that $k_h^{0,-2}$ is an injection (in fact, an isomorphism), as claimed. □

2e Comparison with $\mathbb{Z}/2$-coefficients

Another advantage of defining the comparison maps as above is that we can easily pass to cohomology groups with torsion coefficients. Although these will not lie within our main focus, it seems worthwhile to point out some simple observations that can be made in the case of $\mathbb{Z}/2$-coefficients.

In general, suppose $E^{*,*}$ is some cohomology theory associated with a spectrum $\mathbb{E}$ in $\mathcal{SH}(k)$. A version of $E^{*,*}$ with $\mathbb{Z}/2$-coefficients can be defined in terms of the spectrum $\mathbb{E}/2$ that fits into the following exact triangle:

$$\begin{array}{ccc}
\mathbb{E} & \xrightarrow{2} & \mathbb{E} \\
 & \nwarrow \quad \swarrow & \\
 & \mathbb{E}/2 &
\end{array}$$

Here, 2 denotes the twofold sum of the identity map on $\mathbb{E}$. The triangle defines $\mathbb{E}/2$ up to isomorphism. It follows from the definition that the associated cohomology groups $E^{*,*}(X;\mathbb{Z}/2)$ fit into the following long exact sequences, which we refer to as Bockstein sequences:

$$\cdots \to E^{p,q}(X) \xrightarrow{\cdot 2} E^{p,q}(X) \to E^{p,q}(X;\mathbb{Z}/2) \to E^{p+1,q}(X) \xrightarrow{\cdot 2} \cdots$$

In particular, we can apply this definition to obtain algebraic and hermitian K-groups with $\mathbb{Z}/2$-coefficients. Moreover, since the same definitions work in topology, we obtain comparison maps

$$\begin{aligned} k_i\colon \quad \mathrm{K}_i(X;\mathbb{Z}/2) &\to \mathrm{K}^{-i}(X;\mathbb{Z}/2) \\ k_h^{p,q}\colon \mathrm{KO}^{p,q}(X;\mathbb{Z}/2) &\to \mathrm{KO}^{p}(X;\mathbb{Z}/2) \end{aligned}$$

These cohomology theories and the maps between them share all the formal properties of their integral counterparts. For example, the existence of Karoubi and Bott sequences with $\mathbb{Z}/2$-coefficients may be deduced from the 3×3-Lemma in triangulated categories [May01, Lemma 2.6], and the comparison maps are again compatible with these.

The behaviour of the comparison maps for K-theory with $\mathbb{Z}/2$-coefficients was predicted by the Quillen-Lichtenbaum conjectures: they are isomorphisms in all degrees $i \geq \dim(X) - 1$ and injective in degree $\dim(X) - 2$. Proofs may be found in [Lev99, Corollary 13.5 and Remark 13.2] and [Voe03b, Theorem 7.10]. In particular, on a complex point we have isomorphisms for all $i \geq 0$. This implies the same statement for the hermitian comparison maps. Namely, it is not difficult to see that the comparison maps on Witt groups with $\mathbb{Z}/2$-coefficients

$$w^i\colon\ \mathrm{W}^i(X;\mathbb{Z}/2) \to (\mathrm{KO}^{2i}/\,\mathrm{K})(X;\mathbb{Z}/2)$$

are also isomorphisms when X is a point. For example, this can be deduced from Proposition III.4.13 below, where we show more generally that these maps are isomorphisms whenever they are isomorphisms on the corresponding integral groups and the odd topological K-groups of X contain no 2-torsion. Thus, the following result may be obtained via Karoubi-induction.

2.7 Corollary. *The comparison maps*

$$k_h^{p,q}\colon\ \mathrm{KO}^{p,q}(\mathrm{point};\mathbb{Z}/2) \to \mathrm{KO}^{p}(\mathrm{point};\mathbb{Z}/2)$$

are isomorphisms in all non-negative degrees, i. e. for all (p,q) with $2q - p \geq 0$.

Chapter III

Curves and Surfaces

In the previous chapter, we introduced comparison maps

$$\begin{aligned} gw^i\colon\ \mathrm{GW}^i(X) &\to \mathrm{KO}^{2i}(X) \\ w^i\colon\ \mathrm{W}^i(X) &\to (\mathrm{KO}^{2i}/\,\mathrm{K})(X) \end{aligned}$$

for any smooth complex variety X and showed that they are isomorphisms when X is a point. Here, we analyse what happens when X has dimension one or two. Our final result, proved in Theorems 4.4 and 4.12, is the following:

Theorem. *If X is a smooth complex curve, the comparison maps gw^i are surjective and the maps w^i are isomorphisms. If X is a smooth complex surface, the same claim holds if and only if every continuous complex line bundle over X is algebraic, i. e. if and only if the natural map* $\mathrm{Pic}(X) \to H^2(X;\mathbb{Z})$ *is surjective.*

Some more detailed statements concerning the situation for surfaces are given in Proposition 4.6. For projective surfaces, the condition on line bundles is equivalent to the condition that the geometric genus $h^{2,0}$ of X is zero.

Our proof of this result is largely computational. Namely, we explicitly compute both the Witt groups $\mathrm{W}^i(X)$ and the groups $(\mathrm{KO}^{2i}/\,\mathrm{K})(X)$ of arbitrary smooth complex varieties of dimension at most two, from which we see that they are abstractly isomorphic in the cases claimed. In degree zero, we then deduce that the isomorphism is induced by the comparison map w^0 from the fact that all elements in $\mathrm{W}^0(X)$ are detected by the first two Stiefel-Whitney classes. The shifted groups $\mathrm{W}^1(X)$ and $\mathrm{W}^2(X)$ require more work. To obtain the assertions concerning the comparison maps, we decompose an arbitrary surface into a union of curves and an affine piece whose Picard group vanishes modulo two.

The computation of the classical Witt group $\mathrm{W}^0(X)$ of a complex curve or surface is not new. It was accomplished by Fernández-Carmena more than 20 years ago [FC87], and, at its core, his calculation is very similar to the one we present. However, we have found it convenient to rewrite the computation in terms of more general machinery developed over the past years by Balmer, Walter and Pardon. Not only does this make more transparent which steps become more difficult in higher dimensions, but also the values of the shifted Witt groups $\mathrm{W}^i(X)$ drop out of the calculation for free. It turns out that we may in fact work over any algebraically closed field of characteristic not two.

The structure of this chapter is as follows. In the first section, we recall some facts about Stiefel-Whitney classes in algebraic geometry and topology, and explain why they coincide for complex varieties. In Section 2, we include a computation of the Grothendieck-Witt groups of smooth complex projective curves. The result is not needed to prove the above comparison theorem, but it completes the picture in this dimension. The main calculations of Witt groups and of the groups $(\mathrm{KO}^{2i}/\mathrm{K})(X)$ are presented in Section 3, leading to a proof of the comparison theorem in Section 4. Finally, we briefly analyse how our result is related to the Quillen-Lichtenbaum conjecture and its analogue for hermitian K-theory, which has recently appeared in [Sch10].

1 Stiefel-Whitney classes

As we will see, all elements in the Witt groups $\mathrm{W}^0(X)$ of a curve or surface can be detected by the rank homomorphism and the first two Stiefel-Whitney classes. We therefore include a brief account of the general theory of these characteristic classes in this section. In short, Stiefel-Whitney classes are invariants of symmetric bundles that take values in the étale cohomology groups of X with $\mathbb{Z}/2$-coefficients. First, we summarize a construction of these classes mimicking Grothendieck's construction of Chern classes via projective bundles, and we state their main properties. We then specialize to the case of Stiefel-Whitney classes over fields, as studied by Milnor. The particular appeal of the first two classes stems from the fact that they can be used to identify certain graded pieces of the Witt group of a field with étale cohomology groups. The existence of similar identifications in higher degrees was one of the claims of the now famous Milnor conjectures. However, we sketch only the barest outlines of this story. Finally, we explain the relation of these classes over complex varieties to the classical Stiefel-Whitney classes used in topology.

Stiefel-Whitney classes over varieties

The following construction of Stiefel-Whitney classes due to Delzant [Del62] and Laborde [Lab76] is detailed in [EKV93, § 5]. It works for any scheme X over $\mathbb{Z}[\frac{1}{2}]$.

The first Stiefel-Whitney class w_1 of a symmetric line bundle over X is defined by the correspondence of isometry classes of such bundles with elements in $H^1_{\mathrm{et}}(X;\mathbb{Z}/2)$ (see Example I.1.1). For an arbitrary symmetric bundle $(\mathcal{E},\varepsilon)$, one considers the scheme $\mathbb{P}_{\mathrm{nd}}(\mathcal{E},\varepsilon)$ given by the complement of the quadric in $\mathbb{P}(\mathcal{E})$ defined by ε. The restriction of the universal line bundle $\mathcal{O}(-1)$ on $\mathbb{P}(\mathcal{E})$ to $\mathbb{P}_{\mathrm{nd}}(\mathcal{E},\varepsilon)$ carries a canonical symmetric form. Let w be its first Stiefel-Whitney class in $H^1_{\mathrm{et}}(\mathbb{P}_{\mathrm{nd}}(\mathcal{E},\varepsilon);\mathbb{Z}/2)$. Then the cohomology of $\mathbb{P}_{\mathrm{nd}}(\mathcal{E},\varepsilon)$ can be decomposed as

$$H^*_{\mathrm{et}}(\mathbb{P}_{\mathrm{nd}}(\mathcal{E},\varepsilon);\mathbb{Z}/2) = \bigoplus_{i=0}^{r-1} p^* H^*_{\mathrm{et}}(X;\mathbb{Z}/2)\cdot w^i \qquad (1)$$

where p is the projection of $\mathbb{P}_{\mathrm{nd}}(\mathcal{E},\varepsilon)$ onto X. In particular, w^r is a linear combination of smaller powers of w, so that for certain coefficients

$$w_i(\mathcal{E},\varepsilon) \in H^i(X;\mathbb{Z}/2)$$

we have

$$w^r + p^*w_1(\mathcal{E},\varepsilon)\cdot w^{r-1} + p^*w_2(\mathcal{E},\varepsilon)\cdot w^{r-2} + \cdots + p^*w_r(\mathcal{E},\varepsilon) = 0$$

These coefficients $w_i(\mathcal{E},\varepsilon)$ are defined to be the Stiefel-Whitney classes of $(\mathcal{E},\varepsilon)$. They are characterized by the following axiomatic description [EKV93, § 1]:

Normalization. The first Stiefel-Whitney class of a symmetric line bundle w_1 is as defined above.

Boundedness. $w_i(\mathcal{E},\varepsilon) = 0$ for all $i > \mathrm{rk}(\mathcal{E})$

Naturality. For any morphism $f\colon Y \to X$, we have $f^*w_i(\mathcal{E},\varepsilon) = w_i(f^*(\mathcal{E},\varepsilon))$.

Whitney sum formula. The total Stiefel-Whitney class $w_t := 1 + \sum_{i\geq 1} w_i(\mathcal{E},\varepsilon)t^i$ satisfies

$$w_t((\mathcal{E},\varepsilon)\perp(\mathcal{F},\varphi)) = w_t(\mathcal{E},\varepsilon)\cdot w_t(\mathcal{F},\varphi)$$

The Stiefel-Whitney classes of a metabolic bundle only depend on the Chern classes of its Lagrangian. More precisely, Proposition 5.5 in [EKV93] gives the following formula for a metabolic bundle $(\mathcal{M},\mu)$ with Lagrangian $\mathcal{L}$:

$$w_t(\mathcal{M},\mu) = \sum_{j=0}^{\mathrm{rk}(\mathcal{L})} (1+(-1)t)^{n-j} c_j(\mathcal{L}) t^{2j} \tag{2}$$

For example, for the first two Stiefel-Whitney classes we have

$$w_1(\mathcal{M},\mu) = \mathrm{rk}(\mathcal{L})(-1) \tag{3}$$

$$w_2(\mathcal{M},\mu) = \binom{\mathrm{rk}(\mathcal{L})}{2}(-1,-1) + c_1(\mathcal{L}) \tag{4}$$

It follows that w_t descends to a well-defined homomorphism from the Grothendieck-Witt group of X to the multiplicative group of invertible elements in $\bigoplus_i H^i_{\mathrm{et}}(X;\mathbb{Z}/2)t^i$:

$$w_t\colon \mathrm{GW}^0(X) \longrightarrow \left(\bigoplus_i H^i_{\mathrm{et}}(X;\mathbb{Z}/2)t^i\right)^\times$$

In particular, the individual classes descend to well-defined maps

$$w_i\colon\ \mathrm{GW}^0(X) \to H^i_{\mathrm{et}}(X;\mathbb{Z}/2)$$

In general, none of the individual Stiefel-Whitney classes apart from w_1 define homomorphisms on $\mathrm{GW}^0(X)$. It does follow from the Whitney sum formula, however, that w_2 restricts to a homomorphism on the kernel of w_1, and in general w_i restricts to a homomorphism on the kernel of (the restriction of) w_{i-1}.

It is not generally true either that the Stiefel-Whitney classes factor through the Witt group of X: the right-hand side of (2) may be non-zero. However, we may deduce from (3) that the restriction of the first Stiefel-Whitney class to the reduced group $\widetilde{\mathrm{GW}}^0(X)$ factors through $\widetilde{\mathrm{W}}^0(X)$, yielding a map

$$\overline{w}_1 \colon \widetilde{\mathrm{W}}^0(X) \longrightarrow H^1_{\mathrm{et}}(X;\mathbb{Z}/2)$$

We use a different notation to emphasize that the values of w_1 and $\overline{w}_1$ on a given symmetric bundle may differ. That is, if $(\mathcal{E},\varepsilon)$ is a symmetric bundle of even rank defining an element Ψ in $\widetilde{\mathrm{W}}^0(X)$, then in general $\overline{w}_1(\Psi) \neq w_1(\mathcal{E},\varepsilon)$. Rather, $\overline{w}_1$ needs to be computed on a lift of $(\mathcal{E},\varepsilon)$ to $\widetilde{\mathrm{GW}}^0(X)$. Explicitly, if $\mathbb{H}$ denotes the constant hyperbolic bundle $(\mathcal{O}\oplus\mathcal{O}, (\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}))$, then

$$\begin{aligned} \overline{w}_1(\Psi) &= w_1\left((\mathcal{E},\varepsilon) - \tfrac{\mathrm{rk}(\mathcal{E})}{2}\mathbb{H}\right) \\ &= w_1(\mathcal{E},\varepsilon) - \tfrac{\mathrm{rk}(\mathcal{E})}{2}(-1) \end{aligned}$$

We see similarly from (4) that w_2 induces a well-defined map

$$\overline{w}_2 \colon \widetilde{\mathrm{W}}^0(X) \longrightarrow H^2_{\mathrm{et}}(X;\mathbb{Z}/2)\big/\mathrm{Pic}(X)$$

As before, $\overline{w}_1$ is a surjective homomorphism, while $\overline{w}_2$ restricts to a homomorphism on $\ker(\overline{w}_1)$.

Stiefel-Whitney classes over fields

When X is a field F (of characteristic not 2), the Stiefel-Whitney classes factor through Milnor's K-groups modulo two, which are commonly denoted $k_i^M(F)$. We will denote the classes with values in $k_i^M(F)$ by w_i^M:

$$\begin{array}{ccc} \mathrm{GW}^0(F) & \xrightarrow{\quad w_i \quad} & H^i_{\mathrm{et}}(F;\mathbb{Z}/2) \\ & {\scriptstyle w_i^M}\searrow \quad \nearrow{\scriptstyle \alpha} & \\ & k_i^M(F) & \end{array}$$

Both the groups $k_i^M(F)$ and the classes w_i^M were constructed in [Mil69]. In the same paper, Milnor asked whether the map α appearing in the factorization was an isomorphism, a question that later became known as one of the Milnor conjectures. For $i \leq 2$, which will be the range mainly relevant for us, an affirmative answer was given by Merkurjev [Mer81]. A general affirmation of the conjecture was found more recently by Voevodsky [Voe03b].

A second conjecture of Milnor, also to be found in [Mil69], concerned the relation of $\mathrm{W}^0(F)$ to $k_i^M(F)$. To state it, we introduce the fundamental filtration. If we view $\mathrm{W}^0(F)$ as a ring, then the reduced Witt group $\widetilde{\mathrm{W}}^0(F)$ becomes an ideal inside $\mathrm{W}^0(F)$, which is traditionally written as $I(F)$. The powers of this ideal yield a filtration

$$\mathrm{W}^0(F) \supset I(F) \supset I^2(F) \supset I^3(F) \supset \dots$$

on the Witt ring of F, known as the fundamental filtration. Milnor conjectured that the associated graded ring was isomorphic to $k_*^M(F) := \oplus_i k_i^M(F)$. As a first step towards a proof, he constructed maps $k_i^M(F) \to I^i(F)/I^{i+1}(F)$ in one direction. Moreover, in degrees one and two, Milnor could show that these are isomorphisms, with explicit inverses induced by the Stiefel-Whitney classes w_1^M and w_2^M. In combination with the isomorphisms α above, one obtains the following identifications:

$$\begin{aligned} \mathrm{rk}\colon\ \mathrm{W}^0(F)\big/I(F) &\xrightarrow{\cong} H^0_{\mathrm{et}}(F;\mathbb{Z}/2)\\ \overline{w}_1\colon\ I(F)\big/I^2(F) &\xrightarrow{\cong} H^1_{\mathrm{et}}(F;\mathbb{Z}/2)\\ \overline{w}_2\colon\ I^2(F)\big/I^3(F) &\xrightarrow{\cong} H^2_{\mathrm{et}}(F;\mathbb{Z}/2) \end{aligned}$$

[Mil69, § 4]. Today, these isomorphisms are commonly denoted e^0, e^1, e^2. It was clear from the outset, however, that the higher isomorphisms

$$e^i\colon I^i(F)\big/I^{i+1}(F) \xrightarrow{\cong} H^i_{\mathrm{et}}(F;\mathbb{Z}/2) \tag{5}$$

conjectured by Milnor could not be induced by higher Stiefel-Whitney classes. Their existence was ultimately proved in [OVV07]. Unlike in the case of Stiefel-Whitney classes, it does not seem to be clear how these maps may be generalized to varieties.

Stiefel-Whitney classes over complex varieties

Here, we show that over a complex variety the étale Stiefel-Whitney classes of symmetric vector bundles are compatible with the Stiefel-Whitney classes of real vector bundles used in topology.

Suppose first that Y is an arbitrary CW complex. If we follow the construction of Stiefel-Whitney classes described above, with singular cohomology in place of étale cohomology, we obtain classes $w_i(\mathcal{E},\varepsilon)$ in $H^i(Y;\mathbb{Z}/2)$ for every complex symmetric bundle $(\mathcal{E},\varepsilon)$ over Y. On the other hand, given a real vector bundle $\mathcal{F}$ over Y, we have the usual Stiefel-Whitney classes $w_i(\mathcal{F})$. We claim that these classes are compatible in the following sense. Recall from Corollary I.2.4 that we have a one-to-one correspondence

$$\Re\colon \begin{pmatrix}\text{isometry classes of}\\ \text{complex symmetric}\\ \text{bundles over } Y\end{pmatrix} \to \begin{pmatrix}\text{isomorphism classes}\\ \text{of real vector bundles}\\ \text{over } Y\end{pmatrix}$$

1.1 Lemma. *For any complex symmetric vector bundle $(\mathcal{E}, \varepsilon)$ over a CW complex Y, the classes $w_i(\mathcal{E}, \varepsilon)$ and $w_i(\Re(\mathcal{E}, \varepsilon))$ in $H^i(Y; \mathbb{Z}/2)$ agree.*

Proof. For a complex symmetric line bundle $(\mathcal{L}, \lambda)$ over Y, the first Stiefel-Whitney classes of $(\mathcal{L}, \lambda)$ and $\Re(\mathcal{L}, \lambda)$ in $H^1(Y; \mathbb{Z}/2)$ agree because both bundles have the same associated principal $\mathbb{Z}/2$-bundle. In general, the Stiefel-Whitney classes of a real vector bundle $\mathcal{F}$ may be defined in the same way as the classes $w_i(\mathcal{E}, \varepsilon)$, using its real projectivization $\mathbb{RP}(\mathcal{F})$ in place of the space $\mathbb{P}_{\mathrm{nd}}(\mathcal{E}, \varepsilon)$ [Hus94, Theorem 2.5 and Definition 2.6]. Thus, the lemma follows from the considerations of Section I.2a. Indeed, the spaces $\mathbb{P}_{\mathrm{nd}}(\mathcal{E}, \varepsilon)$ are instances of the non-degenerate Grassmannians introduced there. In particular, for any complex symmetric bundle $(\mathcal{E}, \varepsilon)$, the space $\mathbb{P}_{\mathrm{nd}}(\mathcal{E}, \varepsilon)$ contains $\mathbb{RP}(\Re(\mathcal{E}, \varepsilon))$ as a homotopy equivalent subspace. Let us denote the inclusion and the bundle projections as in the following diagram:

$$\begin{array}{ccc} \mathbb{RP}(\Re(\mathcal{E}, \varepsilon)) & \overset{j}{\hookrightarrow} & \mathbb{P}_{\mathrm{nd}}(\mathcal{E}, \varepsilon) \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle p_{\mathrm{nd}}} & \\ & Y & \end{array}$$

The universal symmetric bundle $\mathcal{O}(-1)_{\mathrm{nd}}$ over $\mathbb{P}_{\mathrm{nd}}(\mathcal{E}, \varepsilon)$ restricts to the universal real bundle $\mathcal{O}(-1)$ over $\mathbb{RP}(\Re(\mathcal{E}, \varepsilon))$ in the sense that $\Re(j^*\mathcal{O}(-1)_{\mathrm{nd}}) = \mathcal{O}(-1)$. In particular, by the naturality of Stiefel-Whitney classes and the above observation concerning line bundles, we have

$$j^* w_1(\mathcal{O}(-1)_{\mathrm{nd}}) = w_1(\mathcal{O}(-1))$$

Let $r := \mathrm{rk}(\mathcal{E})$. Then the Stiefel-Whitney classes of $(\mathcal{E}, \varepsilon)$ are defined by the following relations in the cohomology groups $H^r(\mathbb{P}_{\mathrm{nd}}(\mathcal{E}, \varepsilon); \mathbb{Z}/2)$ and $H^r(\mathbb{RP}(\Re(\mathcal{E}, \varepsilon); \mathbb{Z}/2)$, respectively:

$$w_1(\mathcal{O}(-1)_{\mathrm{nd}})^r = \sum_{i=1}^{r} w_1(\mathcal{O}(-1)_{\mathrm{nd}})^{r-i} p_{\mathrm{nd}}^* w_i(\mathcal{E}, \varepsilon)$$
$$w_1(\mathcal{O}(-1))^r = \sum_{i=1}^{r} w_1(\mathcal{O}(-1))^{r-i} p^* w_i(\Re(\mathcal{E}, \varepsilon))$$

The claim follows by applying j^* to the first line and comparing it with the second. □

Now let X be a complex variety. Then the above lemma implies that the étale and the topological Stiefel-Whitney classes are compatible. Moreover, if we specialize equation (2) for the Stiefel-Whitney classes of a metabolic bundle $(\mathcal{M}, \mu)$ with Lagrangian $\mathcal{L}$ to the case of a complex variety, then since -1 is a square in $\mathbb{C}$ we find that

$$w_{2i}(\mathcal{M},\mu) = c_i(\mathcal{L}) \in H^{2i}_{\mathrm{et}}(X;\mathbb{Z}/2)$$

and all odd Stiefel-Whitney classes of $(\mathcal{M},\mu)$ vanish. This corresponds to the well-known fact in topology that the even Stiefel-Whitney classes of a complex vector bundle agree with its Chern classes modulo two, whereas its odd Stiefel-Whitney classes are zero [MS74, Problem 14.B]. It follows in particular that the odd Stiefel-Whitney classes factor through the (reduced) Witt group of X, while the even classes induce maps

$$\overline{w}_{2i}\colon \widetilde{\mathrm{W}}^0(X) \longrightarrow \frac{H^{2i}_{\mathrm{et}}(X;\mathbb{Z}/2)}{\mathrm{im}(c_i)}$$

We summarize the situation as follows.

1.2 Proposition. *Let X be a complex variety. The Stiefel-Whitney classes factor through the reduced Grothendieck-Witt and KO-group of X to yield commutative diagrams*

$$\begin{array}{ccc} \widetilde{\mathrm{GW}}^0(X) & \xrightarrow{w_i} & H^i_{et}(X;\mathbb{Z}/2) \\ \downarrow & & \downarrow \cong \\ \widetilde{\mathrm{KO}}^0(X) & \xrightarrow{w_i} & H^i(X;\mathbb{Z}/2) \end{array}$$

Moreover, for all odd i we have induced maps

$$\begin{array}{ccc} \widetilde{\mathrm{W}}^0(X) & \xrightarrow{\overline{w}_i} & H^i_{et}(X;\mathbb{Z}/2) \\ \downarrow & & \downarrow \cong \\ (\widetilde{\mathrm{KO}}^0/\widetilde{\mathrm{K}})(X) & \xrightarrow{\overline{w}_i} & H^i(X;\mathbb{Z}/2) \end{array}$$

and for even i we have induced maps

$$\begin{array}{ccc} \widetilde{\mathrm{W}}^0(X) & \xrightarrow{\overline{w}_i} & \dfrac{H^i_{et}(X;\mathbb{Z}/2)}{\mathrm{im}(c_i)} \\ \downarrow & & \downarrow \\ (\widetilde{\mathrm{KO}}^0/\widetilde{\mathrm{K}})(X) & \xrightarrow{\overline{w}_i} & \dfrac{H^i(X;\mathbb{Z}/2)}{\mathrm{im}(c_i)} \end{array}$$

2 Curves

In the next section, we will compute the Witt groups and the groups $(\mathrm{KO}^{2i}/\mathrm{K})(X)$ for smooth varieties of dimension at most two. Here, we briefly summarize the results we will obtain in the case of curves. Moreover, we give a complete description of their

Grothendieck-Witt groups. This description will not be used in the proofs of the main comparison theorems of this chapter but is included merely for the purposes of illustration and completeness.

2a Grothendieck-Witt groups of curves

Let C be a smooth curve over an algebraically closed field k of characteristic not 2, and let $\mathrm{Pic}(C)$ be its Picard group.

If C is projective, say of genus g, we may write $\mathrm{Pic}(C)$ as $\mathbb{Z} \oplus \mathrm{Jac}(C)$. The free summand $\mathbb{Z}$ is generated by a line bundle $\mathcal{O}(p)$ associated with a point p on C, while $\mathrm{Jac}(C)$ denotes (the closed points of) the Jacobian of C, a g-dimensional abelian variety parametrizing line bundles of degree zero over C. As a group, $\mathrm{Jac}(C)$ is two-divisible, and $\mathrm{Jac}(C)[2]$ has rank $2g$ (e. g. [Mil08, Chapter 14]). In particular, when C is projective, we have

$$H^1_{\mathrm{et}}(C; \mathbb{Z}/2) = \mathrm{Pic}(C)[2] = (\mathbb{Z}/2)^{2g}$$
$$H^2_{\mathrm{et}}(C; \mathbb{Z}/2) = \mathrm{Pic}(C)/2 = \mathbb{Z}/2$$

If C is not projective, it is affine and may be obtained from a smooth projective curve by removing a finite number of points. Note that when we remove a single point p from a projective curve $\overline{C}$, the Picard group is reduced to $\mathrm{Pic}(\overline{C} - p) = \mathrm{Jac}(\overline{C})$. It follows that the Picard group of any affine curve is two-divisible. In particular, for any affine curve C we have

$$H^2_{\mathrm{et}}(C; \mathbb{Z}/2) = \mathrm{Pic}(C)/2 = 0$$

The reduced algebraic K-group of a smooth curve may be identified with its Picard group via the first Chern class, so that we have an isomorphism

$$\mathrm{K}_0(C) \cong \mathbb{Z} \oplus \mathrm{Pic}(C) \tag{6}$$

The following proposition shows that, similarly, the Grothendieck-Witt and Witt groups of C are completely determined by $\mathrm{Pic}(C)$ and the group $H^1_{\mathrm{et}}(C; \mathbb{Z}/2)$ of symmetric line bundles.

2.1 Proposition. *Let C be a smooth curve over an algebraically closed field of characteristic not 2. The Grothendieck-Witt and Witt groups of C are as follows:*

$$\begin{aligned}
\mathrm{GW}^0(C) &= [\mathbb{Z}] \oplus H^1_{et}(C; \mathbb{Z}/2) \oplus H^2_{et}(C; \mathbb{Z}/2) & \mathrm{W}^0(C) &= [\mathbb{Z}/2] \oplus H^1_{et}(C; \mathbb{Z}/2)\\
\mathrm{GW}^1(C) &= \mathrm{Pic}(C) & \mathrm{W}^1(C) &= H^2_{et}(C; \mathbb{Z}/2)\\
\mathrm{GW}^2(C) &= [\mathbb{Z}] & \mathrm{W}^2(C) &= 0\\
\mathrm{GW}^3(C) &= [\mathbb{Z}/2] \oplus \mathrm{Pic}(C) & \mathrm{W}^3(C) &= 0
\end{aligned}$$

Here, the summands in square brackets are the trivial ones coming from a point, i. e. those that disappear when passing to reduced groups. In particular, for a projective curve of genus g we obtain:

$$\begin{aligned}
\mathrm{GW}^0(C) &= [\mathbb{Z}] \oplus (\mathbb{Z}/2)^{2g+1} & \mathrm{W}^0(C) &= [\mathbb{Z}/2] \oplus (\mathbb{Z}/2)^{2g} \\
\mathrm{GW}^1(C) &= \mathbb{Z} \oplus \mathrm{Jac}(C) & \mathrm{W}^1(C) &= \mathbb{Z}/2 \\
\mathrm{GW}^2(C) &= [\mathbb{Z}] & \mathrm{W}^2(C) &= 0 \\
\mathrm{GW}^3(C) &= [\mathbb{Z}/2] \oplus \mathbb{Z} \oplus \mathrm{Jac}(C) & \mathrm{W}^3(C) &= 0
\end{aligned}$$

For affine curves, no non-trivial twists are possible. When C is projective, the groups twisted by a generator $\mathcal{O}(p)$ of $\mathrm{Pic}(C)/2$ are as follows:

$$\begin{aligned}
\mathrm{GW}^0(C,\mathcal{O}(p)) &= \mathbb{Z} \oplus (\mathbb{Z}/2)^{2g} & \mathrm{W}^0(C,\mathcal{O}(p)) &= (\mathbb{Z}/2)^{2g} \\
\mathrm{GW}^1(C,\mathcal{O}(p)) &= \mathbb{Z} \oplus \mathrm{Jac}(C) & \mathrm{W}^1(C,\mathcal{O}(p)) &= 0 \\
\mathrm{GW}^2(C,\mathcal{O}(p)) &= \mathbb{Z} & \mathrm{W}^2(C,\mathcal{O}(p)) &= 0 \\
\mathrm{GW}^3(C,\mathcal{O}(p)) &= \mathbb{Z} \oplus \mathrm{Jac}(C) & \mathrm{W}^3(C,\mathcal{O}(p)) &= 0
\end{aligned}$$

2.2 Remark (Explicit generators). The isomorphism between $\widetilde{\mathrm{W}}^0(C)$ and $H^1_{\mathrm{et}}(X;\mathbb{Z}/2)$ is the obvious one, i. e. $\widetilde{\mathrm{W}}^0(C)$ is generated by symmetric line bundles (see Example I.1.1). When C is projective of genus g, the group $\mathrm{W}^1(C)$ has a single generator Ψ_g, which for $g = 0$ we may take to be the 1-symmetric complex given in Example I.1.6. When $g \geq 1$, a generator may be constructed as follows: Pick two distinct points p and q on C. Let s and t be sections $\mathcal{O} \to \mathcal{O}(p)$ and $\mathcal{O} \to \mathcal{O}(q)$ of the associated line bundles that vanish at p and q, respectively. Choose a line bundle $\mathcal{L} \in \mathrm{Pic}(C)$ that squares to $\mathcal{O}(p-q)$. Then

$$\Psi_g := \left(\begin{array}{ccc} \mathcal{L} \otimes \mathcal{O}(-p) & \xrightarrow{\;s\;} & \mathcal{L} \\ \downarrow t & & \downarrow -t \\ \mathcal{L} \otimes \mathcal{O}(q-p) & \xrightarrow{\;-s\;} & \mathcal{L} \otimes \mathcal{O}(q) \end{array} \right) \tag{7}$$

is a 1-symmetric complex over C generating $\mathrm{W}^1(C)$.

Proof of Proposition 2.1, assuming Proposition 3.1:
The values of the untwisted Witt groups may be read off directly from Proposition 3.1 below: we simply note that $c_1\colon \mathrm{Pic}(C) \to H^2_{\mathrm{et}}(C;\mathbb{Z}/2)$ is surjective and induces an isomorphism $\mathrm{Pic}(C)/2 \cong H^2_{\mathrm{et}}(C;\mathbb{Z}/2)$. The twisted groups $\mathrm{W}^i(C,\mathcal{O}(p))$ of a projective curve C may then be calculated from the localization sequence (I.5) associated with the inclusion of the complement of a point p of C into C. Indeed, since the line bundle $\mathcal{O}(p)$ is trivial over $C-p$, this sequences takes the following form:

$$\cdots \to \mathrm{W}^{i-1}(p) \to \mathrm{W}^i(C,\mathcal{O}(p)) \to \mathrm{W}^i(C-p) \to \mathrm{W}^i(p) \to \cdots$$

We now deduce the values of the Grothendieck-Witt groups, concentrating on the case when C is projective. The affine case is very similar, except that all extension problems disappear.

Untwisted case. To compute the untwisted Grothendieck-Witt groups of a projective curve C, we use the Karoubi sequences (I.4) and their restrictions to reduced groups:

$$\widetilde{\mathrm{GW}}^{i-1}(C) \xrightarrow{F} \widetilde{\mathrm{K}}_0(C) \xrightarrow{H^i} \widetilde{\mathrm{GW}}^i(C) \twoheadrightarrow \widetilde{\mathrm{W}}^i(C) \to 0$$

Note first that the identification of the K-group of our projective curve C given in (6) may be written explicitly as

$$\begin{aligned} \mathrm{K}_0(C) &\xleftrightarrow{\cong} \mathbb{Z} \oplus \mathbb{Z} \oplus \mathrm{Jac}(C) \\ [\mathcal{E}] &\mapsto (\mathrm{rk}\mathcal{E}, d, \det \mathcal{E}(-d)) \\ [\mathcal{L}(d)] + (r-1)[\mathcal{O}] &\leftarrow\!\shortmid (r, d, \mathcal{L}) \end{aligned}$$

Here, in the map from left to right, we have written d for the degree of the determinant line bundle of $\mathcal{E}$. Given this description, the endomorphisms FH^i of $\mathrm{K}_0(C)$ may easily be computed directly. For the restrictions to the reduced K-group we obtain:

$$\mathrm{im}(FH^i) = \begin{cases} 0 & \subset \widetilde{\mathrm{K}}_0(C) \quad \text{when } i \text{ is even} \\ 2\mathbb{Z} \oplus \mathrm{Jac}(C) & \subset \widetilde{\mathrm{K}}_0(C) \quad \text{when } i \text{ is odd} \end{cases} \tag{8}$$

To compute $\widetilde{\mathrm{GW}}^0(C)$, we first observe that $\mathrm{W}^3(C) = 0$ implies that $\widetilde{\mathrm{K}}^0(C)$ surjects onto $\widetilde{\mathrm{GW}}^3(C)$. Therefore, the image of $\widetilde{\mathrm{GW}}^3(C)$ in $\widetilde{\mathrm{K}}_0(C)$ coincides with the image of FH^3, and by (8) we obtain a short exact sequence

$$0 \to \mathbb{Z}/2 \longrightarrow \widetilde{\mathrm{GW}}^0(C) \longrightarrow \underbrace{\widetilde{\mathrm{W}}^0(C)}_{= \mathrm{Jac}(C)[2]} \to 0$$

This sequence splits. To define a splitting of the surjection, pick line bundles $\mathcal{L}$ of order 2 defining additive generators $e_{\mathcal{L}} := (\mathcal{L}, \mathrm{id}) - (\mathcal{O}, \mathrm{id})$ of $\widetilde{\mathrm{W}}^0(C)$. We claim that the homomorphism obtained by lifting these to the corresponding elements in $\mathrm{GW}^0(C)$ is well-defined, i. e. that $2e_{\mathcal{L}}$ vanishes in $\mathrm{GW}^0(C)$. Indeed, by Lemma I.1.9 we have $2e_{\mathcal{L}} = H^0(\mathcal{L} - \mathcal{O})$, and H^0 vanishes on $\mathcal{L} - \mathcal{O}$ since by (8) the latter is contained in $F(\widetilde{\mathrm{GW}}^3(X))$.

Next, the image of $\widetilde{\mathrm{GW}}^0(C)$ in $\widetilde{\mathrm{K}}_0(C)$ is $\mathrm{Jac}(C)[2]$. Since the quotient $\mathrm{Jac}(C)/\mathrm{Jac}(C)[2]$ is isomorphic to $\mathrm{Jac}(C)$, we obtain a short exact sequence

$$0 \to \mathbb{Z} \oplus \mathrm{Jac}(C) \xrightarrow{H^1} \widetilde{\mathrm{GW}}^1(C) \longrightarrow \underbrace{\widetilde{\mathrm{W}}^1(C)}_{= \mathbb{Z}/2} \to 0$$

Let Ψ_g be the 1-symmetric complex given by (I.3) or (7). As an element of $\widetilde{\mathrm{K}}_0(C) =$

$\mathbb{Z} \oplus \operatorname{Jac}(C)$, its underlying complex is equivalent to $(1, \mathcal{O})$. Thus, by Lemma I.1.9 again, we have $H^1(1, \mathcal{O}) = H^1(F(\Psi_g)) = 2\Psi_g$. It follows that the sequence does not split and that Ψ_g descends to a generator of $\widetilde{\mathrm{W}}^1(C)$. Thus, $\widetilde{\mathrm{GW}}^1(C) = \mathbb{Z} \oplus \operatorname{Jac}(C)$.

Carrying on to compute $\widetilde{\mathrm{GW}}^2(C)$, we first note that by (8) the image of $\widetilde{\mathrm{GW}}^1(C)$ in $\widetilde{\mathrm{K}}^0(C)$ contains $2\mathbb{Z} \oplus \operatorname{Jac}(C)$. Moreover, $F(\Psi) = \psi = (1, \mathcal{O})$, so in fact $\widetilde{\mathrm{GW}}^1(C)$ surjects onto $\widetilde{\mathrm{K}}_0(C)$. It follows that $\widetilde{\mathrm{GW}}^2(C) = 0$. Finally, since $\widetilde{\mathrm{W}}^3(C)$ also vanishes, we see that H^3 gives an isomorphism between $\widetilde{\mathrm{K}}_0(C)$ and $\widetilde{\mathrm{GW}}^3(C)$. This completes the computations in the untwisted case.

Twisted case. To compute the Grothendieck-Witt groups of C twisted by a generator $\mathcal{O}(p)$ of $\operatorname{Pic}(C)/2$ it again seems easiest to compare them with those of the affine curve $C-p$, over which $\mathcal{O}(p)$ is trivial. More specifically, we will compare the respective Karoubi sequences via the following commutative diagram:

$$\begin{array}{ccccccccc}
\mathrm{GW}^{i-1}(C-p) & \xrightarrow{F} & \mathrm{K}_0(C-p) & \xrightarrow{H^i} & \mathrm{GW}^i(C-p) & \longrightarrow & \mathrm{W}^i(C-p) & \longrightarrow & 0 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
\mathrm{GW}^{i-1}(C,\mathcal{O}(p)) & \xrightarrow{F} & \mathrm{K}_0(C) & \xrightarrow{H^i_{\mathcal{O}(p)}} & \mathrm{GW}^i(C,\mathcal{O}(p)) & \to & \mathrm{W}^i(C,\mathcal{O}(p)) & \to & 0
\end{array} \tag{9}$$

The restriction map $\mathrm{K}_0(C) \to \mathrm{K}_0(C-p)$ is the projection $\mathbb{Z} \oplus \mathbb{Z} \oplus \operatorname{Jac}(C) \twoheadrightarrow \mathbb{Z} \oplus 0 \oplus \operatorname{Jac}(C)$ killing the free summand corresponding to line bundles of non-zero degrees. The twisted version of (8) reads as follows:

$$\operatorname{im}(FH^i_{\mathcal{O}(p)}) = \begin{cases} \mathbb{Z} \cdot (2,1) \oplus 0 & \subset \mathrm{K}_0(C) \quad \text{when } i \text{ is even} \\ 0 \oplus \quad \mathbb{Z} \oplus \operatorname{Jac}(C) & \subset \mathrm{K}_0(C) \quad \text{when } i \text{ is odd} \end{cases} \tag{10}$$

Now consider (9) with $i = 0$. Arguing as in the untwisted case, we may compute the cokernels of the forgetful maps F to reduce the diagram to a comparison of two short exact sequences:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\binom{2}{0}} & \mathbb{Z} \oplus \operatorname{Jac}(C)[2] & \longrightarrow & (\mathbb{Z}/2) \oplus \operatorname{Jac}(C)[2] & \longrightarrow & 0 \\
 & & \Vert & & \uparrow & & \uparrow{\scriptstyle \binom{0}{1}} & & \\
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathrm{GW}^0(C,\mathcal{O}(p)) & \longrightarrow & \operatorname{Jac}(C)[2] & \longrightarrow & 0
\end{array}$$

Since the outer vertical maps are injective, so is the central vertical map. It follows from a diagram chase that the image of a generator of $\mathrm{K}_0(C)$ in $\mathrm{GW}^0(C, \mathcal{O}(p))$ cannot be divisible by 2. Thus, the lower sequence must split, yielding $\mathrm{GW}^0(C, O(p)) = \mathbb{Z} \oplus \operatorname{Jac}(C)[2]$.

Next, we see by (10) that the free summand in $\mathrm{GW}^0(C, O(p))$ has image $\mathbb{Z} \cdot (2,1) \oplus 0$ in $\mathrm{K}_0(C)$. On $\operatorname{Jac}(C)[2]$, on the other hand, F must be the inclusion $\operatorname{Jac}(C)[2] \hookrightarrow \operatorname{Jac}(C)$, by comparison with the case of $C-p$. This gives $\mathrm{GW}^1(C, \mathcal{O}(p)) = \mathbb{Z} \oplus (\operatorname{Jac}(C)/\operatorname{Jac}(C)[2]) \cong \mathbb{Z} \oplus \operatorname{Jac}(C)$.

The image of $\mathrm{GW}^1(C, \mathcal{O}(p))$ in $\mathrm{K}_0(C)$ may be read off directly from (10), and we may deduce that $\mathrm{GW}^2(C, \mathcal{O}(p))$ is a free abelian group generated by the twisted hyperbolic bundle $H^2_{\mathcal{O}(p)}(\mathcal{O})$. Equally directly, we see that $\mathrm{GW}^3(C, \mathcal{O}(p)) = \mathbb{Z} \oplus \mathrm{Jac}(C)$. This completes the computation of all Grothendieck-Witt groups of C. □

2b KO-groups of curves

Now suppose that C is a smooth curve defined over $\mathbb{C}$. Its topological KO-groups may be computed from the Atiyah-Hirzebruch spectral sequence.

2.3 Proposition. *Let C be a smooth projective complex curve of genus g as above. Then its KO-groups are given by*

$$\begin{aligned}
\mathrm{KO}^0(C) &= [\mathbb{Z}] \oplus (\mathbb{Z}/2)^{2g+1} & \mathrm{KO}^{-1}(C) &= [\mathbb{Z}/2] \oplus (\mathbb{Z}/2)^{2g} \\
\mathrm{KO}^2(C) &= \mathbb{Z} & \mathrm{KO}^1(C) &= \mathbb{Z}^{2g} \oplus \mathbb{Z}/2 \\
\mathrm{KO}^4(C) &= [\mathbb{Z}] & \mathrm{KO}^3(C) &= 0 \\
\mathrm{KO}^6(C) &= [\mathbb{Z}/2] \oplus \mathbb{Z} & \mathrm{KO}^5(C) &= \mathbb{Z}^{2g}
\end{aligned}$$

Here, the square brackets again indicate which summands vanish when passing to reduced groups. □

In Section 3b, we will explain in detail how the values of the groups $(\mathrm{KO}^{2i}/\mathrm{K})(C)$ may also be obtained from the spectral sequence. They may be read off as special cases from Proposition 3.8. In fact, we can also read off the values of the twisted groups $(\mathrm{KO}^{2i}/\mathrm{K})(C; \mathcal{O}(p))$ by applying the same proposition to the Thom space $\mathrm{Thom}_C(\mathcal{O}(p))$ (see Definition I.2.9). We then find that these groups agree with the corresponding Witt groups in Proposition 2.1:

$$\begin{aligned}
(\mathrm{KO}^0/\mathrm{K})(C) &= [\mathbb{Z}/2] \oplus (\mathbb{Z}/2)^{2g} & (\mathrm{KO}^0/\mathrm{K})(C, \mathcal{O}(p)) &= (\mathbb{Z}/2)^{2g} \\
(\mathrm{KO}^2/\mathrm{K})(C) &= \mathbb{Z}/2 & (\mathrm{KO}^2/\mathrm{K})(C, \mathcal{O}(p)) &= 0 \\
(\mathrm{KO}^4/\mathrm{K})(C) &= 0 & (\mathrm{KO}^4/\mathrm{K})(C, \mathcal{O}(p)) &= 0 \\
(\mathrm{KO}^6/\mathrm{K})(C) &= 0 & (\mathrm{KO}^6/\mathrm{K})(C, \mathcal{O}(p)) &= 0
\end{aligned}$$

3 Surfaces

This section contains the main calculations of this chapter: the computation of the Witt groups of smooth curves and surfaces on the one hand, and the computation of the corresponding topological groups ($\mathrm{KO}^{2i}/\mathrm{K}$) on the other. The comparison of the results will be postponed until the next section.

3a Witt groups of surfaces

Consider the algebraic K-group $\mathrm{K}_0(X)$ of a smooth variety X, and let c_i be the Chern classes $c_i\colon \mathrm{K}_0(X) \to \mathrm{CH}^i(X)$ with values in the Chow groups of X. Filter $\mathrm{K}_0(X)$ by $\mathrm{K}_0(X) \supset \widetilde{\mathrm{K}}_0(X) \supset \ker(c_1)$. If X has dimension at most two, then the map (rk, c_1, c_2) induces an isomorphism [Ful98, Example 15.3.6]:

$$\mathrm{gr}^*(\mathrm{K}_0(X)) \cong \mathbb{Z} \oplus \mathrm{CH}^1(X) \oplus \mathrm{CH}^2(X) \tag{11}$$

A similar statement holds for the Witt group.

3.1 Proposition. *Let k be an algebraically closed field of characteristic $char(k) \neq 2$, and let X be a smooth variety over k of dimension at most two. Filter $\mathrm{W}^0(X)$ by $\mathrm{W}^0(X) \supset \widetilde{\mathrm{W}}^0(X) \supset \ker(\overline{w}_1)$. Then the map $(\mathrm{rk}, \overline{w}_1, \overline{w}_2)$ gives an isomorphism*

$$\mathrm{gr}^*(\mathrm{W}^0(X)) \cong \mathbb{Z}/2 \oplus H^1_{et}(X;\mathbb{Z}/2) \oplus \left(H^2_{et}(X;\mathbb{Z}/2) \big/ \mathrm{Pic}(X) \right)$$

Moreover, if we write S^1 for the squaring map $\mathrm{CH}^1(X)/2 \to \mathrm{CH}^2(X)/2$, then the shifted Witt groups are as follows:

$$\begin{aligned} \mathrm{W}^1(X) &\cong \ker(S^1) \oplus H^3_{et}(X;\mathbb{Z}/2) \\ \mathrm{W}^2(X) &\cong \mathrm{coker}(S^1) \\ \mathrm{W}^3(X) &= 0 \end{aligned}$$

3.2 Example. Suppose X is a smooth complex projective surface of Picard number ρ. Write b_i for the Betti numbers of X and ν for the rank of $H^2(X;\mathbb{Z})[2]$. Then the above result shows that the Witt groups of X are as follows:

$$\begin{aligned} \mathrm{W}^0(X) &= [\mathbb{Z}/2] \oplus (\mathbb{Z}/2)^{b_1+b_2-\rho+2\nu} \\ \mathrm{W}^1(X) &= \begin{cases} (\mathbb{Z}/2)^{b_1+\rho+2\nu} & \text{if } S^1 = 0 \\ (\mathbb{Z}/2)^{b_1+\rho+2\nu-1} & \text{if } S^1 \neq 0 \end{cases} \\ \mathrm{W}^2(X) &= \begin{cases} \mathbb{Z}/2 & \text{if } S^1 = 0 \\ 0 & \text{if } S^1 \neq 0 \end{cases} \\ \mathrm{W}^3(X) &= 0 \end{aligned}$$

Our computation will follow a route described in [Tot03]. Namely, there is a subtle relationship between Witt groups and étale cohomology groups with $\mathbb{Z}/2$-coefficients, encoded by three spectral sequences:

$$E_{2,\mathrm{BO}}^{s,t}(X) = H^s(X, \mathcal{H}^t) \quad \Rightarrow \quad H_{\mathrm{et}}^{s+t}(X; \mathbb{Z}/2) \tag{BO}$$

$$E_{2,\mathrm{Par}}^{s,t}(X) = H^s(X, \mathcal{H}^t) \quad \Rightarrow \quad H^s(X, \mathcal{W}) \tag{P}$$

$$E_{2,\mathrm{GW}}^{s,t}(X) = H^s(X, \mathcal{W}^t) \quad \Rightarrow \quad \mathrm{W}^{s+t}(X) \tag{GW}$$

Here, $\mathcal{H}$ is the Zariski sheaf attached to the presheaf that sends an open subset $U \subset X$ to $H_{\mathrm{et}}^j(U; \mathbb{Z}/2)$. Similarly, $\mathcal{W}$ denotes the Zariski sheaf attached to the presheaf sending U to $\mathrm{W}^0(U)$, and we set

$$\mathcal{W}^t = \begin{cases} \mathcal{W} & \text{for } t \equiv 0 \mod 4 \\ 0 & \text{otherwise} \end{cases}$$

Before specializing to the case of a surface, we give a short discussion of each of these spectral sequences for an arbitrary dimensional variety X. For simplicity, we assume that X is a smooth variety over a field of characteristic not 2 throughout. More general statements may be found in the references.

The Bloch-Ogus spectral sequence (BO). The first spectral sequence is the well-known Bloch-Ogus spectral sequence [BO74]. This is a first quadrant spectral sequence with differentials in the usual directions, i. e. of bidegree $(r, -r+1)$ on the r^{th} page. The E_2-page is concentrated above the main diagonal $s = t$, and the groups along the diagonal may be identified with the Chow groups of X modulo two [BO74, Corollary 6.2 and proof of Theorem 7.7]:

$$H^s(X, \mathcal{H}^s) \cong \mathrm{CH}^s(X)/2 \tag{12}$$

When the ground field is algebraically closed, the sheaves $\mathcal{H}^t$ vanish for all $t > \dim(X)$, so that the spectral sequence is concentrated in rows $0 \leq t \leq \dim(X)$.

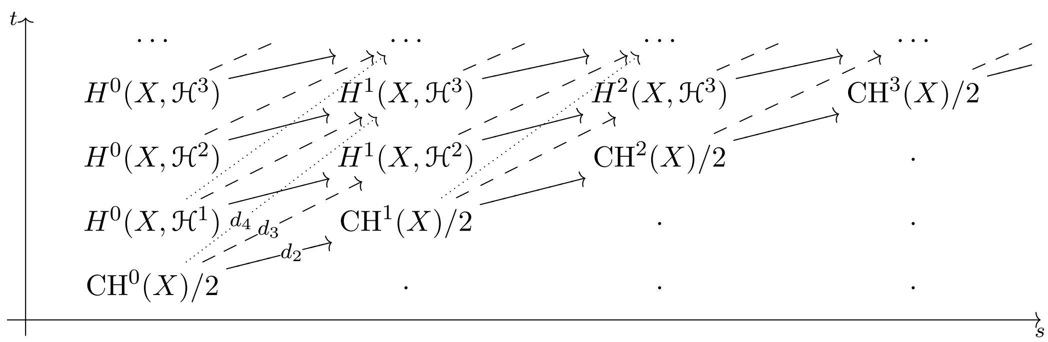


Figure 1: Pardon's spectral sequence. The entries below the diagonal are zero.

Pardon's spectral sequence (P). Pardon's spectral sequence can be indexed to have the same E_2-page as the Bloch-Ogus spectral sequence. The differentials on the r^{th} page then have bidegree $(1, r-1)$, as illustrated in Figure 1. In [Tot03], Totaro shows that, under the identifications (12), the differential d_2 on the main diagonal of the E_2-page corresponds to the Steenrod operation $S^1\colon \mathrm{CH}^s(X)/2 \to \mathrm{CH}^{s+1}(X)/2$, as defined by Brosnan and Voevodsky [Bro03, Voe03a].

The sequence converges to the cohomology of X with respect to $\mathcal{W}$ in the usual sense that the i^{th} column of the E_∞-page is isomorphic to the associated graded module of $H^i(X, \mathcal{W})$ with respect to some filtration. In order to describe the filtration on $H^0(X, \mathcal{W}) = \mathcal{W}(X)$, we briefly summarize how the spectral sequence arises.

Let X be as above, and let K be its function field. We denote the residue field of a scheme-theoretic point x of X by $k(x)$. In [Par04], Pardon shows that the sheaf $\mathcal{W}$ has a flasque resolution by a Gersten-Witt complex $\mathbb{W}$ which on open subsets $U \subset X$ takes the form

$$\mathbb{W}(U)\colon \quad \mathrm{W}^0(K) \to \coprod_{x \in U^{(1)}} \mathrm{W}^0(k(x)) \to \cdots \to \coprod_{x \in U^{(n)}} \mathrm{W}^0(k(x)) \to 0 \tag{13}$$

Here, $U^{(i)}$ denotes the set of codimension i points of X contained in U, and $\mathrm{W}^0(k(x))$ is to be viewed as a constant sheaf supported on the closure of x in X. In particular, $\mathcal{W}$ is a subsheaf of the constant sheaf $\mathrm{W}^0(K)$. Let $I(K) \subset \mathrm{W}^0(K)$ be the fundamental ideal of K (see Section 1). The fundamental filtration on $\mathrm{W}^0(K)$, given by the powers of $I(K)$, induces a filtration on $\mathcal{W}$ which we denote by

$$\mathcal{I}_t := \mathcal{W} \cap I^t(K) \tag{14}$$

Pardon shows more generally that the fundamental filtrations of the Witt groups $\mathrm{W}^0(k(x))$ give rise to flasque resolutions of the sheaves $\mathcal{I}_t$ of the form

$$\mathbb{I}_t(U)\colon \quad I^t(K) \to \coprod_{x \in U^{(1)}} I^{t-1}(k(x)) \to \coprod_{x \in U^{(2)}} I^{t-2}(k(x)) \to \dots$$

Applying the standard construction of the spectral sequence of a filtered complex of abelian groups [GM03, Chapter III.7, Section 5] to the filtration of $\mathbb{W}(X)$ by $\mathbb{I}_t(X)$, we obtain a spectral sequence

$$H^s(X, \mathcal{I}_t/\mathcal{I}_{t+1}) \Rightarrow H^s(X, \mathcal{W})$$

To conclude, Pardon uses the affirmation of the Milnor conjectures (see (5)) to identify the sheaves $\mathcal{I}_t/\mathcal{I}_{t+1}$ with the sheaves $\mathcal{H}^t$. The following lemma is a direct consequence of this construction.

3.3 Lemma. *The filtration on $\mathcal{W}(X)$ appearing on the zeroth column of the E_∞-page of Pardon's spectral sequence is given by the global sections of the sheaves $\mathcal{I}_t$ defined in* (14)*:*

$$E^{0,t}_{\infty,Par}(X) \cong \mathcal{I}_t(X)/\mathcal{I}_{t+1}(X)$$

The edge homomorphism including this column into the zeroth column of the E_2-page is induced by the sheafification map from the presheaf quotient of $\mathcal{I}_t$ by $\mathcal{I}_{t+1}$ to the quotient sheaf $\mathcal{I}_t/\mathcal{I}_{t+1}$.

The group $\mathcal{W}(X)$, viewed as a subgroup of $\mathrm{W}^0(K)$, is commonly referred to as the unramified Witt group of X.

The Gersten-Witt spectral sequence (GW). The Gersten-Witt spectral sequence of Balmer and Walter, third on the list above, may be thought of as an analogue of the Bloch-Ogus spectral sequence for Witt groups. In [BW02], Balmer and Walter, too, construct a Gersten-Witt complex of the form (13). Although it seems likely that their construction agrees with Pardon's, this does not yet seem to have been established. It is known in any case, however, that their complex also provides a flasque resolution of $\mathcal{W}$ [BGPW02, Lemma 4.2 and Theorem 6.1]. The spectral sequence constructed in [BW02] may therefore be rewritten in the form above:

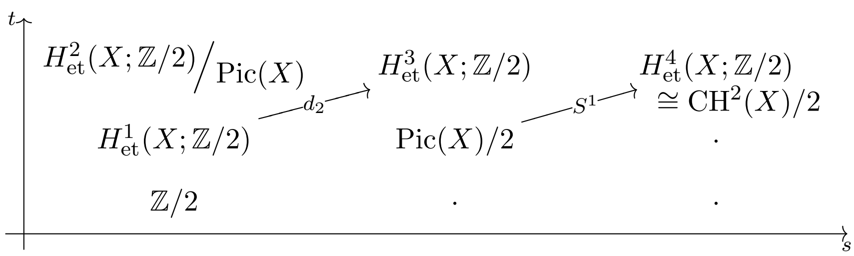
3.4 Theorem. [BW02] *Let X be a smooth variety over a field of characteristic not* 2*. There is a spectral sequence with E_2-page*

$$E_2^{s,t} = \begin{cases} H^s(\mathcal{W}) & \textit{in rows } t \equiv 0 \mod 4 \\ 0 & \textit{elsewhere} \end{cases}$$

The sequence has differentials d_r of bidegree $(r, -r+1)$ and converges to $\mathrm{W}^{s+t}(X)$.

Proof of Proposition 3.1:
We now specialize to the case when $\dim(X) \leq 2$ and the ground field is algebraically closed. Then the Bloch-Ogus spectral sequence (BO) collapses immediately and we obtain descriptions of the groups $H^s(X; \mathcal{H}^t)$ in terms of the étale cohomology of X with $\mathbb{Z}/2$-coefficients.

The E_2-page of Pardon's spectral sequence therefore takes the following form:



On the other hand, we see that the Gersten-Witt spectral sequence also collapses, so that $\mathrm{W}^i(X) \cong H^i(X, \mathcal{W})$. Thus, the i^{th} column of Pardon's spectral sequence simply converges to $\mathrm{W}^i(X)$. In particular, if we write $I_t(X)$ for the filtration on $\mathrm{W}^0(X)$ corresponding to the filtration $\mathcal{I}_t(X)$ of $\mathcal{W}(X)$ defined in (14), then by Lemma 3.3 we have $E^{0,t}_{\infty,\mathrm{Par}} \cong \mathrm{gr}^I_t \mathrm{W}^0(X)$.

3.5 Lemma. *Let X be as above. Then, for $t = 1$ or 2, the edge homomorphisms*

$$\mathrm{gr}^I_t \mathrm{W}^0(X) \cong E^{0,t}_{\infty,Par} \hookrightarrow E^{0,t}_{2,Par}$$

can be identified with the Stiefel-Whitney classes $\overline{w}_t$. More precisely, $I_1(X) = \widetilde{\mathrm{W}}^0(X)$, $I_2(X) = \ker(\overline{w}_1)$, and we have commutative diagrams

$$\begin{array}{ccc}
E^{0,1}_{\infty,Par} & \hookrightarrow & E^{0,1}_{2,Par} \\
\| & & \| \\
\widetilde{\mathrm{W}}^0(X)\big/\ker(\overline{w}_1) & \xrightarrow{\overline{w}_1} & H^1_{et}(X;\mathbb{Z}/2)
\end{array}$$

$$\begin{array}{ccc}
E^{0,2}_{\infty,Par} & \hookrightarrow & E^{0,2}_{2,Par} \\
\| & & \| \\
\ker(\overline{w}_1) & \xrightarrow{\overline{w}_2} & H^2_{et}(X;\mathbb{Z}/2)\big/\mathrm{Pic}(X)
\end{array}$$

Proof. The statement of the lemma is not surprising: the only non-obvious maps that enter into the construction of Pardon's spectral sequence are the isomorphisms e^t displayed in (5); for $t = 1$ or 2, these can be identified with Stiefel-Whitney classes, and the latter can be defined globally.

In more detail, the various identifications arising from the Bloch-Ogus spectral sequence and Pardon's spectral sequence fit into the following diagram:

$$\begin{array}{ccccc}
E^{0,t}_{\infty,\mathrm{Par}} & \hookrightarrow & E^{0,t}_{2,\mathrm{Par}} & & \\
\| & & \| & & \\
I_t(X)\big/I_{t+1}(X) & \longrightarrow & \left(\mathcal{I}_t\big/\mathcal{I}_{t+1}\right)(X) & \hookrightarrow & I^t(K)\big/I^{t+1}(K) \\
\vdots\, ? & & \downarrow \cong & & e^t \downarrow \cong \\
H^t_{\mathrm{et}}(X;\mathbb{Z}/2) & \xrightarrow{\text{sheafification}} & \mathcal{H}^t(X) & \hookrightarrow & H^t_{\mathrm{et}}(K;\mathbb{Z}/2) \\
\downarrow & & \| & & \\
E^{0,t}_{\infty,\mathrm{BO}} & \underset{\text{(BO collapses)}}{\xrightarrow{\cong}} & E^{0,t}_{2,\mathrm{BO}} & &
\end{array}$$

The claim is that, for $t = 1$ or 2, the diagram commutes if we take the dotted map to be the t^{th} Stiefel-Whitney class. Indeed, e^t can be identified with $\overline{w}_t$ in these cases, and the horizontal compositions across the two middle lines in the diagram are given by pullback along the inclusion of the generic point into X. □

There are two potentially non-zero differentials in Pardon's spectral sequence: the differential $S^1\colon \mathrm{CH}^1(X)/2 \to \mathrm{CH}^2(X)/2$, which is simply the squaring operation, and another differential d from $H^1_{\mathrm{et}}(X;\mathbb{Z}/2)$ to $H^3_{\mathrm{et}}(X;\mathbb{Z}/2)$. Since the first Stiefel-Whitney class always surjects onto $H^1_{\mathrm{et}}(X;\mathbb{Z}/2)$, we see from the previous lemma that $H^1_{\mathrm{et}}(X;\mathbb{Z}/2)$ survives to the E_∞-page. So the differential d must vanish. On the other hand, since $E_2^{0,2} = E_\infty^{0,2}$, the lemma implies that the restriction of the second Stiefel-Whitney class to the kernel of the first gives an isomorphism with $H^2_{\mathrm{et}}(X;\mathbb{Z}/2)/\operatorname{Pic}(X)$. This proves the first statement of the proposition. Since S^1 is the only non-trivial differential left, the values of the shifted Witt groups also follow easily from the spectral sequence. This completes the proof of Proposition 3.1. □

In the following corollary, the Chern and Stiefel-Whitney classes refer to the corresponding maps on reduced groups:

$$\begin{aligned} c_i\colon \quad \widetilde{\mathrm{K}}_0(X) &\longrightarrow \mathrm{CH}^i(X) \\ w_i\colon \widetilde{\mathrm{GW}}^0(X) &\longrightarrow H^i_{\mathrm{et}}(X;\mathbb{Z}/2) \end{aligned}$$

3.6 Corollary. *Let X be as above. Then the second Stiefel-Whitney class is surjective and restricts to an epimorphism*

$$w_2\colon \ker(w_1) \twoheadrightarrow H^2_{et}(X;\mathbb{Z}/2)$$

Its kernel is given by the image of $\ker(c_1)$ *under the hyperbolic map* $H^0\colon \widetilde{\mathrm{K}}_0(X) \to \widetilde{\mathrm{GW}}^0(X)$.

Proof. Note first that $H^0\colon \widetilde{\mathrm{K}}_0(X) \to \widetilde{\mathrm{GW}}^0(X)$ factors through the kernel of w_1. In fact, we can restrict the Karoubi sequence to an exact sequence forming the first row of the following diagram.

$$\begin{array}{ccccccc}
\widetilde{\mathrm{K}}_0(X) & \xrightarrow{H^0} & \ker(w_1) & \twoheadrightarrow & \ker(\overline{w}_1) & \longrightarrow & 0 \\
\downarrow{\scriptstyle c_1} & & \downarrow{\scriptstyle w_2} & & \cong\downarrow{\scriptstyle \overline{w}_2} & & \\
\operatorname{Pic}(X) & \longrightarrow & H^2_{\mathrm{et}}(X;\mathbb{Z}/2) & \twoheadrightarrow & H^2_{\mathrm{et}}(X;\mathbb{Z}/2)/\operatorname{Pic}(X) & \longrightarrow & 0
\end{array}$$

The lower row is obtained from the Kummer sequence, hence also exact, and the diagram commutes. We know from the previous proposition that the map $\overline{w}_2$ on the right is an isomorphism. Since c_1 is surjective, surjectivity of w_2 follows.

It remains to show that the restriction $H^0\colon \ker(c_1) \to \ker(w_2)$ is surjective. By an instance of the Snake Lemma, this is equivalent to showing that the kernel of H^0 on $\widetilde{\mathrm{K}}_0(X)$ maps surjectively to $\operatorname{Pic}(X)^2$ under c_1. The kernel of H^0 is given by the image of $\widetilde{\mathrm{GW}}^3(C)$ under the forgetful map F, which coincides with the image of $\widetilde{\mathrm{K}}_0(X)$ under the composition FH^3. Thus, it suffices to show that c_1 restricts to a surjection $c_1\colon FH^3(\mathrm{K}_0(X)) \to \operatorname{Pic}(X)^2$. This is done in the next lemma. □

3.7 Lemma. *Let X be as above. Then for odd i we have commutative diagrams*

$$\begin{array}{ccccc} \mathrm{K}_0(X) & \xrightarrow{H^i} & \mathrm{GW}^i(X) & \xrightarrow{F} & \mathrm{K}_0(X) \\ \downarrow{\scriptstyle c_1} & & & & \downarrow{\scriptstyle c_1} \\ \mathrm{Pic}(X) & & \longrightarrow & & \mathrm{Pic}(X) \\ \mathcal{L} & & \mapsto & & \mathcal{L}^2 \end{array}$$

For even i, the diagram commutes if we replace the map on $\mathrm{Pic}(X)$ by the constant map.

Proof. This can be checked via a direct calculation. The first Chern class c_1 takes a vector bundle $\mathcal{E}$ to its determinant line bundle $\det \mathcal{E}$. Under FH^i, the class of $\mathcal{E}$ is mapped to the class $\mathcal{E} \oplus (-1)^i \mathcal{E}^\vee$ in $\mathrm{K}_0(X)$, which c_1 then takes to $(\det \mathcal{E})^{\otimes(1-(-1)^i)}$. □

Remark. More generally, the endomorphism of $\mathrm{gr}^*(\mathrm{K}_0(X)) \cong \mathbb{Z} \oplus \mathrm{Pic}(X) \oplus \mathrm{CH}^2(X)$ induced by FH^i is given by $(2,0,2)$ if i is even, and by $(0,2,0)$ if i is odd.

3b KO/K-groups of surfaces

On a topological space X, let us write $\mathrm{Sq}^2_{\mathbb{Z}}$ for the composition

$$H^2(X;\mathbb{Z})/2 \hookrightarrow H^2(X;\mathbb{Z}/2) \xrightarrow{\mathrm{Sq}^2} H^4(X;\mathbb{Z}/2)$$

where Sq^2 is the squaring operation.

3.8 Proposition. *Let X be a connected CW complex of dimension at most four. Filter the group $(\mathrm{KO}^0/\mathrm{K})(X)$ by $(\mathrm{KO}^0/\mathrm{K})(X) \supset (\widetilde{\mathrm{KO}}^0/\widetilde{\mathrm{K}})(X) \supset \ker(\overline{w}_1)$. Then the map $(\mathrm{rk}, \overline{w}_1, \overline{w}_2)$ induces an isomorphism*

$$\mathrm{gr}\left((\mathrm{KO}^0/\mathrm{K})(X)\right) \cong \mathbb{Z}/2 \oplus H^1(X;\mathbb{Z}/2) \oplus \left(H^2(X;\mathbb{Z}/2)\Big/ H^2(X;\mathbb{Z})\right)$$

The remaining groups $(\mathrm{KO}^{2i}/\mathrm{K})(X)$ have the following values:

$$\begin{aligned} (\mathrm{KO}^2/\mathrm{K})(X) &\cong \ker(\mathrm{Sq}^2_{\mathbb{Z}}) \oplus H^3(X;\mathbb{Z}/2) \\ (\mathrm{KO}^4/\mathrm{K})(X) &\cong \mathrm{coker}(\mathrm{Sq}^2_{\mathbb{Z}}) \\ (\mathrm{KO}^6/\mathrm{K})(X) &= 0 \end{aligned}$$

3.9 Example. Suppose X is a compact four-dimensional manifold. Write b_i for its Betti numbers and ν for the rank of $H^2(X;\mathbb{Z})[2]$. Then the above result shows that

$$(\mathrm{KO}^0/\,\mathrm{K})(X) = [\mathbb{Z}/2] \oplus (\mathbb{Z}/2)^{b_1+2\nu}$$
$$(\mathrm{KO}^2/\,\mathrm{K})(X) = \begin{cases} (\mathbb{Z}/2)^{b_1+b_2+2\nu} & \text{if } \mathrm{Sq}^2_{\mathbb{Z}} = 0 \\ (\mathbb{Z}/2)^{b_1+b_2+2\nu-1} & \text{if } \mathrm{Sq}^2_{\mathbb{Z}} \neq 0 \end{cases}$$
$$(\mathrm{KO}^4/\,\mathrm{K})(X) = \begin{cases} \mathbb{Z}/2 & \text{if } \mathrm{Sq}^2_{\mathbb{Z}} = 0 \\ 0 & \text{if } \mathrm{Sq}^2_{\mathbb{Z}} \neq 0 \end{cases}$$
$$(\mathrm{KO}^6/\,\mathrm{K})(X) = 0$$

3.10 Remark. Note that both cases — $\mathrm{Sq}^2_{\mathbb{Z}} = 0$ and $\mathrm{Sq}^2_{\mathbb{Z}}$ onto — can occur even for complex projective surfaces of geometric genus zero. For example, $\mathrm{Sq}^2_{\mathbb{Z}}$ is onto for $\mathbb{P}^2$. On the other hand, the Wu formula [MS74, Theorem 11.14] shows that Sq^2 is given by multiplication with $c_1(X)$ mod 2. So $\mathrm{Sq}^2_{\mathbb{Z}} = 0$ on any projective surface X of geometric genus zero whose canonical divisor is numerically trivial. Concretely, we could take X to be an Enriques surface (a quotient of a K3-surface by a fixed-point free involution): see [Bea96, page 90].

To calculate the groups $(\mathrm{KO}^{2i}/\,\mathrm{K})(X)$, we use the Atiyah-Hirzebruch spectral sequences for the K- and KO-theory of X. We begin with some general facts.

3.11 Lemma. *The differential $d_3^{1,-2}\colon E_3^{1,-2} \longrightarrow E_3^{4,-4}$ is trivial both in the Atiyah-Hirzebruch spectral sequence for K-theory and in the spectral sequence for KO-theory.*

Proof. This is immediate from the descriptions of these differentials in terms of the second Steenrod square given in Section I.2e. □

3.12 Lemma. *Let X be a connected finite-dimensional CW complex. Denote the filtrations on the groups $\mathrm{K}^0(X)$ and $\mathrm{KO}^i(X)$ associated with the Atiyah-Hirzebruch spectral sequences by*

$$\mathrm{K}_0(X) \supset \mathrm{K}_1(X) = \mathrm{K}_2(X) \supset \mathrm{K}_3(X) = \mathrm{K}_4(X) \supset \dots$$
$$\mathrm{KO}^i_0(X) \supset \mathrm{KO}^i_1(X) \supset \mathrm{KO}^i_2(X) \supset \mathrm{KO}^i_3(X) \supset \dots$$

The initial layers of these filtrations have more intrinsic descriptions in terms of Chern and Stiefel-Whitney classes:

- $\mathrm{K}_2(X) = \mathrm{K}_1(X) = \widetilde{\mathrm{K}}^0(X)$, *and* $\mathrm{KO}^i_1(X) = \widetilde{\mathrm{KO}}^i(X)$
- *The map $\mathrm{K}_2(X) \to H^2(X;\mathbb{Z})$ arising from the spectral sequence may be identified with the first Chern class, at least up to a sign. That is, we have the following commutative diagram:*

$$\begin{array}{ccc} E_\infty^{2,-2} & \hookrightarrow & E_2^{2,-2} \\ \uparrow\!\!\uparrow & & \| \\ \widetilde{\mathrm{K}}^0(X) & \xrightarrow{\pm c_1} & H^2(X;\mathbb{Z}) \end{array}$$

In particular, $\mathrm{K}_4(X) = \ker(c_1)$. Moreover, since c_1 is surjective, $E_\infty^{2,-2} = E_2^{2,-2}$.

- *The map* $\mathrm{K}_4(X) \to H^4(X;\mathbb{Z})$ *can be identified with the restriction of the second Chern class to* $\ker(c_1)$*, again up to a sign:*

$$\begin{array}{ccc} E_\infty^{4,-4} & \hookrightarrow & E_2^{4,-4} \\ \uparrow\!\!\uparrow & & \| \\ \ker(c_1) & \xrightarrow{\pm c_2} & H^4(X;\mathbb{Z}) \end{array}$$

Thus, $\mathrm{K}_6(X) = \ker(c_2)$*. (Note that the upper horizontal map in this diagram is indeed an inclusion, by the previous lemma.)*

- *Likewise, the map* $\mathrm{KO}^0_1(X) \to H^1(X;\mathbb{Z}/2)$ *can be identified with the first Stiefel-Whitney class:*

$$\begin{array}{ccc} E_\infty^{1,-1} & \hookrightarrow & E_2^{1,-1} \\ \uparrow\!\!\uparrow & & \| \\ \widetilde{\mathrm{KO}}^0(X) & \overset{w_1}{\twoheadrightarrow} & H^1(X;\mathbb{Z}/2) \end{array}$$

Thus, $\mathrm{KO}^0_2(X) = \ker(w_1)$ *and* $E_\infty^{1,-1} = E_2^{1,-1}$*.*

- *Lastly, the map* $\mathrm{KO}^0_2(X) \to H^2(X;\mathbb{Z}/2)$ *can be identified with the restriction of the second Stiefel-Whitney class, i. e. we have a commutative diagram*

$$\begin{array}{ccc} E_\infty^{2,-2} & \hookrightarrow & E_2^{2,-2} \\ \uparrow\!\!\uparrow & & \| \\ \ker(w_1) & \xrightarrow{w_2} & H^2(X;\mathbb{Z}/2) \end{array}$$

Thus, $\mathrm{KO}^0_4(X) = \ker(w_2)$*.*

Proof. The first statement is clear. The other assertions follow by viewing the maps in question as cohomology operations and computing them for a few spaces. For lack of reference, we include a more detailed proof of the statements concerning KO-theory. The case of complex K-theory can be dealt with analogously.

First, we analyse the map $\widetilde{\mathrm{KO}}^0(X) \to H^1(X;\mathbb{Z}/2)$ arising from the spectral sequence. A priori, we have defined this map only for finite-dimensional CW complexes. But we can extend it to a natural transformation of functors on the homotopy category of all connected CW complexes, using the fact that the canonical map $H^1(X;\mathbb{Z}/2) \to \lim_i H^1(X^i;\mathbb{Z}/2)$ is an isomorphism for any CW complex X with i-skeletons X^i. On the homotopy category of connected CW complexes, the functor $\widetilde{\mathrm{KO}}^0(-)$ is represented by BO. Natural transformations $\widetilde{\mathrm{KO}}^0(-) \to H^1(-;\mathbb{Z}/2)$ are therefore in one-to-one correspondence with elements of $H^1(\mathrm{BO};\mathbb{Z}/2) = \mathbb{Z}/2 \cdot w_1$, where w_1 is the first Stiefel-Whitney class of the universal bundle over BO. Thus, either the map in question is zero, or it is given by w_1 as claimed. Since it is non-zero on S^1, the first case may be discarded.

To analyse the map $\mathrm{KO}^0_2(X) \to H^2(X;\mathbb{Z}/2)$, we note that the previous conclusion yields a functorial description of $\mathrm{KO}^0_2(X)$ as the kernel of w_1 on $\widetilde{\mathrm{KO}}^0(X)$. Moreover, given this description, we may define a natural set-wise splitting of the inclusion of $\mathrm{KO}^0_2(X)$ into $\widetilde{\mathrm{KO}}^0(X)$ for any finite-dimensional CW complex X as follows:

$$\begin{aligned}\widetilde{\mathrm{KO}}^0(X) &\twoheadrightarrow \mathrm{KO}^0_2(X)\\ [\mathcal{E}] - \mathrm{rk}(\mathcal{E})[\mathbb{R}] &\mapsto [\mathcal{E}] - \mathrm{rk}(\mathcal{E})[\mathbb{R}] - [\det\mathcal{E}] + [\mathbb{R}]\end{aligned}$$

The composition $\widetilde{\mathrm{KO}}^0(-) \to H^2(-;\mathbb{Z}/2)$ can be extended to a natural transformation of functors on the homotopy category of connected CW complexes in the same way as before, and it may thus be identified with an element of $H^2(\mathrm{BO};\mathbb{Z}/2) = \mathbb{Z}/2 \cdot w_1^2 \oplus \mathbb{Z}/2 \cdot w_2$. Consequently, its restriction to $\mathrm{KO}^0_2(X)$ is either 0 and w_2. Again, the first possibility may easily be discarded, for example by considering S^2. □

Lemma 3.12 has some immediate implications for low-dimensional spaces. In the following corollary, all characteristic classes refer to the corresponding maps on reduced groups:

$$\begin{aligned}c_i\colon\ \widetilde{\mathrm{K}}^0(X) &\longrightarrow H^{2i}(X;\mathbb{Z})\\ w_i\colon \widetilde{\mathrm{KO}}^0(X) &\longrightarrow H^i(X;\mathbb{Z}/2)\end{aligned}$$

3.13 Corollary. *Let X be a connected CW complex of dimension at most four. Then the second Chern class is surjective and restricts to an isomorphism*

$$c_2\colon\ \ker(c_1) \xrightarrow{\cong} H^4(X;\mathbb{Z})$$

The second Stiefel-Whitney class is also surjective, and restricts to an epimorphism

$$w_2\colon \ker(w_1) \twoheadrightarrow H^2(X;\mathbb{Z}/2)$$

Its kernel is given by the image of $\ker(c_1)$ *under realification. Moreover, the induced map* $\overline{w}_2$ *on* $(\widetilde{\mathrm{KO}}^0/\widetilde{\mathrm{K}})(X)$ *restricts to an isomorphism*

$$\overline{w}_2\colon \ker(\overline{w}_1) \xrightarrow{\cong} \left(H^2(X;\mathbb{Z}/2)\big/ H^2(X;\mathbb{Z})\right)$$

Proof. Consider the Atiyah-Hirzebruch spectral sequences for K- and KO-theory.
Lemma 3.11 shows that the sequence for K-theory collapses. Moreover, in the spectral sequence for KO-theory, no differentials affect the diagonal computing $\mathrm{KO}^0(X)$. This implies the first two claims of the corollary. Next, we compare the two spectral sequences via realification. The description of the kernel of w_2 may be obtained from the following row-exact commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^4(X;\mathbb{Z}) & \longrightarrow & \widetilde{\mathrm{K}}(X) & \xrightarrow{c_1} & H^2(X;\mathbb{Z}) & \longrightarrow & 0 \\
 & & \downarrow \cong & & \downarrow r & & \downarrow \text{reduction mod 2} & & \\
0 & \longrightarrow & H^4(X;\mathbb{Z}) & \longrightarrow & \ker(w_1) & \xrightarrow{w_2} & H^2(X;\mathbb{Z}/2) & \longrightarrow & 0
\end{array}$$

The final claim of the corollary also follows from this diagram, by identifying $\ker(\overline{w}_1)$ with $\ker(w_1)/\widetilde{\mathrm{K}}(X)$. □

The situation for a connected CW complex of dimension at most four may now be summarized as follows. Firstly, by Lemma 3.12, the filtrations on the groups $\mathrm{K}^0(X)$ and $\mathrm{KO}^0(X)$ arising in the Atiyah-Hirzebruch spectral sequences can be written as

$$\mathrm{K}^0(X) \supset \widetilde{\mathrm{K}}^0(X) \supset \ker(c_1)$$
$$\mathrm{KO}^0(X) \supset \widetilde{\mathrm{KO}}^0(X) \supset \ker(w_1)$$

Secondly, Corollary 3.13 implies that the maps (rk, c_1, c_2) and $(\mathrm{rk}, \overline{w}_1, \overline{w}_2)$ on the associated graded groups induce isomorphisms

$$\mathrm{gr}(\mathrm{K}^0(X)) \cong \mathbb{Z} \oplus H^2(X;\mathbb{Z}) \oplus H^4(X;\mathbb{Z}) \tag{15}$$
$$\mathrm{gr}\left((\mathrm{KO}^0/\,\mathrm{K})(X)\right) \cong \mathbb{Z}/2 \oplus H^1(X;\mathbb{Z}/2) \oplus \left(H^2(X;\mathbb{Z}/2)\big/ H^2(X;\mathbb{Z})\right) \tag{16}$$

In particular, we have proved the first part of Proposition 3.8.

Proof of the remaining claims of Proposition 3.8:
To compute $(\mathrm{KO}^{2i}/\,\mathrm{K})(X)$ for general i, we identify this quotient with the image of

$$\eta\colon\ \mathrm{KO}^{2i}(X) \to \mathrm{KO}^{2i-1}(X)$$

This image can be computed at each stage of the filtration

$$\mathrm{KO}^j(X) = \mathrm{KO}^j_0(X) \supset \mathrm{KO}^j_1(X) \supset \cdots \supset \mathrm{KO}^j_4(X)$$

To lighten the notation, we simply write KO^j_k for $\mathrm{KO}^j_k(X)$ in the following, and we write $H^*(X)$ for the singular cohomology of X with integral coefficients.

Since we are assuming that X is at most four-dimensional, there are only three possibly non-zero differentials in the spectral sequence computing $\mathrm{KO}^*(X)$. The first two are the differentials

$$\mathrm{Sq}^2 \circ \pi\colon\quad H^2(X) \longrightarrow H^4(X;\mathbb{Z}/2)$$
$$\mathrm{Sq}^2\colon\quad H^2(X;\mathbb{Z}/2) \to H^4(X;\mathbb{Z}/2)$$

on the E_2-page, depicted in Figure I.1. If $E_2^{4,-2} = H^4(X;\mathbb{Z}/2)$ is not killed by Sq^2, then we have a third possibly non-trivial differential on the E_3-page:

$$d_3^{1,0}\colon H^1(X) \to \operatorname{coker}(\mathrm{Sq}^2)$$

The differential $d_3^{1,-2}$ vanishes by Lemma 3.11.

Computation of $\mathbf{KO}^2/\mathbf{K}$. For the last stage of the filtration, we have a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{KO}^2_4 & \longrightarrow & \mathrm{KO}^2_3 & \longrightarrow & H^3(X;\mathrm{KO}^{-1}(\text{point})) & \longrightarrow & 0 \\
 & & \big\downarrow & & \big\downarrow{\scriptstyle \eta} & & \big\downarrow{\scriptstyle \cong} & & \\
0 & \longrightarrow & 0 & \longrightarrow & \mathrm{KO}^1_3 & = & H^3(X;\mathrm{KO}^{-2}(\text{point})) & \longrightarrow & 0
\end{array}$$

Thus, KO^2_3 maps surjectively to $\mathrm{KO}^1_3 \cong H^3(X;\mathbb{Z}/2)$. Next, we have

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{KO}^2_3 & \longrightarrow & \mathrm{KO}^2_2 & \longrightarrow & \ker(\mathrm{Sq}^2 \circ \pi) & \longrightarrow & 0 \\
 & & \twoheaddownarrow{\scriptstyle \eta} & & \big\downarrow{\scriptstyle \eta} & & \big\downarrow{\scriptstyle \pi} & & \\
0 & \longrightarrow & \mathrm{KO}^1_3 & \longrightarrow & \mathrm{KO}^1_2 & \longrightarrow & \ker(\mathrm{Sq}^2) & \longrightarrow & 0
\end{array}$$

where $\ker(\mathrm{Sq}^2 \circ \pi) \subset H^2(X;\mathrm{KO}^0(\text{point}))$ and $\ker(\mathrm{Sq}^2) \subset H^2(X;\mathrm{KO}^{-1}(\text{point}))$. This gives a short exact sequence of images, which must split since all groups involved are killed by multiplication with 2. Thus, we have

$$\begin{aligned}
\operatorname{im}\Big(\mathrm{KO}^2_2 \xrightarrow{\eta} \mathrm{KO}^1_2\Big) &\cong \mathrm{KO}^1_3 \oplus \operatorname{im}\Big(\ker(\mathrm{Sq}^2 \circ \pi) \to \ker(\mathrm{Sq}^2)\Big) \\
&\cong H^3(X;\mathbb{Z}/2) \oplus \ker(\mathrm{Sq}^2_{\mathbb{Z}})
\end{aligned}$$

where $\mathrm{Sq}^2_{\mathbb{Z}}$ is the composition defined at the beginning of this section. Finally, the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{KO}^2_2 & = & \widetilde{\mathrm{KO}}^2(X) & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle \eta} & & \big\downarrow{\scriptstyle \eta} & & \big\downarrow & & \\
0 & \longrightarrow & \mathrm{KO}^1_2 & \longrightarrow & \widetilde{\mathrm{KO}}^1(X) & \longrightarrow & E_4^{1,0} & \longrightarrow & 0
\end{array}$$

shows that $\operatorname{im}(\widetilde{\mathrm{KO}}^2(X) \xrightarrow{\eta} \widetilde{\mathrm{KO}}^1(X)) \cong \operatorname{im}(\mathrm{KO}^2_2 \xrightarrow{\eta} \mathrm{KO}^1_2)$. So $(\widetilde{\mathrm{KO}}^2/\widetilde{\mathrm{K}})(X)$ is isomorphic to $H^3(X;\mathbb{Z}/2) \oplus \ker(\mathrm{Sq}^2_{\mathbb{Z}})$, as claimed.

Computation of $\mathbf{KO}^4/\mathbf{K}$. Proceeding as in the previous case, we consider the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^4(X) & = & \widetilde{\mathrm{KO}}^4(X) & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \twoheaddownarrow & & \big\downarrow{\scriptstyle \eta} & & \big\downarrow & & \\
0 & \longrightarrow & \mathrm{KO}^3_4 & \longrightarrow & \widetilde{\mathrm{KO}}^3(X) & \longrightarrow & H^3(X) & \longrightarrow & 0
\end{array}$$

where the vertical map on the left is the composition

$$H^4(X;\mathrm{KO}^0(\text{point})) \twoheadrightarrow H^4(X;\mathrm{KO}^{-1}(\text{point})) \twoheadrightarrow \operatorname{coker}(\mathrm{Sq}^2 \circ \pi) = \mathrm{KO}^3_4$$

Since $\mathrm{Sq}^2 \circ \pi$ has the same cokernel as $\mathrm{Sq}^2_{\mathbb{Z}}$, we may deduce that

$$(\widetilde{\mathrm{KO}}^4/\widetilde{\mathrm{K}})(X) \cong \operatorname{im}(\widetilde{\mathrm{KO}}^4(X) \xrightarrow{\eta} \widetilde{\mathrm{KO}}^3(X)) \cong \operatorname{coker}(\mathrm{Sq}^2_{\mathbb{Z}})$$

Computation of $\mathbf{KO}^6/\mathbf{K}$. In this case, the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^2(X) & = & \widetilde{\mathrm{KO}}^6(X) & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle\eta} & & \downarrow & & \\
0 & \longrightarrow & 0 & \longrightarrow & \widetilde{\mathrm{KO}}^5(X) & = & H^1(X) & \longrightarrow & 0
\end{array}$$

demonstrates that the map $\widetilde{\mathrm{KO}}^6(X) \xrightarrow{\eta} \widetilde{\mathrm{KO}}^5(X)$ is zero, so $(\widetilde{\mathrm{KO}}^6/\,\mathrm{K})(X)$ is trivial. This completes the computations of $(\widetilde{\mathrm{KO}}^{2i}/\,\mathrm{K})(X)$. □

4 Comparison

In this section, we finally compare our two sets of results for complex curves and surfaces and prove the comparison theorem mentioned in the introduction to this chapter. First, however, we summarize the situation one finds in K-theory.

K-groups

The algebraic and topological K-groups of a smooth complex curve C are given by

$$\mathrm{K}_0(C) = \mathbb{Z} \oplus \operatorname{Pic}(C)$$
$$\mathrm{K}^0(C) = \mathbb{Z} \oplus H^2(C;\mathbb{Z})$$

The comparison map $\mathrm{K}_0(C) \to \mathrm{K}^0(C)$ is always surjective. Indeed, the map is an isomorphism on the first summand, and for a projective curve the map on reduced groups is simply the projection from $\operatorname{Pic}(C) \cong \mathbb{Z} \oplus \operatorname{Jac}(C)$ onto the free part. It follows that the map is still surjective if we remove a finite number of points from C.

For a smooth complex surface X, we have seen in (11) and (15) that we have filtrations on the K-groups such that

$$\mathrm{gr}^*(\mathrm{K}_0(X)) \cong \mathbb{Z} \oplus \operatorname{Pic}(X) \oplus \mathrm{CH}^2(X)$$
$$\mathrm{gr}^*(\mathrm{K}^0(X)) \cong \mathbb{Z} \oplus H^2(X;\mathbb{Z}) \oplus H^4(X;\mathbb{Z})$$

Both isomorphisms can be written as (rk, c_1, c_2), and the comparison map $\mathrm{K}_0(X) \to \mathrm{K}^0(X)$ corresponds to the usual comparison maps on the filtration. Of course, on the first summand we again have the identity. Moreover, the map $\mathrm{CH}^2(X) \to H^4(X;\mathbb{Z})$ is always surjective:

If X is projective, this follows from the fact that $H^4(X;\mathbb{Z})$ is generated by a point. In general, we can embed any smooth surface X into a projective surface $\overline{X}$ as an open subset with complement a divisor with simple normal crossings (c. f. Lemma 4.8). Then $\mathrm{CH}^2(\overline{X})$ surjects onto $\mathrm{CH}^2(X)$, and similarly $H^4(\overline{X};\mathbb{Z})$ surjects onto $H^4(X;\mathbb{Z})$, so that the claim follows. Since we will need this observation in a moment, we record it as a lemma.

4.1 Lemma. *For any smooth complex surface X, the natural map $\mathrm{CH}^2(X) \to H^4(X;\mathbb{Z})$ is surjective.*

4.2 Corollary. *For any smooth complex surface X, the natural map $\mathrm{K}_0(X) \to \mathrm{K}^0(X)$ is surjective if and only if the natural map $\mathrm{Pic}(X) \to H^2(X;\mathbb{Z})$ is surjective.*

When X is projective, we see from the exponential sequence that we have surjections if and only if X has geometric genus zero. Equivalently, this happens if and only if X has full Picard rank, i. e. if and only if its Picard number ρ agrees with its second Betti number b_2. More generally, the Picard group of any smooth complex surface can be written as

$$\mathrm{Pic}(X) = \mathbb{Z}^\rho \oplus H^2(X;\mathbb{Z})_{\mathrm{tors}} \oplus \mathrm{Pic}^0(X) \tag{17}$$

where $\rho \leq b_2$ is an integer that generalizes the Picard number, $H^2(X;\mathbb{Z})_{\mathrm{tors}}$ is the torsion subgroup of $H^2(X;\mathbb{Z})$, and $\mathrm{Pic}^0(X)$ is a divisible group [PW01, Corollary 6.2.1]. Again, the natural map $\mathrm{Pic}(X) \to H^2(X;\mathbb{Z})$ is surjective if and only if $\rho = b_2$.

4a The classical Witt group

For the classical Witt group $\mathrm{W}^0(X)$, the situation can be analysed in a similar way as in the case of $\mathrm{K}_0(X)$, using our description in terms of Stiefel-Whitney classes.

4.3 Proposition. *For a smooth complex curve C, the map*

$$\begin{aligned} gw^0\colon\ \mathrm{GW}^0(C) &\twoheadrightarrow \mathrm{KO}^0(C) \quad \textit{is surjective, and} \\ w^0\colon\ \mathrm{W}^0(C) &\xrightarrow{\cong} (\mathrm{KO}^0/\,\mathrm{K})(C) \quad \textit{is an isomorphism.} \end{aligned}$$

Similarly, for a smooth complex surface X, both gw^0 and w^0 are surjective. If X is projective, then w^0 is an isomorphism if and only if $\mathrm{Pic}(X)$ surjects onto $H^2(X;\mathbb{Z})$.

Proof. It suffices to show the corresponding statements for reduced groups. Let X be a smooth complex variety of dimension at most two. Since the first Stiefel-Whitney classes are always surjective, we have a row exact commutative diagram of the form

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \ker(w_1) & \longrightarrow & \widetilde{\mathrm{GW}}^0(X) & \xrightarrow{w_1} & H^1_{\mathrm{et}}(X;\mathbb{Z}/2) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle gw^0} & & \downarrow{\scriptstyle \cong} & & \\
0 & \longrightarrow & \ker(w_1^{\mathrm{top}}) & \longrightarrow & \widetilde{\mathrm{KO}}^0(X) & \xrightarrow{w_1^{\mathrm{top}}} & H^1(X;\mathbb{Z}/2) & \longrightarrow & 0
\end{array}$$

Similarly, by Corollaries 3.6 and 3.13 we have a row exact commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^0(\ker(c_1)) & \longrightarrow & \ker(w_1) & \xrightarrow{w_2} & H^2_{\mathrm{et}}(X;\mathbb{Z}/2) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \cong} & & \\
0 & \longrightarrow & r(\ker(c_1^{\mathrm{top}})) & \longrightarrow & \ker(w_1^{\mathrm{top}}) & \xrightarrow{w_2^{\mathrm{top}}} & H^2(X;\mathbb{Z}/2) & \longrightarrow & 0
\end{array}$$

In both diagrams, we have written w_i^{top} and c_i^{top} for the topological characteristic classes to avoid ambiguous notation. The kernels of c_1 and c_1^{top} can be identified with $\mathrm{CH}^2(X)$ and $H^4(X;\mathbb{Z})$, respectively, via the second Chern classes. Thus, Lemma 4.1 implies that the vertical map on the left of the lower diagram is a surjection. The surjectivity of the comparison map gw^0 on $\mathrm{GW}^0(X)$ follows.

If we apply the same analysis to $\mathrm{W}^0(X)$, then the second diagram reduces to a commutative square

$$\begin{array}{ccc}
\ker(\overline{w}_1) & \xrightarrow[\cong]{\overline{w}_2} & H^2_{\mathrm{et}}(X;\mathbb{Z}/2)\big/\mathrm{Pic}(X) \\
\downarrow & & \twoheaddownarrow \\
\ker(\overline{w}_1^{\mathrm{top}}) & \xrightarrow[\cong]{\overline{w}_2^{\mathrm{top}}} & H^2(X;\mathbb{Z}/2)\big/H^2(X;\mathbb{Z})
\end{array}$$

When X is a curve, the vertical arrow on the right is an isomorphism, proving the claim. In general, we have a row-exact commutative diagram of the following form:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Pic}(X)\big/2 & \longrightarrow & H^2_{\mathrm{et}}(X;\mathbb{Z}/2) & \longrightarrow & H^2_{\mathrm{et}}(X;\mathbb{Z}/2)\big/\mathrm{Pic}(X) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \cong} & & \twoheaddownarrow & & \\
0 & \longrightarrow & H^2(X;\mathbb{Z})\big/2 & \longrightarrow & H^2(X;\mathbb{Z}/2) & \longrightarrow & H^2(X;\mathbb{Z}/2)\big/H^2(X;\mathbb{Z}) & \longrightarrow & 0
\end{array}$$

If $\mathrm{Pic}(X) \to H^2(X;\mathbb{Z})$ is surjective, then the two outer maps become isomorphisms and it follows that the comparison map w^0 is also an isomorphism. Using the description of the Picard group of X given by (17), we see that the converse is also true. □

4b Shifted Witt groups

We now generalize Proposition 4.3 to shifted groups. The final result is stated in Theorems 4.4 and 4.12. For curves, there is in fact very little left to be shown. We nevertheless give a detailed proof in preparation for a similar line of arguments in the case of surfaces.

4.4 Theorem. *For any smooth complex curve C, the maps*

$$\begin{array}{rll} gw^i\colon \mathrm{GW}^i(C) \to \mathrm{KO}^i(C) & & \text{are surjective, and the maps} \\ w^i\colon \quad \mathrm{W}^i(C) \to (\mathrm{KO}^i/\,\mathrm{K})(C) & & \text{are isomorphisms.} \end{array}$$

Proof. By Lemma II.2.6, the comparison maps appearing here are isomorphisms on a point, so the claims are equivalent to the corresponding claims involving reduced groups. It suffices to show that the maps $gw^i\colon \widetilde{\mathrm{GW}}^i(C) \to \widetilde{\mathrm{KO}}^i(C)$ are surjective and that the maps $w^i\colon \widetilde{\mathrm{W}}^i(C) \to \widetilde{\mathrm{KO}}^{2i-1}(C)$ are injective.

First, suppose C is affine. Then the cohomology of C is concentrated in degrees 0 and 1 and we see that $\mathrm{W}^1(C)$, $\mathrm{W}^2(C)$, $\mathrm{W}^3(C)$ and $\widetilde{\mathrm{KO}}^2(X)$, $\widetilde{\mathrm{KO}}^4(C)$, $\widetilde{\mathrm{KO}}^6(C)$ all vanish. Thus, the claims are trivially true.

The case that C is projective can be reduced to the affine case. Indeed, if p is any point on C, then $\widetilde{C} := C - p$ is affine. By comparing the localization sequences arising from the inclusion of $\widetilde{C}$ into C, we see that the comparison maps for C must also have the desired properties:

$$\begin{array}{ccccccccccccc} \ldots \to & \mathrm{GW}^{i-1}(p) & \to & \widetilde{\mathrm{GW}}^i(C) & \to & \widetilde{\mathrm{GW}}^i(\widetilde{C}) & \longrightarrow & \mathrm{W}^i(p) & \longrightarrow & \widetilde{\mathrm{W}}^{i+1}(C) & \longrightarrow & \widetilde{\mathrm{W}}^{i+1}(\widetilde{C}) & \to \ldots \\ & \downarrow{\cong} & & \vdots & & \twoheaddownarrow & & \downarrow{\cong} & & \vdots & & \downarrow & \\ \ldots \to & \mathrm{KO}^{2i-2}(p) & \to & \widetilde{\mathrm{KO}}^{2i}(C) & \to & \widetilde{\mathrm{KO}}^{2i}(\widetilde{C}) & \to & \mathrm{KO}^{2i-1}(p) & \to & \widetilde{\mathrm{KO}}^{2i+1}(C) & \to & \widetilde{\mathrm{KO}}^{2i+1}(\widetilde{C}) & \to \ldots \end{array} \qquad (18)$$

□

4.5 Corollary. *Theorem 4.4 also hold for groups with twists in any line bundle.*

Proof. Introducing a twist by the line bundle $\mathcal{O}(p)$ into the localization sequence (18) only affects the groups of C, so we can conclude as before. More generally, given a line bundle $\mathcal{O}(D)$ associated with a divisor $D = \sum n_i p_i$ on C, we can similarly reduce to the case of a trivial line bundle over $C - \bigcup_i p_i$. □

We now want to imitate this proof for surfaces, replacing the role of points on the curve by curves on the surface. We first prove the following.

4.6 Proposition. *For any smooth complex surface X, the comparison maps have the properties indicated by the following arrows:*

$$\begin{cases} gw^0\colon \mathrm{GW}^0(X) \twoheadrightarrow \mathrm{KO}^0(X) \\ w^1\colon \ \mathrm{W}^1(X) \rightarrowtail \mathrm{KO}^1(X) \end{cases} \qquad \begin{cases} gw^2\colon \mathrm{GW}^2(X) \twoheadrightarrow \mathrm{KO}^4(X) \\ w^3\colon \ \mathrm{W}^3(X) \rightarrowtail \mathrm{KO}^5(X) \end{cases}$$

4.7 Lemma. *Proposition 4.6 is true when X is affine and* $\mathrm{Pic}(X)/2$ *vanishes.*

Proof. By the theorem of Andreotti and Frankel, a smooth complex affine variety of dimension n has the homotopy type of a CW complex of real dimension at most n [AF59,Laz04, 3.1]. In particular, its cohomology is concentrated in degrees $\leq n$. For a smooth affine surface X, Pardon's spectral sequence shows immediately that $\mathrm{W}^1(X) = \mathrm{W}^2(X) = \mathrm{W}^3(X) = 0$. Similarly, the Atiyah-Hirzebruch spectral sequence shows that $\widetilde{\mathrm{KO}}^4(X)$ vanishes. Thus, three of the four claims are trivially satisfied. The fact that $gw^0\colon \mathrm{GW}^0(X) \twoheadrightarrow \mathrm{KO}^0(X)$ is surjective was already shown in 4.3. □

Before proceeding with the proof of Proposition 4.6, we make a note of two general facts that we will use. First, we will need the following standard consequence of Hironaka's resolution of singularities:

4.8 Lemma. *In characteristic zero, any smooth variety can be embedded into a smooth compact variety with complement a divisor with simple normal crossings.*

Proof. Let X be a smooth variety over a field of characteristic zero, and let $\overline{X}$ be some compactification. Any singularities of $\overline{X}$ may be resolved without changing the smooth locus [Kol07, Theorem 3.36], so we may assume that $\overline{X}$ is smooth. Applying the Principalization Theorem [Kol07, Theorem 3.26] to (the ideal sheaf of) the complement of X in $\overline{X}$ yields the variety we are looking for. □

Note that in dimension two there is no distinction between compactness and projectivity: any smooth compact surface is projective [Har70, II.4.2]. It follows that an arbitrary smooth surface is at least quasi-projective. Secondly, we will need the following lemma concerning generators of the Picard group.

4.9 Lemma. *Let X be a smooth quasi-projective variety over an algebraically closed field of characteristic zero. Then any element of* $\mathrm{Pic}(X)/2$ *can be represented by a smooth prime divisor (i. e. by a smooth irreducible subvariety of codimension* 1*). If X is projective, we may moreover take the divisor to be very ample (i. e. to be given by a hyperplane section of X for some embedding of X into some $\mathbb{P}^N$).*

Proof. We consider the case when X is projective first. If X is a projective curve, $\mathrm{Pic}(X)/2 \cong \mathbb{Z}/2$ is generated by a point and there is nothing to show. So we may assume $\dim(X) \geq 2$.

Fix a very ample line bundle $\mathcal{L}$ on X. Any element of $\mathrm{Pic}(X)/2$ can be lifted to a line bundle $\mathcal{M}$ on X. Tensoring with any sufficiently high power of $\mathcal{L}$ will yield a very ample line bundle $\mathcal{M} \otimes \mathcal{L}^m$ [Laz04, 1.2.10]. In particular, if we take m large and even we obtain a very ample line bundle that maps to the class of $\mathcal{M}$ in $\mathrm{Pic}(X)/2$. The claim now follows from Bertini's theorem on hyperplane sections in characteristic zero [Har77, Corollary 10.9 and Remark 10.9.1].

In general, if X is quasi-projective, we may embed it as an open subset into a smooth resolution $\overline{X}$ of its projective closure. Then $\mathrm{Pic}(\overline{X})$ surjects onto $\mathrm{Pic}(X)$, and we obtain smooth prime divisors on X generating $\mathrm{Pic}(X)/2$ by restriction. □

4.10 Example. Consider $\widetilde{\mathbb{P}}^2$, the blow-up of $\mathbb{P}^2$ at a point p. Its Picard group is given by $\mathrm{Pic}(\widetilde{\mathbb{P}}^2) = \mathbb{Z}[H] \oplus \mathbb{Z}[E]$, where H is a hyperplane section of $\mathbb{P}^2$ that misses p and E is the exceptional divisor, both isomorphic to $\mathbb{P}^1$. A divisor $a[H] - b[E]$ is ample if and only if $a > b > 0$. Thus, $\{[H], 2[H] - [E]\}$ is a basis of $\mathrm{Pic}(\widetilde{\mathbb{P}}^2)$ consisting of ample divisors. A smooth curve representing $2[H] - [E]$ is given by the birational transform of a smooth conic in $\mathbb{P}^2$ through p.

Proof of Proposition 4.6. Let X be a smooth surface. By Lemma 4.8, we can find a projective surface $\overline{X}$ and smooth curves $D_1, \ldots, D_k$ on $\overline{X}$ whose union $\bigcup D_i$ is the complement of X in $\overline{X}$. On the other hand, by Lemma 4.9, we can find smooth ample curves generating $\mathrm{Pic}(\overline{X})/2$. Let $C_1, \ldots, C_\rho$ be a subset of these curves generating $\mathrm{Pic}(X)/2$, and put

$$U_i := X - C_1 - C_2 - \cdots - C_i$$

For sufficiently large k, the divisors $\sum_j D_j + k(C_1 + \cdots + C_i)$ are ample on $\overline{X}$. Thus, each U_i is affine. Moreover, we see from the exact sequences

$$\mathbb{Z}[C_{i+1}|_{U_i}] \to \mathrm{Pic}(U_i) \to \mathrm{Pic}(U_{i+1}) \to 0$$

and the choice of the C_i that $\mathrm{rk}_{\mathbb{Z}/2}(\mathrm{Pic}(U_i)/2) = \rho - i$. Thus, by Lemma 4.7, Proposition 4.6 holds for U_ρ.

We can now proceed as in the proof of Theorem 4.4, by adding the curves back in to obtain X. Namely, consider the successive open inclusions $U_{i+1} \hookrightarrow U_i$. The closed complements of these are given by restrictions of the curves C_i, so we obtain a diagram similar to (18) with U_i playing the role of C, U_{i+1} in the role of $\widetilde{C}$ and C_{i+1} in the role of a point. If the normal bundle of C_{i+1} in U_i is not trivial, the sequences will in fact involve twisted groups of C_i, but in any case we can conclude using Lemma 4.5. □

4.11 Corollary. *Proposition 4.6 also holds for groups with twists in a line bundle over X.*

Proof. As we have seen, generators of $\mathrm{Pic}(X)/2$ can be represented by smooth curves on X. Thus, we can argue as in the proof of Corollary 4.5. □

4.12 Theorem. *Suppose X is a smooth projective surface for which the natural map $\mathrm{Pic}(X) \to H^2(X;\mathbb{Z})$ is surjective. Then the comparison maps*

$$gw^i\colon \mathrm{GW}^i(X) \to \mathrm{KO}^i(X) \quad \textit{are surjective, and the maps}$$
$$w^i\colon \mathrm{W}^i(X) \to (\mathrm{KO}^{2i}/\,\mathrm{K})(X) \quad \textit{are isomorphisms.}$$

By Proposition 4.3, the assumption on the map $\mathrm{Pic}(X) \to H^2(X;\mathbb{Z})$ is clearly necessary.

Proof. Consider the squaring operations S^1 and $\mathrm{Sq}^2_{\mathbb{Z}}$ appearing in the computations of the Witt and (KO/K)-groups. For any smooth complex variety X, we have a commutative diagram

$$\begin{array}{ccc}
\mathrm{Pic}(X)/2 & \xrightarrow{S^1} & \mathrm{CH}^2(X)/2 \\
\downarrow & & \downarrow \\
H^2(X;\mathbb{Z})/2 & \longrightarrow & H^4(X;\mathbb{Z})/2 \\
\uparrow & & \uparrow \\
H^2(X;\mathbb{Z}/2) & \xrightarrow{\mathrm{Sq}^2} & H^4(X;\mathbb{Z}/2)
\end{array}$$

When X is a surface, the two vertical maps on the right are both isomorphisms, and the horizontal map in the middle is essentially $\mathrm{Sq}^2_{\mathbb{Z}}$. So whenever $\mathrm{Pic}(X)$ surjects onto $H^2(X;\mathbb{Z})$, we may identify S^1 and $\mathrm{Sq}^2_{\mathbb{Z}}$. In particular, we may do so for any a smooth projective surface of geometric genus zero. It then follows by comparison of Propositions 3.1 and 3.8 that the Witt groups of such a surface agree with the groups $(\mathrm{KO}^{2i}/\mathrm{K})(X)$. On the other hand, we see from Proposition 4.6 that each of the maps $w^i\colon \mathrm{W}^i(X) \to (\mathrm{KO}^{2i}/\mathrm{K})(X)$ is either surjective or injective. Given that we are dealing with finite groups, these maps must be isomorphisms. Moreover, since we know from Corollary 4.2 that we also have a surjection from the algebraic to the topological K-group of X, we may deduce via the Karoubi/Bott sequences that the maps $gw^i\colon \mathrm{GW}^i(X) \to \mathrm{KO}^{2i}(X)$ are surjective for all values of i. □

4c Comparison with $\mathbb{Z}/2$-coefficients

As we have seen, the comparison maps on Witt groups are isomorphisms for all surfaces X for which $\mathrm{Pic}(X)$ surjects onto $H^2(X;\mathbb{Z})$. To obtain isomorphisms on the level of Grothendieck-Witt groups, we need to pass to $\mathbb{Z}/2$-coefficients (see Section II.2e) and introduce one additional topological constraint on X.

4.13 Proposition. *Let X be a smooth complex variety, of any dimension. If the odd topological K-groups of X contain no 2-torsion (i. e. if $\mathrm{K}^1(X)[2] = 0$), then the integral comparison maps*

$$\mathrm{W}^i(X) \to (\mathrm{KO}^{2i}/\mathrm{K})(X)$$

are isomorphisms for all i if and only if the comparison maps with $\mathbb{Z}/2$-coefficients

$$\mathrm{W}^i(X;\mathbb{Z}/2) \to (\mathrm{KO}^{2i}/\mathrm{K})(X;\mathbb{Z}/2)$$

are isomorphisms for all i.

Before giving the proof, we make one preliminary observation.

4.14 Lemma. *Let X be a topological space such that $\mathrm{K}^1(X)[2] = 0$. Then multiplication by $\eta \in \mathrm{KO}^{-1}(\text{point})$ induces an isomorphism*

$$(\mathrm{KO}^{2i}/\,\mathrm{K})(X) \xrightarrow[\cong]{\eta} \mathrm{KO}^{2i-1}(X)[2]$$

Proof. In general, since $2\eta = 0$, the following exact sequence may be extracted from the Bott sequence:

$$0 \to (\mathrm{KO}^{2i}/\,\mathrm{K})(X) \xrightarrow{\eta} \mathrm{KO}^{2i-1}(X)[2] \to \mathrm{K}^1(X)[2]$$

This proves the claim. □

Proof of Proposition 4.13. We claim that we have the following row-exact commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \frac{\mathrm{GW}^i(X)}{\mathrm{K}_0(X)} & \longrightarrow & \frac{\mathrm{GW}^i(X;\mathbb{Z}/2)}{\mathrm{K}_0(X;\mathbb{Z}/2)} & \longrightarrow & \mathrm{W}^{i+1}(X) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \frac{\mathrm{KO}^{2i}(X)}{\mathrm{K}^0(X)} & \longrightarrow & \frac{\mathrm{KO}^{2i}(X;\mathbb{Z}/2)}{\mathrm{K}^0(X;\mathbb{Z}/2)} & \longrightarrow & \mathrm{KO}^{2i+1}(X)[2] & \longrightarrow & 0
\end{array} \qquad (*)$$

Indeed, the lower exact row may be obtained by applying the Snake Lemma to the following diagram of short exact sequences induced by the Bockstein sequences for K- and KO-theory:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{K}^0(X)/2 & \xrightarrow{\cong} & \mathrm{K}^0(X;\mathbb{Z}/2) & \longrightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{KO}^{2i}(X)/2 & \longrightarrow & \mathrm{KO}^{2i}(X;\mathbb{Z}/2) & \longrightarrow & \mathrm{KO}^{2i+1}(X)[2] & \longrightarrow & 0
\end{array}$$

The upper row of $(*)$ may be obtained similarly from the Bockstein sequences for algebraic and hermitian K-theory. The vertical maps are induced by the comparison maps in degrees 0, 0 and -1, respectively. Using the canonical identification of $\mathrm{GW}^i(X)/\,\mathrm{K}_0(X)$ with $\mathrm{W}^i(X)$, we may however identify the first vertical map with the usual comparison map w^i in degree -1. Likewise, the second vertical map may be identified with the comparison map w^i for Witt groups with $\mathbb{Z}/2$-coefficients. Lastly, by the previous lemma, the entry in the lower right corner may be identified with $(\mathrm{KO}^{2i+2}/\,\mathrm{K})(X)$. Thus, diagram $(*)$ can be rewritten in a form from which both implications of the proposition may be deduced:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{W}^i(X) & \longrightarrow & \mathrm{W}^i(X;\mathbb{Z}/2) & \longrightarrow & \mathrm{W}^{i+1}(X) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle w^i} & & \downarrow{\scriptstyle w^i} & & \downarrow{\scriptstyle w^{i+1}} & & \\
0 & \longrightarrow & \dfrac{\mathrm{KO}^{2i}(X)}{\mathrm{K}(X)} & \longrightarrow & \dfrac{\mathrm{KO}^{2i}(X;\mathbb{Z}/2)}{\mathrm{K}(X;\mathbb{Z}/2)} & \longrightarrow & \dfrac{\mathrm{KO}^{2i+2}(X)}{\mathrm{K}(X)} & \longrightarrow & 0
\end{array}$$

□

Recall from II.2e that, by the Quillen-Lichtenbaum conjectures, the comparison maps on K-groups of curves with $\mathbb{Z}/2$-coefficients are isomorphisms in all non-negative degrees, while for a smooth complex surface X we have isomorphisms in all positive degrees and an inclusion in degree zero:

$$\mathrm{K}_i(X;\mathbb{Z}/2) \xrightarrow{\cong} \mathrm{K}^{-i}(X;\mathbb{Z}/2) \quad \text{for all } i > 0$$
$$\mathrm{K}_0(X;\mathbb{Z}/2) \hookrightarrow \mathrm{K}^0(X;\mathbb{Z}/2)$$

For these low-dimensional cases, proofs may also be found in [Sus95, 4.7; PW01, Theorem 2.2].

4.15 Lemma. *Let X be a smooth complex variety of dimension at most two. The map $\mathrm{K}_0(X;\mathbb{Z}/2) \hookrightarrow \mathrm{K}^0(X;\mathbb{Z}/2)$ is an isomorphism if and only if the map $\mathrm{Pic}(X) \to H^2(X;\mathbb{Z})$ is surjective and $\mathrm{K}^1(X)[2] = 0$.*

Proof. We see from the description of the Picard group (17) that $\mathrm{Pic}(X) \to H^2(X;\mathbb{Z})$ is surjective if and only if it is surjective after tensoring with $\mathbb{Z}/2$. Moreover, by a similar argument as in Corollary 4.2, the surjectivity of the map $\mathrm{Pic}(X)/2 \to H^2(X;\mathbb{Z})/2$ is equivalent to the surjectivity of the map $\mathrm{K}_0(X)/2 \to \mathrm{K}^0(X)/2$. The claim follows from these equivalences and a commutative diagram induced by the Bockstein sequences:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{K}_0(X)/2 & \xrightarrow{\cong} & \mathrm{K}_0(X;\mathbb{Z}/2) & \longrightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{K}^0(X)/2 & \longrightarrow & \mathrm{K}^0(X;\mathbb{Z}/2) & \longrightarrow & \mathrm{K}^1(X)[2] & \longrightarrow & 0
\end{array}$$

□

If we combine the results on K-theory with our results for Witt groups and Proposition 4.13, we obtain the following corollary via Karoubi induction.

4.16 Corollary. *Let X be a smooth complex variety of dimension at most two. Assume that the natural map $\mathrm{Pic}(X) \to H^2(X;\mathbb{Z})$ is surjective and that $\mathrm{K}^1(X)$ has no 2-torsion. Then the hermitian comparison maps*

$$\mathrm{KO}^{p,q}(X;\mathbb{Z}/2) \longrightarrow \mathrm{KO}^p(X;\mathbb{Z}/2)$$

are isomorphisms in all non-negative degrees, i. e. for all (p,q) such that $2q - p \geq 0$. In particular, for all shifts i we have isomorphisms

$$\mathrm{GW}^i(X;\mathbb{Z}/2) \xrightarrow{\cong} \mathrm{KO}^{2i}(X;\mathbb{Z}/2)$$

For example, the conditions of the corollary are satisfied for any smooth complex curve, and for any simply-connected projective surface of geometric genus zero. The condition that $\mathrm{K}^1(X)[2] = 0$ can be rephrased in terms of the integral cohomology group $H^3(X; \mathbb{Z})$:

4.17 Lemma. *Let X be a smooth complex variety of dimension at most two. Then the torsion of $\mathrm{K}^1(X)$ agrees with the torsion of $H^3(X; \mathbb{Z})$.*

Proof. By Lemma 3.11, the Atiyah-Hirzebruch spectral sequence for the K-theory of X collapses. Since the first integral cohomology group $H^1(X; \mathbb{Z})$ of a smooth complex curve or surface is free, we find that $\mathrm{K}^1(X) \cong H^1(X; \mathbb{Z}) \oplus H^3(X; \mathbb{Z})$, and the lemma follows. □

Remark. Conversely, if we assume the analogue of the Quillen-Lichtenbaum conjecture for hermitian K-theory, i. e. if we assume that the hermitian comparison maps with $\mathbb{Z}/2$-coefficients are isomorphisms in high degrees, then we can recover our comparison theorem for Witt groups for all surfaces X with $\mathrm{K}^1(X)[2] = 0$. Said analogue appeared recently in [Sch10]. However, it does not seem possible to relate our result to the Quillen-Lichtenbaum conjecture for surfaces with 2-torsion in $\mathrm{K}^1(X)$. Such surfaces do exist. In particular, if X is an Enriques surface, then $\mathrm{Pic}(X)$ surjects onto $H^2(X; \mathbb{Z})$ but $\mathrm{K}^1(X)[2] \cong \pi_1(X) \cong \mathbb{Z}/2$ [Bea96, page 90].

Chapter IV

Cellular Varieties

In this chapter, we turn our attention to cellular varieties. As we will see, the comparison maps behave particularly well on these, allowing us to identify all Grothendieck-Witt and Witt groups with the corresponding KO-groups. This will be used to compute the Witt groups of a series of projective homogeneous varieties.

By definition, a smooth cellular variety is a smooth variety X with a filtration by closed subvarieties $\emptyset = Z_0 \subset Z_1 \subset Z_2 \subset \cdots \subset Z_N = X$ such that the complement of Z_k in Z_{k+1} is an open "cell" isomorphic to $\mathbb{A}^{n_k}$ for some n_k. For example, projective n-space $\mathbb{P}^n$ contains an open cell isomorphic to $\mathbb{A}^n$ with closed complement $\mathbb{P}^{n-1}$, so such a filtration can be obtained inductively. It is well-known that, when X is a smooth cellular variety over the complex numbers, the comparison map on K-groups is an isomorphism:

$$\mathrm{K}_0(X) \xrightarrow{\cong} \mathrm{K}^0(X)$$

In fact, both sides are easy to compute. They decompose as direct sums of the K-groups of the cells, each of which is isomorphic to $\mathbb{Z}$. However, the computation of Witt groups of cellular varieties does not seem to be as straight-forward. It is true, of course, that the Witt groups of complex varieties decompose into copies of $\mathbb{Z}/2$, the Witt group of $\mathbb{C}$, but even in the cellular case there is no general understanding of how many copies to expect.

Nonetheless, it is possible to prove by an induction over the number of cells of X that the comparison maps on Grothendieck-Witt and Witt groups are also isomorphisms:

$$\mathrm{GW}^i(X) \xrightarrow{\cong} \mathrm{KO}^{2i}(X)$$
$$\mathrm{W}^i(X) \xrightarrow{\cong} (\mathrm{KO}^{2i}/\,\mathrm{K})(X)$$

The map $(\mathrm{KO}^{2i}/\,\mathrm{K})(X) \xrightarrow{\eta} \mathrm{KO}^{2i-1}(X)$ is an isomorphism in this situation (see Lemma 2.2), so the second line may equivalently be stated as

$$\mathrm{W}^i(X) \xrightarrow{\cong} \mathrm{KO}^{2i-1}(X)$$

Thus, the Grothendieck-Witt and Witt groups of smooth complex cellular varieties may be read off directly from their KO-groups.

As in the previous chapter, we use the homotopy-theoretic approach to the comparison maps. As a by-product of the proof we give, we also obtain partial information on the maps in degrees 1 and −2. A slightly different proof using only the maps in degrees 0 and −1 and those properties that can be seen in a more elementary way may be found in [Zib09, Theorem 3.1]. The argument is sketched in Remark 1.5.

After presenting our proof of the comparison result at the beginning of this chapter, we turn to concrete topological computations. As our main tool will be the Atiyah-Hirzebruch spectral sequence, we collect some general observations concerning its behaviour on cellular varieties in Section 2. We then run through the list of all projective homogeneous varieties that fall within the class of hermitian symmetric spaces. The untwisted KO-groups of these are known by several papers of Kono and Hara, and we extend their computations to twisted KO-groups. Combined with the comparison result, this furnishes us with a complete additive description of the Witt groups of these varieties.

1 The comparison theorem

1.1 Theorem. *For a smooth cellular complex variety X, the following comparison maps are isomorphisms:*

$$\begin{aligned} k\colon \quad \mathrm{K}_0(X) &\xrightarrow{\cong} \mathrm{K}^0(X) \\ gw^q\colon \mathrm{GW}^q(X) &\xrightarrow{\cong} \mathrm{KO}^{2q}(X) \\ w^q\colon \quad \mathrm{W}^q(X) &\xrightarrow{\cong} \mathrm{KO}^{2q-1}(X) \end{aligned}$$

More generally, the maps gw^q and w^q are isomorphisms for the corresponding groups twisted by an arbitrary vector bundle over X (see Section II.2c).

In the proof, we concentrate on the hermitian case. The case of algebraic/complex K-theory could be dealt with similarly, or deduced from the hermitian case using triangle (II.10). As indicated in the introduction, it will be helpful to consider not only the maps $gw^q = k_h^{2q,q}$ and $w^{q+1} = k_h^{2q+1,q}$ in degrees 0 and −1, respectively, but also the maps $k_h^{2q-1,q}$ in degree 1 and the maps $k_h^{2q+2,q}$ in degree −2. In fact, we will prove the following extended version of the theorem, generalizing Proposition II.2.6:

1.2 Theorem. *For X as above, the hermitian comparison maps in degrees 1, 0, −1 and −2 have the properties indicated:*

$$\begin{aligned} \mathrm{KO}^{2q-1,q}(X) &\twoheadrightarrow \mathrm{KO}^{2q-1}(X) \\ \mathrm{KO}^{2q,q}(X) &\xrightarrow{\cong} \mathrm{KO}^{2q}(X) \\ \mathrm{KO}^{2q+1,q}(X) &\xrightarrow{\cong} \mathrm{KO}^{2q+1}(X) \\ \mathrm{KO}^{2q+2,q}(X) &\rightarrowtail \mathrm{KO}^{2q+2}(X) \end{aligned}$$

The analogous statements for twisted groups are also true.

Proof. The proof will proceed by induction over the number of cells of X. To begin the induction, we need to consider the case of only one cell, which immediately reduces to the case of a point by homotopy invariance. This case was dealt with in Proposition II.2.6.

Spheres. Since the theorem holds for a point, the compatibility of the comparison maps with suspensions immediately shows that it is also true for the reduced cohomology of the spheres $(\mathbb{P}^1)^{\wedge d} = S^{2d,d}$. In other words, the following maps in degrees 1, 0, -1 and -2 have the properties indicated:

$$\begin{aligned} \widetilde{\mathrm{KO}}^{2q-1,q}(S^{2d,d}) &\twoheadrightarrow \widetilde{\mathrm{KO}}^{2q-1}(S^{2d}) \\ \widetilde{\mathrm{KO}}^{2q,q}(S^{2d,d}) &\xrightarrow{\cong} \widetilde{\mathrm{KO}}^{2q}(S^{2d}) \\ \widetilde{\mathrm{KO}}^{2q+1,q}(S^{2d,d}) &\xrightarrow{\cong} \widetilde{\mathrm{KO}}^{2q+1}(S^{2d}) \\ \widetilde{\mathrm{KO}}^{2q+2,q}(S^{2d,d}) &\rightarrowtail \widetilde{\mathrm{KO}}^{2q+2}(S^{2d}) \end{aligned}$$

Cellular varieties. Now let X be a smooth cellular variety. By definition, X has a filtration by closed subvarieties $\emptyset = Z_0 \subset Z_1 \subset Z_2 \cdots \subset Z_N = X$ such that the open complement of Z_k in Z_{k+1} is isomorphic to $\mathbb{A}^{n_k}$ for some n_k. In general, the subvarieties Z_k will not be smooth. Their complements $U_k := X - Z_k$ in X, however, are always smooth as they are open in X. So we obtain an alternative filtration $X = U_0 \supset U_1 \supset U_2 \cdots \supset U_N = \emptyset$ of X by smooth open subvarieties U_k. Each U_k contains a closed cell $C_k \cong \mathbb{A}^{n_k}$ with open complement U_{k+1}.

Our inductive hypothesis is that we have already proved the theorem for U_{k+1}, and we now want to prove it for U_k. We can use the following exact triangle in $\mathcal{SH}(\mathbb{C})$:

$$\begin{array}{ccc} \Sigma^\infty((U_{k+1})_+) & \longrightarrow & \Sigma^\infty((U_k)_+) \\ & \nwarrow \quad \swarrow & \\ & \Sigma^\infty \mathrm{Thom}(\mathcal{N}_{C_k \setminus U_k}) & \end{array}$$

As C_k is a cell, the Quillen-Suslin theorem tells us that the normal bundle $\mathcal{N}_{C_k \setminus U_k}$ of C_k in U_k has to be trivial. Thus, $\mathrm{Thom}(\mathcal{N}_{C_k \setminus U_k})$ is $\mathbb{A}^1$-weakly equivalent to $S^{2d,d}$, where d is the codimension of C_k in U_k. Figure 1 on page 85 displays the comparison between the long exact cohomology sequences induced by this triangle. The inductive step is completed by applying the Five Lemma to each dotted map in the diagram.

The twisted case. To obtain the theorem in the case of coefficients in a vector bundle $\mathcal{E}$ over X, we replace the exact triangle above by the triangle

$$\begin{array}{ccc} \Sigma^\infty \mathrm{Thom}(\mathcal{E}_{|U_{k+1}}) & \longrightarrow & \Sigma^\infty \mathrm{Thom}(\mathcal{E}_{|U_k}) \\ & \nwarrow \quad \swarrow & \\ & \Sigma^\infty \mathrm{Thom}(\mathcal{E}_{|C_k} \oplus \mathcal{N}_{C_k \setminus U_k}) & \end{array}$$

The existence of this exact triangle is shown in the next lemma. The Thom space on the right is again just a sphere, so we can proceed as in the untwisted case. □

1.3 Lemma. *Given a smooth subvariety Z of a smooth variety X with complement U, and given any vector bundle $\mathcal{E}$ over X, we have an exact triangle*

$$\begin{array}{ccc} \Sigma^\infty \operatorname{Thom}(\mathcal{E}_{|U}) & \longrightarrow & \Sigma^\infty \operatorname{Thom} \mathcal{E} \\ & \nwarrow \quad \swarrow & \\ & \Sigma^\infty \operatorname{Thom}(\mathcal{E}_{|Z} \oplus \mathcal{N}_{Z \setminus X}) & \end{array}$$

Proof. From the Thom isomorphism theorem we know that the Thom space of a vector bundle over a smooth base is $\mathbb{A}^1$-weakly equivalent to the quotient of the vector bundle by the complement of the zero section. Consider the closed embeddings $U \hookrightarrow (\mathcal{E} - Z)$, $X \hookrightarrow \mathcal{E}$ and $Z \hookrightarrow \mathcal{E}$. Computing the normal bundles, we obtain

$$\begin{aligned} (\mathcal{E} - Z)/(\mathcal{E} - X) &\cong \operatorname{Thom}_U(\mathcal{E}_{|U}) \\ \mathcal{E}/(\mathcal{E} - X) &\cong \operatorname{Thom}_X \mathcal{E} \\ \mathcal{E}/(\mathcal{E} - Z) &\cong \operatorname{Thom}_Z(\mathcal{E}_{|Z} \oplus \mathcal{N}_{Z \setminus X}) \end{aligned}$$

The claim follows by passing to the stable homotopy category and applying the octahedral axiom to the composition of the embeddings $(\mathcal{E} - X) \subseteq (\mathcal{E} - Z) \subseteq \mathcal{E}$. □

1.4 Remark (Comparison in degree -2). As we saw in Section II.2d, the comparison map in degrees 1 and -2 are not isomorphisms even when X is a point, and in degrees below -2 the comparison maps are necessarily zero. For cellular varieties, the map in degree -2 may be identified with the inclusion of the 2-torsion subgroup of $\mathrm{KO}^{2q+2}(X)$ into $\mathrm{KO}^{2q+2}(X)$. This follows from the theorem and the description of the KO-groups of cellular varieties given in Lemma 2.2.

1.5 Remark. We indicate briefly how Theorem 1.1 can alternatively be obtained by working only with the maps in degrees 0 and -1 defined by more elementary means. The basic strategy — comparing the localization sequences arising from the inclusion of a closed cell C_k into the union of "higher" cells U_k — still works. But we cannot deduce that the comparison maps are isomorphisms on U_k from the fact that they are isomorphisms on U_{k+1} because the parts of the sequences that we can actually compare are now too short. We can, however, still deduce that the maps in degree 0 with domains the Grothendieck-Witt groups of U_k are surjective, and that the maps in degree -1 with domains the Witt groups of U_k are injective. The inductive step can then be completed with the help of the Bott/Karoubi sequences. This argument works even without assuming that the comparison maps are compatible with the boundary maps in localization sequences in general: in the relevant cases the cohomology groups involved are so simple that this property can be checked by hand.

$$
\begin{array}{ccc}
\downarrow & \cdots & \downarrow \\
\widetilde{\mathrm{KO}}^{2q-1,q}(S^{2d,d}) & \twoheadrightarrow & \widetilde{\mathrm{KO}}^{2q-1}(S^{2d}) \\
\downarrow & & \downarrow \\
\mathrm{KO}^{2q-1,q}(U_k) & \overset{k_h^{2q-1,q}}{\dashrightarrow} & \mathrm{KO}^{2q-1}(U_k) \\
\downarrow & & \downarrow \\
\mathrm{KO}^{2q-1,q}(U_{k+1}) & \twoheadrightarrow & \mathrm{KO}^{2q-1}(U_{k+1}) \\
\downarrow & & \downarrow \\
\widetilde{\mathrm{KO}}^{2q,q}(S^{2d,d}) & \xrightarrow{\cong} & \widetilde{\mathrm{KO}}^{2q}(S^{2d}) \\
\downarrow & & \downarrow \\
\mathrm{KO}^{2q,q}(U_k) & \overset{k_h^{2q,q}}{\dashrightarrow} & \mathrm{KO}^{2q}(U_k) \\
\downarrow & & \downarrow \\
\mathrm{KO}^{2q,q}(U_{k+1}) & \xrightarrow{\cong} & \mathrm{KO}^{2q}(U_{k+1}) \\
\downarrow & & \downarrow \\
\widetilde{\mathrm{KO}}^{2q+1,q}(S^{2d,d}) & \xrightarrow{\cong} & \widetilde{\mathrm{KO}}^{2q+1}(S^{2d}) \\
\downarrow & & \downarrow \\
\mathrm{KO}^{2q+1,q}(U_k) & \overset{k_h^{2q+1,q}}{\dashrightarrow} & \mathrm{KO}^{2q+1}(U_k) \\
\downarrow & & \downarrow \\
\mathrm{KO}^{2q+1,q}(U_{k+1}) & \xrightarrow{\cong} & \mathrm{KO}^{2q+1}(U_{k+1}) \\
\downarrow & & \downarrow \\
\widetilde{\mathrm{KO}}^{2q+2,q}(S^{2d,d}) & \rightarrowtail & \widetilde{\mathrm{KO}}^{2q+2}(S^{2d}) \\
\downarrow & & \downarrow \\
\mathrm{KO}^{2q+2,q}(U_k) & \overset{k_h^{2q+2,q}}{\dashrightarrow} & \mathrm{KO}^{2q+2}(U_k) \\
\downarrow & & \downarrow \\
\mathrm{KO}^{2q+2,q}(U_{k+1}) & \rightarrowtail & \mathrm{KO}^{2q+2}(U_{k+1}) \\
\downarrow & \cdots & \downarrow
\end{array}
$$

Figure 1: The inductive step.

2 The Atiyah-Hirzebruch spectral sequence for cellular varieties

We now prepare the ground for the discussion of the KO-theory of some examples of cellular spaces in the next section. Our main tool will be the Atiyah-Hirzebruch spectral sequence described in Section I.2e. For cellular varieties, or more generally for CW complexes with only even-dimensional cells, this spectral sequence becomes simple enough to make some general deductions.

So let X be a CW complex with cells only in even dimensions. Then the cohomology of X is free on generators given by the cells and concentrated in even degrees. The Atiyah-Hirzebruch spectral sequence for the K-theory of such a space collapses immediately: the entries of all columns and rows of odd degrees are zero, so there cannot be any non-zero differentials. We arrive at the following well-known lemma:

2.1 Lemma. *Let X be a CW complex with cells only in even dimensions. Then $\mathrm{K}^0(X)$ is a free abelian group of rank equal to the number of cells of X. The K-groups of odd degrees, on the other hand, vanish.*

After this preliminary observation, we turn to KO-theory. The Atiyah-Hirzebruch spectral sequence for the KO-theory of a CW complex with cells only in even dimensions does not necessarily collapse, but one can still make some general deductions. We summarize some lemmas of Hoggar and Kono and Hara.

2.2 Lemma. [Hog69, 2.1 and 2.2] *Let X be a CW complex with only even-dimensional cells. Then:*

- *The ranks of the free parts of $\mathrm{KO}^0 X$ and $\mathrm{KO}^4 X$ are equal to the number t_0 of cells of X of dimension a multiple of 4.*
- *The ranks of the free parts of $\mathrm{KO}^2 X$ and $\mathrm{KO}^6 X$ are equal to the number t_1 of cells of X of dimension 2 modulo 4.*
- *The groups of odd degrees are two-torsion, i. e. $\mathrm{KO}^{2i-1} X = (\mathbb{Z}/2)^{s_i}$ for some s_i.*
- *$\mathrm{KO}^{2i} X$ is isomorphic to the direct sum of its free part and $\mathrm{KO}^{2i+1} X$.*

Moreover, multiplication by $\eta \in \mathrm{KO}^{-1}(\text{point})$ induces an isomorphism

$$\eta\colon (\mathrm{KO}^{2i}/\,\mathrm{K})(X) \xrightarrow{\cong} \mathrm{KO}^{2i-1}(X)$$

The bullet points are summarized by Table 2 in Section 3a.

Proof. The first two statements can be seen directly from the Atiyah-Hirzebruch spectral sequence for KO-theory (e. g. after tensoring with $\mathbb{Q}$). The remaining statements may then be deduced from the above description of the K-groups and the Bott sequence (I.20). □

The free part of KO^* is thus very simple. In good cases, the spectral sequence also provides a nice description of the 2-torsion. To see this, note that $\mathrm{Sq}^2\,\mathrm{Sq}^2 = \mathrm{Sq}^3\,\mathrm{Sq}^1$ must vanish when the cohomology of X with $\mathbb{Z}/2$-coefficients is concentrated in even degrees. So we can view $(H^*(X;\mathbb{Z}/2), \mathrm{Sq}^2)$ as a differential graded algebra over $\mathbb{Z}/2$. To lighten notation, we will write

$$H^*(X, \mathrm{Sq}^2) := H^*(H^*(X;\mathbb{Z}/2), \mathrm{Sq}^2)$$

for the cohomology of this algebra. We keep the same grading as before, so that it is concentrated in even degrees. The row $q \equiv -1$ on the E_3-page is given by $H^*(X, \mathrm{Sq}^2) \cdot \eta$, where η is the generator of $\mathrm{KO}^{-1}(\text{point})$. Since it is the only row that contributes to KO^* in odd degrees, we arrive at the following lemma, which will be central to our computations.

2.3 Lemma. *Let X be as above. If the Atiyah-Hirzebruch spectral sequence of $\mathrm{KO}^*(X)$ degenerates on the E_3-page then*

$$\mathrm{KO}^{2i-1}(X) \cong \bigoplus_k H^{2i+8k}(X, \mathrm{Sq}^2)$$

Now suppose that $\mathcal{E}$ is a complex vector bundle over X. Then the twisted KO-groups $\mathrm{KO}^*(X;\mathcal{E})$ are computed by the Atiyah-Hirzebruch spectral sequence of the Thom space $\mathrm{Thom}\,\mathcal{E}$. Recall that the reduced cohomology of this space is isomorphic to the cohomology of X up to a shift in degrees. By Lemma I.2.11, the Steenrod square on $H^*(\mathrm{Thom}\,\mathcal{E};\mathbb{Z}/2)$ is given by $\mathrm{Sq}^2 + c_1(\mathcal{E})$.

If, as before, X has cells only in even dimensions, then $\mathrm{Sq}^2 + c_1$ may again be viewed as a differential on $H^*(X;\mathbb{Z}/2)$ for any c_1 in $H^2(X;\mathbb{Z}/2)$. Extending our previous notation, we denote cohomology with respect to this differential by

$$H^*(X, \mathrm{Sq}^2 + c_1) := H^*(H^*(X;\mathbb{Z}/2), \mathrm{Sq}^2 + c_1) \tag{1}$$

Lemma 2.3 has the following corollary.

2.4 Corollary. *Let X and $\mathcal{E}$ be as above. If the Atiyah-Hirzebruch spectral sequence of $\widetilde{\mathrm{KO}}^*(\mathrm{Thom}\,\mathcal{E})$ degenerates on the E_3-page, then*

$$\mathrm{KO}^{2i-1}(X;\mathcal{E}) \cong \bigoplus_k H^{2i+8k}(X, \mathrm{Sq}^2 + c_1\mathcal{E})$$

In all the examples we consider below, the spectral sequence does indeed degenerate at this stage. However, showing that it does can be tricky. One step in the right direction is the following observation of Kono and Hara [KH91, Proposition 1].

2.5 Lemma. *Let X be as above. If the differentials d_3, d_4, …, d_{r-1} are trivial and d_r is* *non-trivial, then* $r \equiv 2 \mod 8$*. In other words, the first non-trivial differential after* d_2 *can only appear on a page* E_r *with page number* $r \equiv 2 \mod 8$*.*

Such a differential is non-zero only on rows $q \equiv 0$ *and* $q \equiv -1$ *mod* 8*. If it is non-zero on some* x *in row* $q \equiv 0$*, then it is also non-zero on* ηx *in row* $q \equiv -1$*. Conversely, if it is non-zero on some* y *in row* $q \equiv -1$*, there exists some* x *in row* $q \equiv 0$ *such that* $y = x\eta$ *and* d_r *is non-zero on* x*.*

Proof. We see from the spectral sequence of a point that $d_r\eta = 0$ for all differentials. Thus, multiplication by η gives a map of bidegree $(0, -1)$ on the spectral sequence that commutes with the differentials. On the E_2-page this map is mod-2 reduction from row $q \equiv 0$ to row $q \equiv -1$ and the identity between rows $q \equiv -1$ and $q \equiv -2$. It follows that on the E_3-page multiplication by η induces a surjection from row $q \equiv 0$ to row $q \equiv -1$ and an injection of row $q \equiv -1$ into row $q \equiv -2$. This implies all statements above. □

We derive a corollary that we will use to deduce that the spectral sequence collapses for certain Thom spaces:

2.6 Corollary. *Suppose we have a continuous map* $p\colon X \to T$ *of CW complexes with only even-dimensional cells. Suppose further that the Atiyah-Hirzebruch spectral sequence for* $\mathrm{KO}^*(X)$ *collapses on the* E_3*-page, and that* p^* *induces an injection in row* $q \equiv -1$*:*

$$p^*\colon H^*(T, \mathrm{Sq}^2) \hookrightarrow H^*(X, \mathrm{Sq}^2)$$

Then the spectral sequence for $\mathrm{KO}^*(T)$ *also collapses at this stage.*

Proof. Write d_r for the first non-trivial higher differential, so $r \equiv 2 \mod 8$. Then, for any element x in row $q \equiv 0$, we have $p^*(d_r x) = d_r p^*(x) = 0$ since the spectral sequence for X collapses. From our assumption on p^* we can deduce that $d_r x = 0$. By the preceding lemma, this is all we need to show. □

3 Examples

We now turn to the study of projective homogeneous varieties, that is, varieties of the form G/P for some complex simple linear algebraic group G with a parabolic subgroup P. Any such variety has a cell decomposition [BGG73, Proposition 5.1], so that our comparison theorem applies. As far as we are only interested in the topology of G/P, we may alternatively view it as a homogeneous space for the compact real Lie group G^c corresponding to G:

3.1 Proposition. *Let P be a parabolic subgroup of a simple complex algebraic group G. Then we have a diffeomorphism*

$$G\big/P \cong G^c\big/K$$

where K is a compact subgroup of maximal rank in a maximal compact subgroup G^c of G. More precisely, K is a maximal compact subgroup of a Levi subgroup of P.

Proof. The Iwasawa decomposition for G viewed as a real Lie group implies that we have a diffeomorphism $G \cong G^c \cdot P$ [GOV94, Ch. 6, Prop. 1.7], inducing a diffeomorphism of quotients as claimed for $K = G^c \cap P$. Since $G^c \hookrightarrow G$ is a homotopy equivalence, so is the inclusion $G^c \cap P \hookrightarrow P$. On the other hand, if L is a Levi subgroup of P then $P = U \rtimes L$, where U is unipotent and hence contractible. So the inclusion $L \hookrightarrow P$ is also a homotopy equivalence. It follows that any maximal compact subgroup L^c of L is also maximal compact in P, and conversely that any maximal compact subgroup of P will be contained as a maximal compact subgroup in some Levi subgroup of P. We may therefore assume that $K \subset L^c \subset L \subset P$ and conclude that $K \hookrightarrow L^c$ is a homotopy equivalence. Since both groups are compact, it follows that in fact $K \cong L^c$. □

The KO-theory of homogeneous varieties has been studied intensively. In particular, the papers [KH91] and [KH92] of Kono and Hara provide complete computations of the (untwisted) KO-theory of all compact irreducible hermitian symmetric spaces, which we list in Table 1. For the convenience of the reader, we indicate how each of these arises as a quotient of a simple complex algebraic group G by a parabolic subgroup P, describing the latter in terms of marked nodes on the Dynkin diagram of G as in [FH91, § 23.3]. The last column gives an alternative description of each space as a quotient of a compact real Lie group.

On the following pages, we will run through this list of examples and, in each case, extend Kono and Hara's computations to include KO-groups twisted by a line bundle. Since each of these spaces is a "Grassmannian" in the sense that the parabolic subgroup P in G is maximal, its Picard group is free on a single generator. Thus, there is exactly one non-trivial twist that we need to consider. In most cases, we — reassuringly — recover results for Witt groups that are already known. In a few other cases, we consider our results as new.

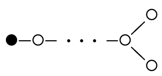
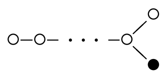
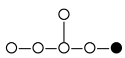
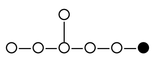
	G/P	G	Diagram of P	G^c/K
Grassmannians (AIII)	$\mathrm{Gr}_{m,n}$	SL_{m+n}	○– ··· –○–●–○– ··· –○ (1 … n … $n+m-1$)	$\frac{\mathrm{U}(m+n)}{\mathrm{U}(m)\times\mathrm{U}(n)}$
Maximal symplectic Grassmannians (CI)	X_n	Sp_{2n}	○–○– ··· –○–○=<=●	$\mathrm{Sp}(n)/\mathrm{U}(n)$
Projective quadrics of dimension $n\geq 3$ (BDI)	Q^n	SO_{n+2}	●–○– ··· –○=>=○ (n odd)  (n even)	$\frac{\mathrm{SO}(n+2)}{\mathrm{SO}(n)\times\mathrm{SO}(2)}$
Spinor varieties (DIII)	S_n	SO_{2n}		$\mathrm{SO}(2n)/\mathrm{U}(n)$
Exceptional hermitian symmetric spaces:	EIII	E_6		$\frac{E_6^c}{\mathrm{Spin}(10)\cdot S^1}$ $(\mathrm{Spin}(10)\cap S^1=\mathbb{Z}/4)$
	EVII	E_7		$\frac{E_7^c}{E_6^c\cdot S^1}$ $(E_6^c\cap S^1=\mathbb{Z}/3)$

Table 1: List of irreducible compact hermitian symmetric spaces. The symbols AIII, CI, ... refer to E. Cartan's classification. In the description of $\mathrm{Gr}_{m,n}$ we use $\mathrm{U}(m+n)$ instead of $G^c=\mathrm{SU}(m+n)$.

The untwisted KO-theory of complete flag varieties is also known in all three classical cases thanks to Kishimoto, Kono and Ohsita. We do not reproduce their result here but instead refer the reader directly to [KKO04]. By a recent result of Calmès and Fasel, all Witt groups with non-trivial twists vanish for these varieties [CF11].

3a Notation

Topologically, a cellular variety is a CW complex with cells only in even (real) dimensions. For such a CW complex X the KO-groups can be written in the form displayed in Table 2 below. This was shown in Section 2 in the case when the twist $\mathcal{L}$ is trivial, and the general case follows: if X is a CW complex with only even-dimensional cells, so is the Thom space of any complex vector bundle over X [MS74, Lemma 18.1].

In the following examples, results on KO^* will be displayed by listing the values of the t_i and s_i. Since the t_i are just given by counting cells, and since the numbers of odd- and even-dimensional cells of a Thom space $\mathrm{Thom}_X\mathcal{E}$ only depend on X and the rank of $\mathcal{E}$, the t_i are in fact independent of $\mathcal{L}$. The s_i, on the other hand, certainly will depend on the twist, and we will sometimes acknowledge this by writing $s_i(\mathcal{L})$.

$\mathrm{KO}^6(X;\mathcal{L})$	$= \mathbb{Z}^{t_1} \oplus (\mathbb{Z}/2)^{s_0}$	$=$	$\mathrm{GW}^3(X;\mathcal{L})$
$\mathrm{KO}^7(X;\mathcal{L})$	$= (\mathbb{Z}/2)^{s_0}$	$=$	$\mathrm{W}^0(X;\mathcal{L})$
$\mathrm{KO}^0(X;\mathcal{L})$	$= \mathbb{Z}^{t_0} \oplus (\mathbb{Z}/2)^{s_1}$	$=$	$\mathrm{GW}^0(X;\mathcal{L})$
$\mathrm{KO}^1(X;\mathcal{L})$	$= (\mathbb{Z}/2)^{s_1}$	$=$	$\mathrm{W}^1(X;\mathcal{L})$
$\mathrm{KO}^2(X;\mathcal{L})$	$= \mathbb{Z}^{t_1} \oplus (\mathbb{Z}/2)^{s_2}$	$=$	$\mathrm{GW}^1(X;\mathcal{L})$
$\mathrm{KO}^3(X;\mathcal{L})$	$= (\mathbb{Z}/2)^{s_2}$	$=$	$\mathrm{W}^2(X;\mathcal{L})$
$\mathrm{KO}^4(X;\mathcal{L})$	$= \mathbb{Z}^{t_0} \oplus (\mathbb{Z}/2)^{s_3}$	$=$	$\mathrm{GW}^2(X;\mathcal{L})$
$\mathrm{KO}^5(X;\mathcal{L})$	$= (\mathbb{Z}/2)^{s_3}$	$=$	$\mathrm{W}^3(X;\mathcal{L})$

Table 2: Notational conventions in the examples. Only the s_i depend on $\mathcal{L}$.

3b Projective spaces

Complex projective spaces are perhaps the simplest examples for which Theorem 1.1 asserts something non-trivial, so we describe the results here separately before turning to complex Grassmannians in general. The computations of the Witt groups of projective spaces were landmark events in the history of the theory. In 1980, Arason was able to show that the Witt group $\mathrm{W}^0(\mathbb{P}^n)$ of $\mathbb{P}^n$ over a field k agrees with the Witt group of k [Ara80]. The shifted Witt groups of projective spaces, and more generally of arbitrary projective bundles, were first computed by Walter in [Wal03b]. Quite recently, Nenashev deduced the same results via different methods [Nen09].

In the topological world, complete computations of $\mathrm{KO}^i(\mathbb{CP}^n)$ were first published in a 1967 paper by Fujii [Fuj67]. It is not difficult to deduce the values of the twisted groups $\mathrm{KO}^i(\mathbb{CP}^n;\mathcal{O}(1))$ from these: the space $\mathrm{Thom}(\mathcal{O}_{\mathbb{CP}^n}(1))$ can be identified with $\mathbb{CP}^{n+1}$, so

$$\begin{aligned}\mathrm{KO}^i(\mathbb{CP}^n;\mathcal{O}(1)) &= \widetilde{\mathrm{KO}}^{i+2}(\mathrm{Thom}(\mathcal{O}(1))) \\ &= \widetilde{\mathrm{KO}}^{i+2}(\mathbb{CP}^{n+1})\end{aligned}$$

Alternatively, we could do all required computations directly following the methods outlined in Section 2. The result, in any case, is displayed in Table 3, coinciding with the known results for the (Grothendieck-)Witt groups.

$\mathrm{KO}^*(\mathbb{CP}^n;\mathcal{L})$			$\mathcal{L} \equiv \mathcal{O}$				$\mathcal{L} \equiv \mathcal{O}(1)$			
	t_0	t_1	s_0	s_1	s_2	s_3	s_0	s_1	s_2	s_3
$n \equiv 0 \mod 4$	$(n/2)+1$	$n/2$	1	0	0	0	1	0	0	0
$n \equiv 1$	$(n+1)/2$	$(n+1)/2$	1	1	0	0	0	0	0	0
$n \equiv 2$	$(n/2)+1$	$n/2$	1	0	0	0	0	0	1	0
$n \equiv 3$	$(n+1)/2$	$(n+1)/2$	1	0	0	1	0	0	0	0

Table 3: KO-groups of projective spaces

3c Grassmannians

We now consider the Grassmannians $\mathrm{Gr}_{m,n}$ of complex m-planes in $\mathbb{C}^{m+n}$. Again both the Witt groups and the untwisted KO-groups are already known: the latter by Kono and Hara [KH91], the former by the work of Balmer and Calmès [BC08]. A detailed comparison of the two sets of results in the untwisted case has been carried out by Yagita [Yag09]. We provide here an alternative topological computation of the twisted groups.

Balmer and Calmès state their result by describing an additive basis of the total Witt group of $\mathrm{Gr}_{m,n}$ in terms of certain "even Young diagrams". This is probably the most elegant approach, but needs some space to explain. We will stick instead to the tabular exposition used in the other examples. Let $\mathcal{O}(1)$ be a generator of $\mathrm{Pic}(\mathrm{Gr}_{m,n})$, say the dual of the determinant line bundle of the universal m-bundle over $\mathrm{Gr}_{m,n}$. The result is displayed in Table 4.

$\mathrm{KO}^*(\mathrm{Gr}_{m,n};\mathcal{L})$			$\mathcal{L}\equiv\mathcal{O}$				$\mathcal{L}\equiv\mathcal{O}(1)$			
	t_0	t_1	s_0	s_1	s_2	s_3	s_0	s_1	s_2	s_3
m and n odd s.t. $m\equiv n$	$\frac{a}{2}$	$\frac{a}{2}$	b	b	0	0	0	0	0	0
m and n odd s.t. $m\not\equiv n$	$\frac{a}{2}$	$\frac{a}{2}$	b	0	0	b	0	0	0	0
$\begin{cases} m\equiv n\equiv 0 \\ m\equiv 0 \text{ and } n \text{ odd} \\ n\equiv 0 \text{ and } m \text{ odd}\end{cases}$	$\frac{a+b}{2}$	$\frac{a-b}{2}$	b	0	0	0	b	0	0	0
$\begin{cases} m\equiv n\equiv 2 \\ m\equiv 2 \text{ and } n \text{ odd} \\ n\equiv 2 \text{ and } m \text{ odd}\end{cases}$	$\frac{a+b}{2}$	$\frac{a-b}{2}$	b	0	0	0	0	0	b	0
$m\equiv 0$ and $n\equiv 2$	$\frac{a+b}{2}$	$\frac{a-b}{2}$	b	0	0	0	b_1	0	b_2	0
$m\equiv 2$ and $n\equiv 0$	$\frac{a+b}{2}$	$\frac{a-b}{2}$	b	0	0	0	b_2	0	b_1	0

All equivalences ($\equiv$) are modulo 4. For the values of a and $b=b_1+b_2$, put $k:=\lfloor m/2\rfloor$ and $l:=\lfloor n/2\rfloor$. Then

$$a:=\binom{m+n}{m}\qquad b:=\binom{k+l}{k}\qquad b_1:=\binom{k+l-1}{k}\qquad b_2:=\binom{k+l-1}{k-1}$$

Table 4: KO-groups of Grassmannians

Our computation is based on the following geometric observation. Let $\mathcal{U}_{m,n}$ and $\mathcal{U}_{m,n}^{\perp}$ be the universal m-bundle and the orthogonal n-bundle over $\mathrm{Gr}_{m,n}$, so that $\mathcal{U}\oplus\mathcal{U}^{\perp}=$

$\mathcal{O}^{\oplus(m+n)}$. We have various natural inclusions between the Grassmannians of different dimensions, of which we fix two:

$\mathrm{Gr}_{m,n-1} \hookrightarrow \mathrm{Gr}_{m,n}$ via the inclusion of the first $m+n-1$ coordinates into $\mathbb{C}^{m+n}$

$\mathrm{Gr}_{m-1,n} \hookrightarrow \mathrm{Gr}_{m,n}$ by sending an $(m-1)$-plane Λ to the m-plane $\Lambda \oplus \langle e_{m+n} \rangle$, where $e_1, e_2, \dots, e_{m+n}$ are the canonical basis vectors of $\mathbb{C}^{m+n}$

3.2 Lemma. *The normal bundle of* $\mathrm{Gr}_{m,n-1}$ *in* $\mathrm{Gr}_{m,n}$ *is the dual* $\mathcal{U}^{\vee}_{m,n-1}$ *of the universal* m*-bundle. Similarly, the normal bundle of* $\mathrm{Gr}_{m-1,n}$ *in* $\mathrm{Gr}_{m,n}$ *is given by* $\mathcal{U}^{\perp}_{m-1,n}$*. In both cases, the embeddings of the subspaces extend to embeddings of their normal bundles, such that one subspace is the closed complement of the normal bundle of the other.*

This gives us two cofibration sequences of pointed spaces:

$$\mathrm{Gr}_{m-1,n+} \overset{i}{\hookrightarrow} \mathrm{Gr}_{m,n+} \overset{p}{\twoheadrightarrow} \mathrm{Thom}(\mathcal{U}^{\vee}_{m,n-1}) \tag{2}$$

$$\mathrm{Gr}_{m,n-1+} \overset{i}{\hookrightarrow} \mathrm{Gr}_{m,n+} \overset{p}{\twoheadrightarrow} \mathrm{Thom}(\mathcal{U}^{\perp}_{m-1,n}) \tag{3}$$

These sequences are the key to relating the untwisted KO-groups to the twisted ones. Following the notation in [KH91], we write $A_{m,n}$ for the cohomology of $\mathrm{Gr}_{m,n}$ with $\mathbb{Z}/2$-coefficients, denoting the Chern classes of $\mathcal{U}$ and $\mathcal{U}^{\perp}$ by a_i and b_i, respectively, and the total Chern classes $1 + a_1 + \cdots + a_m$ and $1 + b_1 + \cdots + b_n$ by a and b:

$$A_{m,n} = \frac{\mathbb{Z}/2\,[a_1, a_2, \dots, a_m, b_1, b_2, \dots b_n]}{a \cdot b = 1}$$

We write d for the differential given by the second Steenrod square Sq^2, and d' for $\mathrm{Sq}^2 + a_1$. To describe the cohomology of $A_{m,n}$ with respect to these differentials, it is convenient to introduce the algebra

$$B_{k,l} = \frac{\mathbb{Z}/2\,[a_2^2, a_4^2, \dots, a_{2k}^2, b_2^2, b_4^2, \dots, b_{2l}^2]}{(1 + a_2^2 + \cdots + a_{2k}^2)(1 + b_2^2 + \cdots + b_{2l}^2) = 1}$$

Note that this subquotient of $A_{2k,2l}$ is isomorphic to $A_{k,l}$ up to a "dilatation" in grading. Proposition 2 in [KH91] tells us that

$$H^*(A_{m,n}, d) = \begin{cases} B_{k,l} & \text{if } (m,n) = (2k, 2l),\ (2k+1, 2l) \text{ or } (2k, 2l+1) \\ B_{k,l} \oplus B_{k,l} \cdot a_m b_{n-1} & \text{if } (m,n) = (2k+1, 2l+1) \end{cases}$$

Here, the algebra structure in the case where both m and n are odd is determined by $(a_m b_{n-1})^2 = 0$.

3.3 Lemma. *The cohomology of $A_{m,n}$ with respect to the twisted differential d' is as follows:*

$$H^*(A_{m,n}, d') = \begin{cases} B_{k,l-1} \cdot a_m \oplus B_{k-1,l} \cdot b_n & \text{if } (m,n) = (2k, 2l) \\ B_{k,l} \cdot a_m & \text{if } (m,n) = (2k, 2l+1) \\ B_{k,l} \cdot b_n & \text{if } (m,n) = (2k+1, 2l) \\ 0 & \text{if } (m,n) = (2k+1, 2l+1) \end{cases}$$

Proof. Let us shift the dimensions in the cofibration sequences (2) and (3) in such a way that we have the Thom spaces of $\mathcal{U}^\vee_{m,n}$ and $\mathcal{U}^\perp_{m,n}$ on the right. Since the cohomologies of the spaces involved are concentrated in even degrees, the associated long exact sequence of cohomology groups falls apart into short exact sequences. Reassembling these, we obtain two short exact sequences of differential $(A_{m,n+1}, d)$- and $(A_{m+1,n}, d)$-modules, respectively:

$$0 \to (A_{m,n}, d') \cdot \theta^\vee \xrightarrow{p^*} (A_{m,n+1}, d) \xrightarrow{i^*} (A_{m-1,n+1}, d) \to 0 \tag{4}$$

$$0 \to (A_{m,n}, d') \cdot \theta^\perp \xrightarrow{p^*} (A_{m+1,n}, d) \xrightarrow{i^*} (A_{m+1,n-1}, d) \to 0 \tag{5}$$

Here, $\theta^\vee$ and $\theta^\perp$ are the respective Thom classes of $\mathcal{U}^\vee_{m,n}$ and $\mathcal{U}^\perp_{m,n}$. The map i^* in the first row is the obvious quotient map annihilating a_m. Its kernel, the image of $A_{m,n}$ under multiplication by a_m, is generated as an $A_{m,n+1}$-module by its unique element in degree $2m$, and thus we must have $p^*(\theta^\vee) = a_m$. Likewise, in the second row we have $p^*(\theta^\perp) = b_n$.

The lemma can be deduced from here case by case. For example, when both m and n are even, i^* maps $H^*(A_{m,n+1}, d) = B_{k,l}$ to the first summand of $H^*(A_{m-1,n+1}, d) = B_{k-1,l} \oplus B_{k-1,l} \cdot a_{m-1} b_n$ by annihilating a_m^2. We know by comparison with the short exact sequences for the $A_{m,n}$ that the kernel of this map is $B_{k,l-1}$ mapping to $B_{k,l}$ under multiplication by a_m^2. Thus, we obtain a short exact sequence

$$0 \to B_{k-1,l} \cdot a_{m-1} b_n \xrightarrow{\partial} H^*(A_{m,n}, d') \cdot \theta^\vee \xrightarrow{p^*} B_{k,l-1} \cdot a_m^2 \to 0 \tag{6}$$

For the Steenrod square Sq^2 of the top Chern class a_m of $\mathcal{U}$, we have $\mathrm{Sq}^2(a_m) = a_1 a_m$. This can be checked, for example, by expressing a_m as the product of the Chern roots of $\mathcal{U}$. Consequently, $d'(a_m) = 0$. Together with the fact that $H^*(A_{m,n}, d')$ is a module over $H^*(A_{m,n+1}, d)$, this shows that we can define a splitting of p^* by sending a_m^2 to $a_m \theta^\vee$. Thus, $H^*(A_{m,n}, d')$ contains $B_{k,l-1} \cdot a_m$ as a direct summand. If instead of working with sequence (4) we work with sequence (5), we see that $H^*(A_{m,n}, d')$ also contains a direct summand $B_{k-1,l} \cdot b_n$. These two summands intersect trivially, and a dimension count shows that together they encompass all of $H^*(A_{m,n}, d')$. Alternatively, one may check explicitly that the boundary map ∂ above sends $a_{m-1} b_n$ to $b_n \theta$. The other cases are simpler. □

3.4 Lemma. *The Atiyah-Hirzebruch spectral sequence for* $\widetilde{\mathrm{KO}}^*(\mathrm{Thom}\,\mathcal{U}^\vee_{m,n})$ *collapses at the* E_3*-page.*

Proof. By Proposition 4 of [KH91] we know that the spectral sequence for $\mathrm{KO}^*(\mathrm{Gr}_{m,n})$ collapses as this stage, for any m and n. Now, if both m and n are even, we have

$$(B_{k,l-1} \cdot a_m \oplus B_{k-1,l} \cdot b_n) \cdot \theta$$

in the $(-1)^{\mathrm{st}}$ row of the E_3-pages of the spectral sequences for $\mathrm{Thom}\,\mathcal{U}^\vee$ and $\mathrm{Thom}\,\mathcal{U}^\perp$, where $\theta = \theta^\vee$ or $\theta^\perp$, respectively. In the case of $\mathcal{U}^\vee$ we see from (6) that p^* maps the second summand injectively to the E_3-page of the spectral sequence for $\mathrm{KO}^*(\mathrm{Gr}_{m,n+1})$. Similarly, in the case of $\mathcal{U}^\perp$, the first summand is mapped injectively to the E_3-page of $\mathrm{KO}^*(\mathrm{Gr}_{m+1,n})$. Since the spectral sequences for $\mathrm{Thom}\,\mathcal{U}^\vee$ and $\mathrm{Thom}\,\mathcal{U}^\perp$ can be identified via Corollary I.2.13, we can argue as in Corollary 2.6 to see that they must collapse at this stage. Again, the cases when at least one of m, n is odd are similar but simpler. □

We may now apply Corollary 2.4. The entries of Table 4 that do not appear in [KH91], i. e. those of the last four columns, follow from Lemma 3.3 by noting that $B_{k,l}$ is concentrated in degrees $8i$ and of dimension $\dim B_{k,l} = \dim A_{k,l} = \binom{k+l}{k}$.

3d Maximal symplectic Grassmannians

The Grassmannian of isotropic n-planes in $\mathbb{C}^{2n}$ with respect to a non-degenerate skew-symmetric bilinear form is given by $X_n = \mathrm{Sp}(n)/U(n)$. The universal bundle $\mathcal{U}$ over the usual Grassmannian $\mathrm{Gr}(n, 2n)$ restricts to the universal bundle over X_n, and so does the orthogonal complement bundle $\mathcal{U}^\perp$. We will continue to denote these restrictions by the same letters. Thus, $\mathcal{U} \oplus \mathcal{U}^\perp \cong \mathbb{C}^{2n}$ over X_n, and the fibres of $\mathcal{U}$ are orthogonal to those of $\mathcal{U}^\perp$ with respect to the standard hermitian metric on $\mathbb{C}^{2n}$. The determinant line bundles of $\mathcal{U}$ and $\mathcal{U}^\perp$ give dual generators $\mathcal{O}(1)$ and $\mathcal{O}(-1)$ of the Picard group of X_n.

3.5 Theorem. *The additive structure of* $\mathrm{KO}^*(X_n; \mathcal{L})$ *is as follows:*

	t_0	t_1	$s_i(\mathcal{O})$	$s_i(\mathcal{O}(1))$
n even	2^{n-1}	2^{n-1}	$\rho(\frac{n}{2}, i)$	$\rho(\frac{n}{2}, i-n)$
n odd	2^{n-1}	2^{n-1}	$\rho(\frac{n+1}{2}, i)$	0

Here, for any $i \in \mathbb{Z}/4$ *we write* $\rho(n,i)$ *for the dimension of the* i*-graded piece of a* $\mathbb{Z}/4$*-graded exterior algebra* $\Lambda_{\mathbb{Z}/2}(g_1, g_2, \ldots, g_n)$ *on* n *homogeneous generators* g_1*,* g_2*, …,* g_n *of degree* 1*, i. e.*

$$\rho(n,i) = \sum_{\substack{d \equiv i \\ \mathrm{mod}\ 4}} \binom{n}{d}$$

A table of the values of $\rho(n,i)$ *can be found in* [KH92, Proposition 4.1].

It turns out to be convenient to work with the vector bundle $\mathcal{U}^\perp \oplus \mathcal{O}$ for the computation of the twisted groups $\mathrm{KO}^*(X_n; \mathcal{O}(1))$. Namely, we have the following analogue of Lemma 3.2.

3.6 Lemma. *There is an open embedding of the bundle $\mathcal{U}^\perp \oplus \mathcal{O}$ over the symplectic Grassmannian X_n into the symplectic Grassmannian X_{n+1} whose closed complement is again isomorphic to X_n.*

Proof. To fix notation, let e_1, e_2 be the first two canonical basis vectors of $\mathbb{C}^{2n+2}$, and embed $\mathbb{C}^{2n}$ into $\mathbb{C}^{2n+2}$ via the remaining coordinates. Assuming X_n is defined in terms of a skew-symmetric form Q_{2n}, define X_{n+1} with respect to the form

$$Q_{2n+2} := \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & Q_{2n} \end{pmatrix}$$

Then we have embeddings i_1 and i_2 of X_n into X_{n+1} sending an n-plane $\Lambda \subset \mathbb{C}^{2n}$ to $e_1 \oplus \Lambda$ or $e_2 \oplus \Lambda$ in $\mathbb{C}^{2n+2}$, respectively.

We extend i_1 to an embedding of $\mathcal{U}^\perp \oplus \mathcal{O}$ by sending an n-plane $\Lambda \in X_n$ together with a vector v in $\Lambda^\perp \subset \mathbb{C}^{2n}$ and a complex scalar z to the graph $\Gamma_{\Lambda,v,z} \subset \mathbb{C}^{2n+2}$ of the linear map

$$\begin{pmatrix} z & Q_{2n}(-,v) \\ v & 0 \end{pmatrix} : \langle e_1 \rangle \oplus \Lambda \to \langle e_2 \rangle \oplus \Lambda^\perp$$

To avoid confusion, we emphasize that v is orthogonal to Λ with respect to a *hermitian* metric on $\mathbb{C}^{2n}$. The value of $Q_{2n}(-,v)$, on the other hand, may well be non-zero on Λ. Consider the above embedding of $\mathcal{U}^\perp \oplus \mathcal{O}$ together with the embedding i_2:

$$\begin{array}{ccccc} \mathcal{U}^\perp \oplus \mathcal{O} & \hookrightarrow & X_{n+1} & \overset{i_2}{\hookleftarrow} & X_n \\ (\Lambda, v, z) & \mapsto & \Gamma_{\Lambda,v,z} & & \\ & & \langle e_2 \rangle \oplus \Lambda & \leftarrow\!\shortmid & \Lambda \end{array}$$

To see that the two embeddings are complementary, take an arbitrary $(n+1)$-plane W in X_{n+1}. If $e_2 \in W$ then we can consider a basis

$$e_2, \begin{pmatrix} a_1 \\ 0 \\ v_1 \end{pmatrix}, \dots, \begin{pmatrix} a_n \\ 0 \\ v_n \end{pmatrix}$$

of W, and the fact that Q_{2n+2} vanishes on W implies that all a_i are zero. Thus W can be identified with $i_2(\langle v_1, \dots, v_n \rangle)$.

If, on the other hand, e_2 is not contained in W then we must have a vector of the form ${}^t(1, z', v')$ in W, for some $z' \in \mathbb{C}$ and $v' \in \mathbb{C}^{2n}$. Extend this vector to a basis of W of the form

$$\begin{pmatrix} 1 \\ z' \\ v' \end{pmatrix}, \begin{pmatrix} 0 \\ b_1 \\ v_1 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ b_n \\ v_n \end{pmatrix}$$

and let $\Lambda := \langle v_1, \ldots, v_n \rangle$. The condition that Q_{2n+2} vanishes on W implies that Q vanishes on Λ and that $b_i = Q_{2n}(v_i, v')$ for each i. In particular, Λ is n-dimensional. Moreover, we can replace the first vector of our basis by a vector ${}^t(1, z, v)$ with $v \in \Lambda^\perp$, by subtracting appropriate multiples of the remaining basis vectors. Since Q vanishes on Λ we have $Q_{2n}(v_i, v') = Q_{2n}(v_i, v)$ and our new basis has the form

$$\begin{pmatrix} 1 \\ z \\ v \end{pmatrix}, \begin{pmatrix} 0 \\ Q(v_1,v) \\ v_1 \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ Q(v_n,v) \\ v_n \end{pmatrix}$$

This shows that $W = \Gamma_{\Lambda,v,z}$. □

3.7 Corollary. *We have a cofibration sequence*

$$X_{n+} \xrightarrow{i} X_{n+1+} \xrightarrow{p} \mathrm{Thom}_{X_n}(\mathcal{U}^\perp \oplus \mathcal{O})$$

The associated long exact cohomology sequence splits into a short exact sequence of $H^*(X_{n+1})$-modules since all cohomology here is concentrated in even degrees:

$$0 \to \widetilde{H}^*(\mathrm{Thom}_{X_n}(\mathcal{U}^\perp \oplus \mathcal{O})) \xrightarrow{p^*} H^*(X_{n+1}) \xrightarrow{i^*} H^*(X_n) \to 0 \tag{7}$$

3.8 Lemma. *Let c_i denote the i^{th} Chern classes of $\mathcal{U}$ on X_n. We have*

$$H^*(X_n, \mathrm{Sq}^2) = \begin{cases} \Lambda(a_1, a_5, a_9, \ldots, a_{4m-3}) & \text{if } n = 2m \\ \Lambda(a_1, a_5, a_9, \ldots, a_{4m-3}, a_{4m+1}) & \text{if } n = 2m+1 \end{cases}$$

$$H^*(X_n, \mathrm{Sq}^2 + c_1) = \begin{cases} \Lambda(a_1, a_5, \ldots, a_{4m-3}) \cdot c_{2m} & \text{if } n = 2m \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

for certain generators a_i of degree $2i$.

Proof. Consider the short exact sequence (7). The mod-2 cohomology of X_n is an exterior algebra on the Chern classes c_i of $\mathcal{U}$,

$$H^*(X_n; \mathbb{Z}/2) = \Lambda(c_1, c_2, \ldots, c_n)$$

and i^* is given by sending c_{n+1} to zero. Thus, p^* is the unique morphism of $H^*(X_{n+1}; \mathbb{Z}/2)$-modules that sends the Thom class θ of $\mathcal{U}^\perp \oplus \mathcal{O}$ to c_{n+1}.

This short exact sequence induces a long exact sequence of cohomology groups with respect to the Steenrod square Sq^2. The algebra $H^*(X_n, \mathrm{Sq}^2)$ was computed in [KH92, 2–2], with the result displayed above, so we already know two thirds of this sequence. Explicitly, we have $a_{4i+1} = c_{2i}c_{2i+1}$,[1] so i^* is the obvious surjection sending a_i to a_i (or to zero). Thus, the long exact sequence once again splits.

[1] In [KH92] the generators are written as $c_{2i}c'_{2i+1}$ with $c'_{2i+1} = c_{2i+1} + c_1c_{2i}$.

If $n = 2m$ we obtain a short exact sequence

$$0 \to H^*(X_{2m}, \mathrm{Sq}^2 + c_1) \cdot \theta \xrightarrow{p^*} \Lambda(a_1, \ldots, a_{4m-3}, a_{4m+1}) \xrightarrow{i^*} \Lambda(a_1, \ldots, a_{4m-3}) \to 0$$

We see that $H^*(X_{2m}, \mathrm{Sq}^2 + c_1) \cdot \theta$ is isomorphic to $\Lambda(a_1, \ldots, a_{4m-3}) \cdot a_{4m+1}$ as a module over $\Lambda(a_1, \ldots, a_{4m+1})$. It is thus generated by a single element, which is the unique element of degree $8m + 2$. Since $p^*(c_{2m}\theta) = a_{4m+1}$, the class of $c_{2m}\theta$ is the element we are looking for, and the result displayed above follows.

If, on the other hand, n is odd, then i^* is an isomorphism and $H^*(X_n, \mathrm{Sq}^2 + c_1)$ must be trivial. □

We see from the proof that p^* induces an injection of $H^*(X_n, \mathrm{Sq}^2 + c_1) \cdot \theta$ into $H^*(X_n, \mathrm{Sq}^2)$. Since we already know from [KH92, Theorem 2.1] that the Atiyah-Hirzebruch spectral sequence for $\mathrm{KO}^*(X_n)$ collapses, we can apply Corollary 2.6 to deduce that the spectral sequence for $\widetilde{\mathrm{KO}}^*(\mathrm{Thom}_{X_n}(\mathcal{U}^\perp \oplus \mathcal{O}))$ collapses at the E_3-page as well. This completes the proof of Theorem 3.5.

3e Quadrics

We next consider smooth complex quadrics Q^n in $\mathbb{P}^{n+1}$. As far as we are aware, the first complete results on (shifted) Witt groups of split quadrics were due to Walter: they are mentioned together with the results for projective bundles in [Wal03a] as the main applications of that paper. Unfortunately, they seem to have remained unpublished. Partial results are also included in Yagita's preprint [Yag04], see Corollary 8.3. More recently, Nenashev obtained almost complete results by considering the localization sequences arising from the inclusion of a linear subspace of maximal dimension [Nen09]. Calmès informs me that the geometric description of the boundary map given in [BC09] can be used to show that these localization sequences split in general, yielding a complete computation. The calculation described here is completely independent of these results.

For $n \geq 3$ the Picard group of Q^n is free on a single generator given by the restriction of the universal line bundle $\mathcal{O}(1)$ on $\mathbb{P}^{n+1}$. We will use the same notation $\mathcal{O}(1)$ for this restriction.

3.9 Theorem. *The KO-theory of a smooth complex quadric Q^n of dimension $n \geq 3$ is as described in Table 5.*

Untwisted KO-groups. Before turning to $\mathrm{KO}^*(Q^n; \mathcal{O}(1))$ we review the initial steps in the computation of the untwisted KO-groups. The integral cohomology of Q^n is well-known:

If n is even, write $n = 2m$. We have a class x in $H^2(Q^n)$ given by a hyperplane section, and two classes a and b in $H^n(Q^n)$ represented by linear subspaces of Q of maximal

$\mathrm{KO}^*(Q^n;\mathcal{L})$			$\mathcal{L}\equiv\mathcal{O}$				$\mathcal{L}\equiv\mathcal{O}(1)$			
	t_0	t_1	s_0	s_1	s_2	s_3	s_0	s_1	s_2	s_3
$n\equiv 0 \mod 8$	$(n/2)+2$	$n/2$	2	0	0	0	2	0	0	0
$n\equiv 1$	$(n+1)/2$	$(n+1)/2$	1	1	0	0	1	1	0	0
$n\equiv 2$	$(n/2)+1$	$(n/2)+1$	1	2	1	0	0	0	0	0
$n\equiv 3$	$(n+1)/2$	$(n+1)/2$	1	1	0	0	0	0	1	1
$n\equiv 4$	$(n/2)+2$	$n/2$	2	0	0	0	0	0	2	0
$n\equiv 5$	$(n+1)/2$	$(n+1)/2$	1	0	0	1	0	1	1	0
$n\equiv 6$	$(n/2)+1$	$(n/2)+1$	1	0	1	2	0	0	0	0
$n\equiv 7$	$(n+1)/2$	$(n+1)/2$	1	0	0	1	1	0	0	1

Table 5: KO-groups of projective quadrics ($n \geq 3$)

dimension. These three classes generate the cohomology multiplicatively, modulo the relations

$$x^m = a+b \qquad\qquad x^{m+1} = 2ax$$

$$ab = \begin{cases} 0 & \text{if } n\equiv 0 \\ ax^m & \text{if } n \equiv 2 \end{cases} \qquad\qquad a^2=b^2=\begin{cases} ax^m & \text{if } n\equiv 0 \mod 4 \\ 0 & \text{if } n\equiv 2 \mod 4 \end{cases}$$

Additive generators can thus be given as follows:

d	0	2	4	$\dots$	$n-2$	n	$n+2$	$n+4$	$\dots$	$2n$
$H^d(Q^n)$	1	x	x^2	$\dots$	x^{m-1}	a,b	ax	ax^2	$\dots$	ax^m

If n is odd, write $n=2m+1$. Then similarly multiplicative generators are given by the class of a hyperplane section x in $H^2(Q^n)$ and the class of a linear subspace a in $H^{n+1}(Q^n)$ modulo the relations $x^{m+1}=2a$ and $a^2=0$.

d	0	2	4	$\dots$	$n-1$	$n+1$	$n+3$	$n+5$	$\dots$	$2n$
$H^d(Q^n)$	1	x	x^2	$\dots$	x^m	a	ax	ax^2	$\dots$	ax^m

The action of the Steenrod square on $H^*(Q^n;\mathbb{Z}/2)$ is also well-known; see for example [Ish92, Theorem 1.4 and Corollary 1.5] or [EKM08, § 78]:

$$\mathrm{Sq}^2(x) = x^2$$

$$\mathrm{Sq}^2(a) = \begin{cases} ax & \text{if } n\equiv 0 \text{ or } 3 \mod 4 \\ 0 & \text{if } n \equiv 1 \text{ or } 2 \end{cases}$$

$$\mathrm{Sq}^2(b) = \mathrm{Sq}^2(a) \quad \text{(for even } n\text{)}$$

As before, we write $H^*(Q^n,\mathrm{Sq}^2)$ for the cohomology of $H^*(Q^n;\mathbb{Z}/2)$ with respect to the differential Sq^2.

3.10 Lemma. *Write $n = 2m$ or $n = 2m+1$ as above. The following table gives a complete list of the additive generators of $H^*(Q^n, \mathrm{Sq}^2)$.*

d	0	$\dots$	$n-1$	n	$n+1$	$\dots$	$2n$	
$H^d(Q^n, \mathrm{Sq}^2)$	1						ax^m	*if* $n \equiv 0 \mod 4$
	1				a			*if* $n \equiv 1$
	1			a, b			ab	*if* $n \equiv 2$
	1		x^m					*if* $n \equiv 3$

The results of Kono and Hara on $\mathrm{KO}^*(Q)$ follow from here provided there are no non-trivial higher differentials in the Atiyah-Hirzebruch spectral sequence. This is fairly clear in all cases except for the case $n \equiv 2 \mod 4$. In that case, the class $a + b = x^m$ can be pulled back from Q^{n+1}, and therefore all higher differentials must vanish on $a+b$. But one has to work harder to see that all higher differentials vanish on a (or b). Kono and Hara proceed by relating the KO-theory of Q^n to that of the spinor variety $S_{\frac{n}{2}+1}$ discussed in Section 3f.

Twisted KO-groups. We now compute $\mathrm{KO}^*(Q^n; \mathcal{O}(1))$.

Let $\theta \in H^2(\mathrm{Thom}_{Q^n}\mathcal{O}(1))$ be the Thom class of $\mathcal{O}(1)$, so that multiplication by θ maps the cohomology of Q^n isomorphically to the reduced cohomology of $\mathrm{Thom}_{Q^n}\mathcal{O}(1)$. The Steenrod square on $\widetilde{H}^*(\mathrm{Thom}_{Q^n}\mathcal{O}(1); \mathbb{Z}/2)$ is determined by Lemma I.2.11: for any $y \in H^*(Q^n; \mathbb{Z}/2)$ we have $\mathrm{Sq}^2(y \cdot \theta) = (\mathrm{Sq}^2 y + xy) \cdot \theta$. We thus arrive at

3.11 Lemma. *A complete list of the additive generators of $\widetilde{H}^*(\mathrm{Thom}_{Q^n}\mathcal{O}(1), \mathrm{Sq}^2)$ is provided by the following table:*

d	$\dots$	$n+1$	$n+2$	$n+3$	$\dots$	$2n+2$	
$\widetilde{H}^d(\dots)$			$a\theta, b\theta$				*if* $n \equiv 0 \mod 4$
		$x^m\theta$				$ax^m\theta$	*if* $n \equiv 1$
							if $n \equiv 2$
				$a\theta$		$ax^m\theta$	*if* $n \equiv 3$

We claim that all higher differentials in the Atiyah-Hirzebruch spectral sequence for $\widetilde{\mathrm{KO}}^*(\mathrm{Thom}_{Q^n}\mathcal{O}(1))$ vanish. For even n this is clear. But for $n = 8k+1$ the differential d_{8k+2} might a priori take $x^m\theta$ to $ax^m\theta$, and for $n = 8k+3$ the differential d_{8k+2} might take $a\theta$ to $ax^m\theta$.

We therefore need some geometric considerations. Namely, the space $\mathrm{Thom}_{Q^n}\mathcal{O}(1)$ can be identified with the projective cone over Q^n embedded in $\mathbb{P}^{n+2}$. This projective cone can be realized as the intersection of a smooth quadric $Q^{n+2} \subset \mathbb{P}^{n+3}$ with its projective tangent space at the vertex of the cone [Har92, p. 283]. Thus, we can consider the following inclusions:

$$Q^n \overset{j}{\hookrightarrow} \mathrm{Thom}_{Q^n}\mathcal{O}(1) \overset{i}{\hookrightarrow} Q^{n+2}$$

The composition is the inclusion of the intersection of Q^{n+2} with two transversal hyperplanes.

3.12 Lemma. *All higher differentials (d_k with $k > 2$) in the Atiyah-Hirzebruch spectral sequence for* $\mathrm{KO}^*(\mathrm{Thom}_{Q^n}\mathcal{O}(1))$ *vanish.*

Proof. We need only consider the cases when n is odd. Write $n = 2m + 1$.

When $n \equiv 1 \mod 4$ we claim that i^* maps x^{m+1} in $H^{n+1}(Q^{n+2}, \mathrm{Sq}^2)$ to $x^m\theta$ in $H^{n+1}(\mathrm{Thom}_{Q^n}\mathcal{O}(1), \mathrm{Sq}^2)$. Indeed, j^*i^* maps the class of the hyperplane section x in $H^2(Q^{n+2})$ to the class of the hyperplane section x in $H^2(Q^n)$. So i^*x in $H^2(\mathrm{Thom}_{Q^n}\mathcal{O}(1))$ must be non-zero, hence equal to θ modulo 2. It follows that $i^*(x^{m+1}) = \theta^{m+1}$. Since $\theta^2 = \mathrm{Sq}^2(\theta) = x\theta$, we have $\theta^{m+1} = x^m\theta$, proving the claim. As we already know that all higher differentials vanish on $H^*(Q^{n+2}, \mathrm{Sq}^2)$, we may now deduce that they also vanish on $H^*(\mathrm{Thom}_{Q^n}\mathcal{O}(1), \mathrm{Sq}^2)$.

When $n \equiv 3 \mod 4$ we claim that i^* maps the element a in $H^{n+3}(Q^{n+2}, \mathrm{Sq}^2)$ to $a\theta$ in $H^{n+3}(\mathrm{Thom}_{Q^n}\mathcal{O}(1), \mathrm{Sq}^2)$. Indeed, a represents a linear subspace of codimension $m + 2$ in Q^{n+2} and is thus mapped to the class of a linear subspace of the same codimension in Q^n: $j^*i^*(a) = ax$ in $H^{n+3}(Q^n)$. Thus, $i^*(a)$ is non-zero in $H^{n+3}(\mathrm{Thom}_{Q^n}\mathcal{O}(1))$, equal to $a\theta$ modulo 2. Again, this implies that all higher differentials vanish on $H^*(\mathrm{Thom}_{Q^n}\mathcal{O}(1), \mathrm{Sq}^2)$ since they vanish on $H^*(Q^{n+2}, \mathrm{Sq}^2)$. □

The additive structure of $\mathrm{KO}^*(Q^n; \mathcal{O}(1))$ thus follows directly from the result for $H^d(Q^n, \mathrm{Sq}^2 + x) = \widetilde{H}^{d+2}(\mathrm{Thom}_{Q^n}\mathcal{O}(1))$ displayed in Lemma 3.11 via Corollary 2.4.

3f Spinor varieties

Let $\mathrm{Gr}_{\mathrm{SO}}(n, N)$ be the Grassmannian of n-planes in $\mathbb{C}^N$ isotropic with respect to a fixed non-degenerate symmetric bilinear form, or, equivalently, the Fano variety of projective $(n-1)$-planes contained in the quadric Q^{N-2}. For each $N > 2n$, this is an irreducible homogeneous variety. In particular, for $N = 2n + 1$ we obtain the spinor variety $S_{n+1} = \mathrm{Gr}_{\mathrm{SO}}(n, 2n+1)$. The variety $\mathrm{Gr}_{\mathrm{SO}}(n, 2n)$ falls apart into two connected components, both of which are isomorphic to S_n. This is reflected by the fact that we can equivalently identify S_n with $\mathrm{SO}(2n-1)/U(n-1)$ or $\mathrm{SO}(2n)/U(n)$.

As for all Grassmannians, the Picard group of S_n is isomorphic to $\mathbb{Z}$; we fix a line bundle $\mathcal{S}$ which generates it. The KO-theory twisted by $\mathcal{S}$ vanishes:

3.13 Theorem. *For all $n \geq 2$ the additive structure of* $\mathrm{KO}^*(S_n; \mathcal{L})$ *is as follows:*

	t_0	t_1	$s_i(\mathcal{O})$	$s_i(\mathcal{S})$
$n \equiv 2 \mod 4$	2^{n-2}	2^{n-2}	$\rho(\frac{n}{2}, 1-i)$	0
otherwise	2^{n-2}	2^{n-2}	$\rho(\lfloor\frac{n}{2}\rfloor, -i)$	0

The values $\rho(n,i)$ are defined as in Theorem 3.5.

Proof. The cohomology of S_n with $\mathbb{Z}/2$-coefficients has simple generators e_2, e_4, …, e_{2n-2}, i. e. it is additively generated by products of distinct elements of this list. Its multiplicative structure is determined by the rule $e_{2i}^2 = e_{4i}$, and the second Steenrod square is given by $\mathrm{Sq}^2(e_{2i}) = ie_{2i+2}$ [Ish92, Proposition 1.1]. In both formulae it is of course understood that $e_{2j} = 0$ for $j \geq n$. What we need to show is that for all $n \geq 2$ we have

$$H^*(S_n, \mathrm{Sq}^2 + e_2) = 0$$

Let us abbreviate $H^*(S_n, \mathrm{Sq}^2 + e_2)$ to (H_n, d'). We claim that we have the following short exact sequence of differential $\mathbb{Z}/2$-modules:

$$0 \to (H_n, d') \overset{\cdot e_{2n}}{\to} (H_{n+1}, d') \to (H_n, d') \to 0 \tag{8}$$

This can be checked by a direct calculation. Alternatively, it can be deduced from the geometric considerations below. Namely, it follows from the cofibration sequence of Corollary 3.15 that we have such an exact sequence of $\mathbb{Z}/2$-modules with maps respecting the differentials given by Sq^2 on all three modules. Since they also commute with multiplication by e_2, they likewise respect the differential $d' = \mathrm{Sq}^2 + e_2$.

The long exact cohomology sequence associated with (8) allows us to argue by induction: if $H^*(H_n, d') = 0$ then also $H^*(H_{n+1}, d') = 0$. Since we can see by hand that $H^*(H_2, d') = 0$, this completes the proof. □

We close with a geometric interpretation of the exact sequence (8), via an analogue of Lemmas 3.2 and 3.6. Let us write $\mathcal{U}$ for the universal bundle over S_n, i. e. for the restriction of the universal bundle over $\mathrm{Gr}(n-1, 2n-1)$ to S_n, and $\mathcal{U}^\perp$ for the restriction of the orthogonal complement bundle, so that $\mathcal{U} \oplus \mathcal{U}^\perp$ is the trivial $(2n-1)$-bundle over S_n. As in Section 3d, we emphasize that under these conventions the fibres of $\mathcal{U}$ and $\mathcal{U}^\perp$ are perpendicular with respect to a hermitian metric on $\mathbb{C}^{2n-1}$ — they are not orthogonal with respect to the chosen symmetric form.

3.14 Lemma. *The spinor variety S_n embeds into the spinor variety S_{n+1} with normal bundle $\mathcal{U}^\perp$ such that the embedding extends to an embedding of this bundle. The closed complement of $\mathcal{U}^\perp$ in S_{n+1} is again isomorphic to S_n.*

3.15 Corollary. *We have a cofibration sequence*

$$S_{n+} \overset{i}{\hookrightarrow} S_{n+1+} \overset{p}{\twoheadrightarrow} \mathrm{Thom}_{S_n} \mathcal{U}^\perp$$

Note however that, unlike in the symplectic case, the first Chern classes of $\mathcal{U}$ and $\mathcal{U}^\perp$ pull back to *twice* a generator of the Picard group of S_n. For example, the embedding of S_2 into $\mathrm{Gr}(1,3)$ can be identified with the embedding of the one-dimensional smooth quadric into the projective plane, of degree 2, and the higher dimensional cases can be reduced to this example. Thus, $c_1(\mathcal{U})$ and $c_1(\mathcal{U}^\perp)$ are trivial in $\mathrm{Pic}(S_n)/2$.

proof of Lemma 3.14. The proof is similar to the proof of Lemma 3.6. Let e_1, e_2 be the first two canonical basis vectors of $\mathbb{C}^{2n+1}$, and let $\mathbb{C}^{2n-1}$ be embedded into $\mathbb{C}^{2n+1}$ via the remaining coordinates. Let S_n be defined in terms of a symmetric form Q on $\mathbb{C}^{2n-1}$, and define S_{n+1} in terms of

$$Q_{2n+1} := \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & Q \end{pmatrix}$$

Let i_1 and i_2 be the embeddings of S_n into S_{n+1} sending an $(n-1)$-plane $\Lambda \subset \mathbb{C}^{2n-1}$ to $e_1 \oplus \Lambda$ or $e_2 \oplus \Lambda$ in $\mathbb{C}^{2n+1}$, respectively. Given an $(n-1)$-plane $\Lambda \in S_n$ together with a vector v in $\Lambda^\perp \subset \mathbb{C}^{2n-1}$, consider the linear map

$$\begin{pmatrix} -\frac{1}{2}Q(v,v) & -Q(-,v) \\ v & 0 \end{pmatrix} : \langle e_1 \rangle \oplus \Lambda \to \langle e_2 \rangle \oplus \Lambda^\perp$$

Sending (Λ, v) to the graph of this function defines an open embedding of $\mathcal{U}^\perp$ whose closed complement is the image of i_2. □

3g Exceptional hermitian symmetric spaces

Lastly, we turn to the exceptional hermitian symmetric spaces EIII and EVII. We write $\mathcal{O}(1)$ for a generator of the Picard group in both cases.

3.16 Theorem. *The KO-groups of the exceptional hermitian symmetric spaces* EIII *and* EVII *are as follows:*

			$\mathcal{L} \equiv \mathcal{O}$				$\mathcal{L} \equiv \mathcal{O}(1)$			
	t_0	t_1	s_0	s_1	s_2	s_3	s_0	s_1	s_2	s_3
$\mathrm{KO}^*(\mathrm{EIII}; \mathcal{L})$	15	12	3	0	0	0	3	0	0	0
$\mathrm{KO}^*(\mathrm{EVII}; \mathcal{L})$	28	28	1	3	3	1	0	0	0	0

Proof. The untwisted KO-groups have been computed in [KH92], the main difficulty as always being to prove that the Atiyah-Hirzebruch spectral sequence collapses. For the twisted groups, however, there are no problems. We quote from § 3 of said paper that the cohomologies of the spaces in question can be written as

$$H^*(\mathrm{EIII}; \mathbb{Z}/2) = \mathbb{Z}/2[t,u] \Big/ (u^2 t,\ u^3 + t^{12})$$
$$H^*(\mathrm{EVII}; \mathbb{Z}/2) = \mathbb{Z}/2[t,v,w] \Big/ (t^{14},\ v^2,\ w^2)$$

with t of degree 2 in both cases, and u, v and w of degrees $8, 10$ and 18, respectively. The Steenrod squares are determined by $\mathrm{Sq}^2 u = ut$ and $\mathrm{Sq}^2 v = \mathrm{Sq}^2 w = 0$. Thus, we find

$$\begin{aligned} H^*(\mathrm{EIII}, \mathrm{Sq}^2 + t) &= \mathbb{Z}/2 \cdot u \oplus \mathbb{Z}/2 \cdot u^2 \oplus \mathbb{Z}/2 \cdot u^3 \\ H^*(\mathrm{EVII}, \mathrm{Sq}^2 + t) &= 0 \end{aligned}$$

By Lemma 2.5, the Atiyah-Hirzebruch spectral sequence for EIII must collapse. This gives the result displayed above.

□

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