Aufgabe 1. Gram-Schmidt Orthonormalization.

(a) This should be a straight-forward calculation:

$$\beta_A(\begin{pmatrix}-2\\-2/3\\1\end{pmatrix},\begin{pmatrix}-2\\-2/3\\1\end{pmatrix}) = \begin{pmatrix}-2 & -2/3 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 2\\0 & 3 & 2\\2 & 2 & 10\end{pmatrix}\begin{pmatrix}-2\\-2/3\\1\end{pmatrix}$$
$$= \begin{pmatrix}-2 & -2/3 & 1\end{pmatrix}\begin{pmatrix}0\\0\\14/3\end{pmatrix}$$
$$= \frac{14}{3}.$$

(b) I assume these kids start with the standard basis. The calculations then are as follows.

(i) *First basis vector:* All that is needed is that we normalize $\begin{pmatrix}1 & 0 & 0\end{pmatrix}^T$ with respect to β_A. We find:

$$\beta_A(\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}1\\0\\0\end{pmatrix}) = \begin{pmatrix}1 & 0 & 0\end{pmatrix}\begin{pmatrix}1 & 0 & 2\\0 & 3 & 2\\2 & 2 & 10\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}1 & 0 & 0\end{pmatrix}\begin{pmatrix}1\\0\\2\end{pmatrix} = 1,$$

meaning that it already is normalized with respect to β_A, and we can write down $\begin{pmatrix}1 & 0 & 0\end{pmatrix}^T$ as our first basis vector.

(ii) *Second basis vector:* We start with $\begin{pmatrix}0 & 1 & 0\end{pmatrix}^T$. We find

$$\beta_A(\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix}) = \begin{pmatrix}1 & 0 & 0\end{pmatrix}\begin{pmatrix}1 & 0 & 2\\0 & 3 & 2\\2 & 2 & 10\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}1 & 0 & 0\end{pmatrix}\begin{pmatrix}0\\3\\2\end{pmatrix} = 0.$$

In other words, $\begin{pmatrix}0 & 1 & 0\end{pmatrix}^T$ is already orthogonal to $\begin{pmatrix}1 & 0 & 0\end{pmatrix}^T$ with respect to β_A, and so we only need to normalize. We find:

$$\beta_A(\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix}) = \begin{pmatrix}1 & 0 & 0\end{pmatrix}\begin{pmatrix}1 & 0 & 2\\0 & 3 & 2\\2 & 2 & 10\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}0 & 1 & 0\end{pmatrix}\begin{pmatrix}0\\3\\2\end{pmatrix} = 3.$$

Therefore, our second basis vector is $\frac{1}{\sqrt{3}}\begin{pmatrix}0 & 1 & 0\end{pmatrix}^T$.

(iii) *Third basis vector:* We find:

$$\beta_A(\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}) = \begin{pmatrix}1 & 0 & 0\end{pmatrix}\begin{pmatrix}1 & 0 & 2\\0 & 3 & 2\\2 & 2 & 10\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}1 & 0 & 0\end{pmatrix}\begin{pmatrix}2\\2\\10\end{pmatrix} = 2$$

and

$$\beta_A(\frac{1}{\sqrt{3}}\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}) = \frac{1}{\sqrt{3}}\begin{pmatrix}0 & 1 & 0\end{pmatrix}\begin{pmatrix}1 & 0 & 2\\0 & 3 & 2\\2 & 2 & 10\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix} = \frac{1}{\sqrt{3}}\begin{pmatrix}0 & 1 & 0\end{pmatrix}\begin{pmatrix}2\\2\\10\end{pmatrix} = \frac{2}{\sqrt{3}}.$$

We thus construct a vector $\vec{v}$ from $\begin{pmatrix}0 & 0 & 1\end{pmatrix}^T$ that is orthogonal to our first and second basis vectors as follows:

$$\vec{v} = \begin{pmatrix}0\\0\\1\end{pmatrix} - 2\begin{pmatrix}1\\0\\0\end{pmatrix} - \frac{2}{\sqrt{3}}\frac{1}{\sqrt{3}}\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}-2\\-2/3\\1\end{pmatrix}.$$

We saw how one normalizes this vector $\vec{v}$ in (a), and thus our third and final basis vector is $\sqrt{\frac{14}{3}}\begin{pmatrix}-2 & -2/3 & 1\end{pmatrix}^T$.

Aufgabe 2.

(a) (Marcus no doubt had to make some sort of statement in the midst of the exam that there was a typo here, and it should be „*Definieren Sie, was es bedeutet, dass c ein größter gemeinsamer Teiler von a und b ist.*")

We say that c is a **greatest common divisor** of a and b if (i) c is a divisor of both a and b, and (ii) if d is any other divisor of both a and b, then d is also a divisor of c.

(b) This question requires use of both polynomial long division and the Euclidean algorithm. Expect students to make serious mistakes here. We have

$$A - B = X^5 + X^2 + X + 1 - (X^5 + 1) = X^2 + X = C$$

and

$$\begin{array}{r|l}
 & X^3 - X^2 + X - 1 \\ \hline
X^2 + X & X^5 \qquad\qquad\qquad\qquad + 1 \\
 & -X^5 - X^4 \\ \hline
 & -X^4 \\
 & X^4 + X^3 \\ \hline
 & X^3 \\
 & -X^3 - X^2 \\ \hline
 & -X^2 \\
 & X^2 + X \\ \hline
 & X + 1
\end{array}$$

and thus we have $B = (X^3 - X^2 + X - 1)C + D$, where $D = (X + 1)$. Next, we have

$$\begin{array}{r|l}
 & X \\ \hline
X + 1 & X^2 + X \\
 & -X^2 - X \\ \hline
 & 0
\end{array}$$

so $C = XD$ with no remainder. No need for further polynomial long division. By the Euclidean algorithm:

$$\begin{aligned}
D = (X + 1) &= B - (X^3 - X^2 + X - 1)C \\
&= B - (X^3 - X^2 + X - 1)(A - B) \\
&= -(X^3 - X^2 + X - 1)A + (X^3 - X^2 + X)B,
\end{aligned}$$

and so, $P = -X^3 + X^2 - X + 1$ and $Q = X^3 - X^2 + X$.

Aufgabe 3.

(a) It is possible to put a matrix in Jordan Normal Form if and only if its characteristic polynomial decomposes into linear factors.

(b) For the first matrix, we find the characteristic polynomial to be

$$\begin{aligned}\chi_A(\lambda) = \begin{vmatrix} 11-\lambda & -4 \\ 25 & -9-\lambda \end{vmatrix} &= (11-\lambda)(-9-\lambda)+100 \\ &= (-99-2\lambda+\lambda^2)+100 \\ &= \lambda^2-2\lambda+1 \\ &= (\lambda-1)^2.\end{aligned}$$

The only two candidates for μ_A are $(\lambda-1)$ and $(\lambda-1)^2$. Since clearly $A-\mathbb{1}$ is not the zero matrix, it follows that we must have $\mu_A(\lambda)=(\lambda-1)^2$.

For the second matrix, we find the characteristic polynomial to be

$$\begin{aligned}\chi_B(\lambda) = \begin{vmatrix} 1-\lambda & -2 \\ 2 & -1-\lambda \end{vmatrix} &= -(1-\lambda)(1+\lambda)+4 \\ &= \lambda^2-1+4 \\ &= \lambda^2+3.\end{aligned}$$

Since we are working over the real numbers, this polynomial is irreducible, and hence it follows that $\mu_B=\chi_B$.

For the third matrix, we find the characteristic polynomial to be

$$\begin{aligned}\chi_C(\lambda) = \begin{vmatrix} 5-\lambda & 4 & 2 \\ 0 & 1-\lambda & -1 \\ -1 & -1 & 3-\lambda \end{vmatrix} &= (5-\lambda)((1-\lambda)(3-\lambda)-1)-(-4-2(1-\lambda)) \\ &= (5-\lambda)(3-4\lambda+\lambda^2-1)-(-6+2\lambda) \\ &= (5-\lambda)(\lambda^2-4\lambda+2)(6-2\lambda) \\ &= -\lambda^3+9\lambda^2-24\lambda+16.\end{aligned}$$

Turns out that $\chi_C(1)=-1+9-24+16=25-25=0$ (*let us hope that the students have seen that!*), so $(\lambda-1)$ is a factor. Doing a little more polynomial long division:

$$\begin{array}{r|l} & -X^2+8X-16 \\ \hline X-1 & -X^3+9X^2-24X+16 \\ & X^3-X^2 \\ \hline & 8X^2-24X \\ & -8X^2+8X \\ \hline & -16X+16 \\ & 16X-16 \\ \hline & 0 \end{array}$$

and we can easily see that $-\lambda^2+8\lambda-16=-(\lambda-4)^2$. Hence, $\chi_C(\lambda)=-(\lambda-1)(\lambda-4)^2$. The only two options for μ_C are $(\lambda-1)(\lambda-4)$ and $(\lambda-1)(\lambda-4)^2$. We find

$$(C-\mathbb{1})(C-4\mathbb{1}) = \begin{pmatrix} 4 & 4 & 2 \\ 0 & 0 & -1 \\ -1 & -1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 4 & 2 \\ 0 & -3 & -1 \\ -1 & -1 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 2 & 2 \\ 1 & 1 & 1 \\ -3 & -3 & -3 \end{pmatrix},$$

and seeing this is decidedly not the zero matrix, we conclude that $\mu_C(\lambda)=(\lambda-1)(\lambda-4)^2$.

(c) With reference to the answer to (a), we cannot put the matrix B in Jordan Normal Form. Since the factor $(\lambda - 1)$ appears squared in the minimal polynomial for A, it follows that the Jordan Normal Form of A is $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Since the factor $(\lambda - 4)$ appears squared in the minimal polynomial for C, it follows that the Jordan Normal Form of C is $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 4 \end{pmatrix}$.

Aufgabe 4.

(a) We find:

$$\chi_A(\lambda) = \begin{vmatrix} -2-\lambda & 1 & 1 \\ -1 & -\lambda & 2 \\ 0 & 0 & -2-\lambda \end{vmatrix} = (-2-\lambda)(-\lambda)(-2-\lambda) - (-1)(-2-\lambda) + 0$$
$$= -\lambda(2+\lambda)(2+\lambda) - (2+\lambda)$$
$$= (-\lambda^2 - 2\lambda - 1)(2+\lambda)$$
$$= -(\lambda+1)^2(\lambda+2)$$

To find the minimal polynomial, we note that it must either be $(\lambda{+}1)(\lambda{+}2)$ or $(\lambda{+}1)^2(\lambda{+}2)$. To see if it's the former, we may compute $(A+\mathbb{1})(A+2\mathbb{1})$ and check if we get the zero matrix:

$$\begin{pmatrix} -1 & 1 & 1 \\ -1 & 1 & 2 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ -1 & 2 & 2 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 1 & 1 \\ -1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

and since this is not the zero matrix, we conclude that $\mu_A(\lambda) = (\lambda+1)^2(\lambda+2)$. (*A clever student will of course recognize that they can jump to that conclusion the moment they have computed that the upper-left element in the matrix is distinct from 0. After all, it's not like the values of those other entries are going to somehow change the value of the first.*)

(b) This was implicitly done already in (a). The two eigenvalues are $\lambda = -1$ and $\lambda = -2$.

(c) Seeing $\lambda = -2$ occurs with multiplicity 1, finding $\mathrm{Hau}(A; -2)$ is equivalent to finding the eigenvector associated with that eigenvalue. We need to solve

$$\begin{pmatrix} -2 & 1 & 1 \\ -1 & 0 & 2 \\ 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = -2 \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -2a+b+c \\ -a+2c \\ -2c \end{pmatrix}$$

The third row is of course a tautology. From the top row, we deduce that $b + c = 0$, or $c = -b$. Then the second row, we obtain $-2b = -a - 2b$, meaning that $a = 0$. Thus we conclude that $\begin{pmatrix} 0 & -1 & 1 \end{pmatrix}^T$ is the eigenvector we seek, and

$$\mathrm{Hau}(A; -2) = \langle \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \rangle.$$

The factor $(\lambda + 1)$ occurs squared in the minimal polynomial, so to find $\mathrm{Hau}(A; -1)$, we need to find $\ker((A + \mathbb{1})^2)$. We have

$$(A+\mathbb{1})^2 = \begin{pmatrix} -1 & 1 & 1 \\ -1 & 1 & 2 \\ 0 & 0 & -1 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{pmatrix},$$

so the kernel of $(A + \mathbb{1})^2$ consists of vectors $\begin{pmatrix} x & y & z \end{pmatrix}^T$ for which $z = 0$. In other words, the plane spanned by $\begin{pmatrix} 1 & 0 & 0 \end{pmatrix}^T$ and $\begin{pmatrix} 0 & 1 & 0 \end{pmatrix}^T$, and so we have

$$\mathrm{Hau}(A; -1) = \langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \rangle.$$

(d) We have our first basis vector in the eigenvector associated with $\lambda = -2$, noted above. Next, we need to find the eigenvector associated with $\lambda = -1$. We find:

$$\begin{pmatrix} -2 & 1 & 1 \\ -1 & 0 & 2 \\ 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = - \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -2a+b+c \\ -a+2c \\ -2c \end{pmatrix}$$

We may conclude from the bottom row that $c = 0$, and using this information, the middle row gives us that $a = b$. Thus the eigenvector is $\begin{pmatrix}1 & 1 & 0\end{pmatrix}^T$.

Max's solution for the final part:

Now, we simply need to find $\vec{w} = \begin{pmatrix}x & y & z\end{pmatrix}^T \in \mathrm{Hau}(A; -1)$ such that

$$\begin{pmatrix}-2 & 1 & 1\\ -1 & 0 & 2\\ 0 & 0 & -2\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix} = \begin{pmatrix}1\\ 1\\ 0\end{pmatrix} - \begin{pmatrix}x\\ y\\ z\end{pmatrix}.$$

Since

$$\begin{pmatrix}-2 & 1 & 1\\ -1 & 0 & 2\\ 0 & 0 & -2\end{pmatrix}\begin{pmatrix}1\\ 0\\ 0\end{pmatrix} = \begin{pmatrix}-2\\ -1\\ 0\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}-2 & 1 & 1\\ -1 & 0 & 2\\ 0 & 0 & -2\end{pmatrix}\begin{pmatrix}0\\ 1\\ 0\end{pmatrix} = \begin{pmatrix}1\\ 0\\ 0\end{pmatrix},$$

we simply need to solve

$$a\begin{pmatrix}-2\\ -1\\ 0\end{pmatrix} + b\begin{pmatrix}1\\ 0\\ 0\end{pmatrix} = \begin{pmatrix}1\\ 1\\ 0\end{pmatrix} - a\begin{pmatrix}1\\ 0\\ 0\end{pmatrix} - b\begin{pmatrix}0\\ 1\\ 0\end{pmatrix},$$

that is,

$$-a + b = 1,$$

for which $a = 0, b = 1$ is a perfectly valid solution. Then

$$\vec{w} = \begin{pmatrix}0\\ 1\\ 0\end{pmatrix},$$

and so in fine, our Jordan Normal Basis is

$$\left\{\begin{pmatrix}0\\ -1\\ 1\end{pmatrix}, \begin{pmatrix}1\\ 1\\ 0\end{pmatrix}, \begin{pmatrix}0\\ 1\\ 0\end{pmatrix}\right\}.$$

Marcus's solution for the final part:

Now, we simply need to find $\vec{w} = \begin{pmatrix}x & y & z\end{pmatrix}^T \in \mathrm{Hau}(A; -1)$ such that

$$\ker((A + \mathbb{1})^2) = \langle \vec{w} \rangle \oplus \ker(A + \mathbb{1}),$$

in other words,

$$\langle \begin{pmatrix}1\\ 0\\ 0\end{pmatrix}, \begin{pmatrix}0\\ 1\\ 0\end{pmatrix} \rangle = \langle \vec{w} \rangle \oplus \langle \begin{pmatrix}1\\ 1\\ 0\end{pmatrix} \rangle.$$

Here, $\vec{w} = \begin{pmatrix}0\\ 1\\ 0\end{pmatrix}$ works just fine. To then get the final vector, we just need to multiply $\vec{w}$ from the left with $A + \mathbb{1}$, so

$$\begin{pmatrix}-1 & 1 & 1\\ -1 & 1 & 2\\ 0 & 0 & -1\end{pmatrix}\begin{pmatrix}0\\ 1\\ 0\end{pmatrix} = \begin{pmatrix}1\\ 1\\ 0\end{pmatrix},$$

and so our Jordan Normal Form is

$$\left\{\begin{pmatrix}0\\ -1\\ 1\end{pmatrix}, (A + \mathbb{1})\vec{w}, \vec{w}\right\} = \left\{\begin{pmatrix}0\\ -1\\ 1\end{pmatrix}, \begin{pmatrix}1\\ 1\\ 0\end{pmatrix}, \begin{pmatrix}0\\ 1\\ 0\end{pmatrix}\right\}.$$

In either event, the Jordan Normal Form that the matrix assumes in this basis is

$$\begin{pmatrix} -2 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}.$$

Aufgabe 5

Kreuzen Sie in den folgenden sieben Aufgabenteilen alle Aussagen an, die richtig sind.

Es ist pro Aufgabenteil mindestens eine *Aussage richtig. Manchmal sind* mehrere *Aussagen richtig. Als Gesamtpunktzahl erhalten Sie die Differenz aus der Anzahl aller richtig gesetzten Kreuze und aller falsch gesetzten Kreuze, mindestens aber 0 Punkte und höchstens 10 Punkte.*

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(1) Ein Endomorphismus f eines endlich-dimensionalen Vektorraums V ist *genau dann* diagonalisierbar, wenn gilt:

- ■ Das Minimalpolynom von f zerfällt in paarweise verschiedene Linearfaktoren.
- ☐ Das Minimalpolynom von f stimmt bis auf ein Vorzeichen mit dem charakteristischen Polynom von f überein.
- ☐ Das charakteristische Polynom von f zerfällt in paarweise verschiedene in Linearfaktoren.

(2) Sei f ein Endomorphismus eines endlich-dimensionalen Vektorraums V mit Minimalpolynom $\mu_f = (X-a)^m \cdot P(X)$, wobei $m \geq 1$ und P ein zu $X-a$ teilerfremdes Polynom ist. Dann gilt:

- ■ a ist ein Eigenwert von f.
- ☐ $P(a) = 0$.
- ☐ Der Eigenraum $\mathrm{Eig}(f; a)$ hat Dimension m.

(3) Welche der folgenden Matrizen definieren ein Skalarprodukt auf $\mathbb{R}^2$?

☐ $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ ■ $\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$ ☐ $\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$

(4) Für eine **Isometrie** $f\colon V \to V$ eines euklidischen Vektorraums V gilt:

- ■ Alle Eigenwerte haben Betrag 1.
- ■ Eigenvektoren zu verschiedenen Eigenwerten stehen senkrecht zueinander.
- ☐ f ist diagonalisierbar.

(5) Für jede reelle symmetrische Matrix $A \in \mathrm{Mat}_{\mathbb{R}}(n \times n)$ gilt:

- ■ A ist diagonalisierbar.
- ☐ Alle Eigenwerte von A sind positiv.
- ■ Eigenvektoren zu verschiedenen Eigenwerten von A stehen senkrecht zueinander (bezüglich des Standardskalarprodukt auf $\mathbb{R}^n$).

(6) Die folgenden Ringe sind Integritätsringe:

- ■ $\mathbb{Z}[X, Y] := (\mathbb{Z}[X])[Y]$ (Polynomring in zwei Variablen)
- ☐ $\mathbb{Z}/22\mathbb{Z}$
- ■ $\mathbb{C}$

(7) Sei V ein endlich-dimensionalre Vektorraum, und U ein Untervektorraum, und $i\colon U \to V$ die Inklusion. Für die Dualräume V^* und U^* gilt:

- ■ $\dim(U^*) = \dim U$
- ☐ $\dim(U^*) = \dim(V)$ und $\dim(U) = \dim(V^*)$.
- ☐ Die Abbildung $i^*\colon V^* \to U^*$ ist injektiv.

Aufgabe 6.

And, finally, we have Marcus's signature "creative question." I'm curious how the students fared on this one.

(a) A matrix M is said to be **nilpotent** if there exists $k \in \mathbb{N}$ such that $M^k = 0$.
(b) A field does not contain any zero-divisors. Hence, since the domain of 1×1-matrices over a field is identical to the field itself, only the zero element is nilpotent.
(c) The way that subquestion (d) is formulated ought give the students reason to suspect that the answer is no, and that they are expected to construct a counterexample. The one I came up with went as follows. We have

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

but we have

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

so the proposition does not hold true.
(d) First we prove that $N_1 N_2$ is nilpotent. Since N_1 and N_2 are both nilpotent, there exist $k_1, k_2 \in \mathbb{N}$ such that $N_1^{k_1} = 0$ and $N_2^{k_2} = 0$. Since N_1 and N_2 commute, $(N_1 N_2)^{k_1 k_2} = N_1^{k_1 k_2} N_2^{k_1 k_2} = 0 \cdot 0 = 0$. Therefore, $N_1 N_2$ is nilpotent.

Next, we prove that $N_1 + N_2$ is nilpotent. Using the binomial theorem and commutativity, we have

$$(N_1+N_2)^{k_1k_2} = N_1^{k_1k_2} + \binom{k_1k_2}{1} N_1^{k_1k_2-1} N_2 + \binom{k_1k_2}{2} N_1^{k_1k_2-2} N_2^2 + \cdots + \binom{k_1k_2}{k_1k_2-1} N_1^1 N_2^{k_1k_2-1} + N_2^{k_1k_2}.$$

If k_1 and k_2 are integers greater than 1, then it follows that $\dfrac{k_1k_2}{2} \geq k_1, k_2$. Therefore,

$$(N_1 + N_2)^{k_1k_2} = 0 + 0 + 0 + \cdots + 0 + 0 = 0.$$

Therefore, $N_1 + N_2$ is nilpotent.