Topologie I, Blatt 0

1. Verdickung.

Seien X, Y und Z topologische Räume, $f : X \times Y \to Z$ eine stetige Abbildung, $L \subset Y$ eine kompakte und $O \subset Z$ eine offene Teilmenge. Dann existiert zu jedem Punkt $x \in X$ mit

$$f(x \times L) \subset O$$

eine Umgebung U von x, sodass sogar $f(U \times L)$ ganz in O enthalten ist.

Lösung.

Recall that the basis of product topology on $X \times Y$ is given by the collection of open sets $\mathcal{U}_X \times \mathcal{V}_Y$, where $\mathcal{U}_X$ (resp. $\mathcal{V}_Y$) is an open set of X (resp., of Y). The subset L of Y is compact by the assumption, and so is the subset $x \times L$ of $X \times Y$. Consider any point $l \in L$. Since the map f is continuous, there exists a neighbourhood $\mathcal{U}_{x \times l}$ of $x \times l$ (an open set containing $x \times l$), such that $f(\mathcal{U}_{x \times l})$ is contained in O. We can choose the neighbourhood $\mathcal{U}_{x \times l}$, shrinking it, if necessary, to be of the form $\mathcal{U}_x^l \times \mathcal{V}_l$, where $\mathcal{U}_x^l$ is a neighbourhood of x, and $\mathcal{V}_l$ is a neighbourhood of $l \in L$ [i]. The open sets $\mathcal{U}_x^l \times \mathcal{V}_l$ form a covering of $x \times L$, and since it is compact, we can find a finite subcovering. Let $i = 1, \dots, N$ be the set of indices that corresponds to that finite sucovering. We thus obtain an inclusion $x \times L \subset \bigcup_{i=1}^{i=N} \mathcal{U}_x^{l_i} \times \mathcal{V}_{l_i}$. Consider the set $\bigcap_{i=1}^{i=N} \mathcal{U}_x^{l_i}$. It is non-empty, because each of open sets $\mathcal{U}_x^{l_i}$ contains the point x, and it is open, being a finite intersection of open sets. Denote $\mathcal{U}_x := \bigcap_{i=1}^{i=N} \mathcal{U}_x^{l_i}$. Then $\mathcal{U}_x$ satisfies the condition required in the task. Indeed, $\mathcal{U}_x \times L$ is contained in $\bigcup_{i=1}^{i=N} \mathcal{U}_x^{l_i} \times \mathcal{V}_{l_i}$ by the construction, and for each $i = 1, \dots, N$ we have $f(\mathcal{U}_x^{l_i} \times \mathcal{V}_{l_i}) \subset O$. Therefore,

$$f(\mathcal{U}_x \times L) \subset f(\bigcup_{i=1}^{i=N} \mathcal{U}_x^{l_i} \times \mathcal{V}_{l_i}) = \bigcup_{i=1}^{i=N} f(\mathcal{U}_x^{l_i} \times \mathcal{V}_{l_i}) \subset O.$$

□

[i] The superscript l at $\mathcal{U}_x^l$ is just to underline dependence of $\mathcal{U}_x^l$ on the point l.