The Milnor Conjectures

Oberseminar Algebra, Geometrie und Zahlentheorie WS 2015/16

The Milnor Conjectures relate the following three algebraic invariants of a field: the Witt group of quadratic forms, étale cohomology and Milnor K-theory. Our aims in this seminar will be very modest: we would like to understand the statements of the conjectures and their verifications in the numerous cases dealt with in Milnor's original paper [Mil69]. The aim is *not* to understand Voevodsky et al.'s proof. Even Merkurjew's proof of the conjectures in degree two will at most be sketched in one of the last talks.

Dates

All talks will take place Fridays in 25.22.03.73, beginning at 12:30 and lasting for up to 90 minutes.

Talk 1:	*23 Oct*	Marcus Zibrowius
	30 Okt	Extrakolloquien – *kein Seminar*
Talk 2:	*6 Nov*	Tobias Hemmert
Talk 3:	*13 Nov*	Anitha Thillaisundaram
Talk 4:	*20 Nov*	Matthias Riepe
Talk 5:	*27 Nov*	Leif Zimmermann
Talk 6:	*4 Dec*	Steffen Kionke
Talk 7:	*11 Dec*	André Schell
Talk 8:	*18 Dec*	Benedikt Schilson
Talk 9:	*8 Jan*	Saša Novaković
Talk 10:	*15 Jan*	Kevin Langlois
Talk 11:	*22 Jan*	Matteo Vannacci
Talk ⋆12:	*29 Jan*	Stefan Schröer
Talk ⋆15:	*5 Feb*	Alexander Samokhin

Contents

The ⋆ in the last four talks means "pick and mix" – these talks are mutually independent.

Talk 1: Introduction

Present an overview over (a) the history of the conjectures and (b) the seminar. In particular, give names and symbols to all key players:

$$\begin{tikzcd} & k_*(F) \arrow[dl, "s"'] \arrow[dr, "h"] & \\ \mathrm{gr}^* W(F) \arrow[rr, "e"'] & & h^*(F) \end{tikzcd}$$

At the top, we have the Milnor K-theory of our field F, or more precisely its Milnor K-theory modulo 2: $k_i(F) := K_i(F)/2$. On the left, we have the graded ring associated with the Witt ring $W(F)$, and on the right we have the étale cohomology ring with $\mathbb{Z}/2$-coefficients $h^i(F) := H^i_{\mathrm{et}}(F, \mathbb{Z}/2)$. Moreover, we have three ring homomorphisms s, h, e connecting them. Milnor's Conjectures say that these are isomorphisms.

Convey a sense of surprise (c. f. [Pfi00]).

Part I: Basic theory

Talk 2: The Witt ring of quadratic forms

Discuss all material in [QM11, §§2.1–2.3]. More details can be found in [Lam05]:

- Witt's cancellation theorem: [Lam05, I.2]
- The Grothendieck-Witt ring $\hat{W}(F)$ and the Witt ring $W(F)$: [Lam05, II.1], [Mil69, §2]
- Pfister forms, the fundamental ideal $I(F)$ and its powers, and the graded Witt ring $\mathrm{gr}^*W(F) = \bigoplus_n I^n(F)/I^{n+1}(F)$: [Lam05, X.1]

(A warning concerning notation: the Grothendieck-Witt ring, also known as Witt-Grothendieck ring, is usually denoted either $GW(F)$ or $\hat{W}(F)$. In [QM11], however, $GW(F)$ is used to denote the *graded* Witt ring, which we will write as $\mathrm{gr}^*W(F)$. The graded Grothendieck-Witt ring coincides with the graded Witt ring in all positive degrees [Mil69, first paragraph of §4].)

Talk 3: Galois cohomology

Begin by discussing group cohomology for a finite group G. In particular, explain how to compute the cohomology of a cyclic group $\mathbb{Z}/n$ [GS06, 3.2.9].

Next, pass to profinite groups [SerCG, I; GS06, 4.2]. Explain how to compute the cohomology of $\hat{\mathbb{Z}}$ [SerCG, I §2.2, Prop. 8; SerCL, XIII §1, Prop. 2].

Finally, discuss the basic properties of Galois cohomology, including the identification of $H^*(F,\mu_n)$ with $F^\times/(F^\times)^n$ via Hilbert's Theorem 90 [SerCL, X §1, Prop. 2; GS06, 4.3.6]. In particular, using our notation $h^i(F) := H^i(F,\mathbb{Z}/2)$ and noting that $\mathbb{Z}/2 \cong \mu_2$ (canonically), we find $h^1(F) \cong F^\times/(F^\times)^2$.

In the end, you should have covered all material in [QM11, §4 and §§5.1–5.2], except the very last statement (Thm 5.5).

Talk 4: Quaternion algebras

Present [Lam05, III.1 and 2] and explain:

(a) How to view a quaternion algebra as a quadratic space, and, in fact, as a two-fold Pfister form.

(b) How to view a quaternion algebra as an element of $h^2(F) = H^2(F, \mathbb{Z}/2)$ using the second identification below:

$$\begin{aligned} H^2(F, F_s^\times) &\cong \mathrm{Br}(F) \\ h^2(F) &\cong {}_2\mathrm{Br}(F) \end{aligned}$$

Here, $\mathrm{Br}(F)$ is the Brauer group of F, and ${}_2\mathrm{Br}(F)$ is its two-torsion subgroup. The first isomorphism appeared in last semester's seminar. [SerCL, X §5, Prop. 9; GS06, 4.4] The second follows via Hilbert's Theorem 90.

Talk 5: Milnor's K-theory and his conjectures

Introduce Milnor K-theory [Mil69, §1; QM11, 1.1] and discuss its relation with symbols [Kup, 2.1–2.3] For an example calculation, perhaps use [Mil69, 1.4] = [Lam05, Thm X.6.10] on non-real fields.

Next, explain how to obtain the surjective ring homomorphism s [Mil69, 4.1; QM11, §2.4] and the ring homomorphism h [Mil69, 6.1; QM11, 5.5]. State Milnor's implicit conjectures: [Mil69, 4.3 and sentence above 6.2] and explain why they hold in degrees zero and one.

As for degree two, we will see an elementary proof that s_2 is bijective in the next talk. Discuss here the significance of the bijectivity of h_2 (Merkurjev's Theorem), as discussed in [Lam05, V.6].

Talk 6: Stiefel-Whitney classes

Discuss Milnor's Stiefel-Whitney classes w_i for quadratic forms with values in k_* [Mil69, §3]. Use these classes to prove that s_2 is bijective [Mil69, 4.1]. Also, explain Milnor's proof that s_n is an isomorphism in all degrees for global fields [Mil69, 4.2 and 4.5]. Simply assume Tate's computations of $k_*(F)$ for now—we may return to them later (see talk on global fields below).

By composing with h, we obtain Stiefel-Whitney classes with values in Galois cohomology. In degrees one and two, these coincide with certain classical invariants on the Grothendieck-Witt ring: w_1 is the discriminant (i. e. the determinant modulo two), and w_2 is the Hasse-invariant [Lam05, V.3.17].

Finally, point out that while, by definition, w_1 is e_1 and w_2 is e_2, the higher invariants $e_i := h_i s_i^{-1}$ are much more mysterious.

Part II: Examples

Talk 7: Algebraically closed fields and real closed fields

- For $\mathbb{C}$, all the rings above are isomorphic to $\mathbb{Z}/2$, concentrated in degree zero. More generally, this is true for any algebraically closed field.

To prove this for k_*, use [Lam05, V.6.15] together with the fact that k_* is generated in degree one.

- For $\mathbb{R}$, all the rings above are isomorphic to $\mathbb{Z}/2[x]$, with x of degree one. More generally, this is true for any real closed field.

For k_*, see [Mil69, eg. 1.6] or [Lam05, X.6.9]. From here, the fact that s is an isomorphism could again be deduced using [Mil69, 4.2], thus yielding a computation of gr^*W. Alternatively, gr^*W can be computed directly: note that any real closed field is euclidean [Lam05, VIII.1.8] and use [Lam05, II.3.2].

Talk 8: Finite fields

Show that the Milnor Conjectures hold for finite fields [Mil69, 4.6 and 6.2] through direct computations of all invariants involved. These can be found in:

- $W^*(\mathbb{F}_q)$: [Lam05]
- $h^*(\mathbb{F}_q)$: $Gal(\mathbb{F}_q) = \hat{\mathbb{Z}}$ [GS06, 4.1.5], whose cohomology we have already computed (c. f. talk on Galois cohomology)
- $k^*(\mathbb{F}_q)$: [Mil69, eg. 1.5]

Also mention the following result:

- Let $F_1 \subset F_2 \subset F_3 \subset \dots$ be a sequence of subfields of F, with union F. If the Milnor Conjectures hold for each F_i, they also hold for F.

The idea here is simply to check that all three invariants "commute with colimits". (For h^*, this follows from [SerCG, I §2.2, Prop. 8], which has already been used in the computation of $\hat{\mathbb{Z}}$.) Thus, for example, the conjectures hold not just for the algebraic closure $\bar{\mathbb{F}}_p = \bigcup_n \mathbb{F}_{p^n}$ but also for $\bigcup_n \mathbb{F}_{p^{2^n}}$, $\bigcup_n \mathbb{F}_{p^{3^n}}$ etc.

Talk 9: Complete valued fields

By a complete valued field we mean a field complete with respect to a discrete valuation, for example the field of formal Laurent series $\bar{F}((t))$ over an arbitrary field $\bar{F}$. Discuss the following result on s:

- Let F be a complete valued field with residue field $\bar{F}$. If s is an isomorphism for $\bar{F}$, then it is also an isomorphism for F. [Mil69, 5.2]

It will easily follow from two versions of "Springer's Theorem":

- Springer's Theorem for the Witt ring [Lam05, VI.1]
- Milnor's "Springer's Theorem for K_*" [Mil69, 2.6]

Milnor also proves a similar result for h, but only in a special case:

- If h is an isomorphism for a field $\bar{F}$, then h is also isomorphism for $\bar{F}((t))$. [Mil69, 6.3]

Talk 10: Local fields

Both Lam and Milnor understand "local fields" to be non-archimedean, so we will follow the same convention here. Thus, a local field is either of the form $\mathbb{F}((t))$ with $\mathbb{F}$ finite or it is a finite extension of $\mathbb{Q}_p$.[1] These local fields are precisely the complete valued fields whose residue fields are finite. Thus, the conjectures have mostly been verified in Talks 8 and 9: the conjecture concerning s has been verified for all local fields, and the conjecture concerning h has been verified at least for $\mathbb{F}((t))$.

Here, we complete the verification of Milnor's conjectures for all local fields by explicitly calculating:

(a) $\mathrm{gr}^2 W(F) \cong k_2(F) \cong h^2(F) \cong \mathbb{Z}/2$

(b) $\mathrm{gr}^* W(F)$, $k_*(F)$ and $h^*(F)$ vanish in degrees > 2.

For the degree two statement (a), we need:

- There exists a unique quaternion division algebra over each local field. [Lam05, VI.2.10]
- The Brauer group of a local field is $\mathrm{Br}(F) = \mathbb{Q}/\mathbb{Z}$, so $h^2(F) = {}_2\mathrm{Br}(F) = \mathbb{Z}/2$. [SerCL, XII §3, Thm 2]

Indeed, the first bullet point implies that the image of h_2 is $\mathbb{Z}/2$, so combined with the second we find that h_2 is surjective. Moreover:

- The dimension and the Stiefel-Whitney classes w_1 and w_2 with values in Galois cohomology are a complete set of invariants for quadratic forms over local fields. [Lam05, VI.2.12].

[1] The only other two fields usually included under the heading "local" are $\mathbb{R}$ and $\mathbb{C}$.

In particular, $w_2 = e_2$ is injective, and thus the degree two statement (a) follows.

For statement (b), we first deduce from the previous considerations that $\mathrm{gr}^*W \cong k_*$ vanishes in higher degrees. Then all we are left to show is that $cd_2 F = 2$ for a local field of char $\neq 2$. This is a special case of [SerCG, II §4.3, Prop. 12].

Talk 11: Transcendental extensions

For a purely transcendental extension $F(t)$, Milnor obtains the following result:

- If s is an isomorphism for every finite extension of F, then it is also an isomorphism for $F(t)$. [Mil69, 5.8]

For example, it follows that s is an isomorphism for the fields $\mathbb{F}_q(t)$. This will be generalized in the talk on global fields.

Present Milnor's proof of the above result. In particular, go over the constructions of the short exact sequences of the form:

$$0 \to K_n(F) \to K_n(F(t)) \to \bigoplus_\pi K_{n-1}(F_\pi) \to 0$$
$$0 \to W(F) \to W(F(t)) \to \bigoplus_\pi W(F_\pi) \to 0$$

For the first, see [Mil69, §2]; for the second, see [Mil69, 5.3]/[Lam05, IX.3].

Mention the analogous results for $\mathbb{Q}$. [Lam05, VI.4]

Talk 12: *Global fields

A global field is either a number field (i. e. a finite extension of $\mathbb{Q}$) or a function field of a curve over a finite field (i. e. a finite extension of $\mathbb{F}_q(t)$). Tate showed that for any such field F, we have a short exact sequence and isomorphisms as follows:

$$0 \to k_2(F) \to \bigoplus_\nu k_2(F_\nu) \to \mathbb{Z}/2 \to 0$$
$$k_n(F) \cong \bigoplus_\nu k_n(F_\nu) \qquad \forall\, n \geq 3$$

Here, F_ν runs over all completions of F. Each such completion is either a local field or isomorphic to $\mathbb{R}$ or $\mathbb{C}$. By our previous computations, only the real completions are relevant for $n \geq 3$.

Explain where Tate's results and corresponding results for $\mathrm{gr}^*W(F)$ and $h^*(F)$ come from, and why the Milnor Conjectures hold. For s, there are several options:

Option Milnor-Tate*:* Discuss Tate's original computation, and remind us of Milnor's quick corollary that s is an isomorphism (c. f. talk on Stiefel-Whitney classes above). The computations are presented in the appendix to Milnor's paper:

- The statement on k_2 is [Mil69, A1]. As we already know that $k_2F \cong \mathrm{gr}^2W(F)$ for any field F, we can alternatively obtain this statment via a computation of gr^2W (c. f. next option, in particular [Lam05, VI.3.11 and the following remark]).
- The isomorphisms in higher degrees are stated in [Mil69, A2]. They are proved using the theory of ideles.

OPTION ELMAN-LAM*:* Discuss Elman and Lam's proof that s is an isomorphism for all local and global fields, compute $\mathrm{gr}^*W(F)$, and obtain Tate's result on k_* "for free". The main ingredients are:

- The Hasse-Minkowski principal for quadratic forms. [Lam05, VI.3.1]
  — This principal allows us to compute $\mathrm{gr}^*W(F)$. [Lam05, VI.3.9, 3.11 and the following remark; X.6.28]
- Global and local fields are linked. [Lam05, VI.3.6]
  — This statement obviously requires some background on linkage of quaternion algebras, in particular Albert's Theorem. [Lam05, III.4]
- The map s is an isomorphism for linked fields. [Lam05, X.6.27]
  — The proof uses the theory of linkage of Pfister forms. [Lam05, X.5]

Either option is probably enough to fill a talk, but if your adrenalin level is high, you can go on to discuss $h^*(F)$ and the map h. (Follow the references in [Mil69, proof of 6.2].)

Part III: Complements

Talk 13: *Merkurjev's Theorem

Discuss some aspect(s) of Merkurjev's proof that h_2 is an isomorphism, for example "Hilbert 90 for K_2". See for example [Wad86], [Ker90] or [GS06].

Talk 14: *Arason and Pfister's Hauptsatz

There is another "conjecture" in [Mil69]: Milnor asks whether the filtration of $W(F)$ by powers of the fundamental ideal $I(F)$ is Hausdorff, i. e. whether $\bigcap_n I(F)^n = 0$. This is indeed the case, and was proved soon after the paper had appeared. It is an immediate consequence of the following "Haupsatz" of Arason and [AP71]:

- A positive-dimensional anisotropic form $q \in I^n(F)$ has dimension at least 2^n.

Illustrate this statement using the examples we have seen, and discuss its proof (c. f. [Lam05, X.4 and X.5]).

Talk 15: *Milnor K-theory versus Quillen's K-theory

In degrees greater than two, Milnor K-theory and Quillen K-theory generally differ. Discuss some examples (see for example [Kup, §2]) and explain why "Milnor K-theory is the simplest part of [Quillen] K-theory" [Tot92].

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