Blatt 1, Aufgabe 4

4. Skelett. Every category is equivalent to a category, in which there is a unique object in each isomorphism class.

Proof. Recall that a functor $\mathsf{F} : \mathcal{C} \to \mathcal{D}$ between two categories is an *equivalence of categories* if there exist a functor $\mathsf{G} : \mathcal{D} \to \mathcal{C}$ and isomorphisms of functors:

$$\mathsf{G} \circ \mathsf{F} \simeq \mathrm{id}_{\mathcal{C}}, \quad \mathsf{F} \circ \mathsf{G} \simeq \mathrm{id}_{\mathcal{D}}. \tag{0.1}$$

Two categories $\mathcal{C}$ and $\mathcal{D}$ are said to be equivalent, if there exists an equivalence $\mathsf{F} : \mathcal{C} \to \mathcal{D}$.

Further, a functor $\mathsf{F} : \mathcal{C} \to \mathcal{D}$ is called *faithful*, if for any two objects $X, Y \in \mathcal{C}$ the induced map

$$\mathrm{H}om_{\mathcal{C}}(X, Y) \to \mathrm{H}om_{\mathcal{D}}(\mathsf{F}(X), \mathsf{F}(Y))$$

is injective, and F is called *full*, if the above induced map is *surjective*. If F is both full and faithful, then such a functor is called *fully faithful*.

Let $\mathcal{C}$ be a category. Define the category $\mathsf{sk}(\mathcal{C})$ (skeleton of $\mathcal{C}$). Let $[\mathbf{A}]$ be an isomorphism class of objects in $\mathcal{C}$; then take one object $\mathbf{a}$ in this isomorphism class (for this one needs the *Axiom of Choice*). Define the set of morphisms between two objects $\mathbf{a}$ and $\mathbf{b}$ of $\mathsf{sk}(\mathcal{C})$ to be the set of morphisms between these objects in $\mathcal{C}$. We obtain an inclusion functor $\mathsf{F} : \mathsf{sk}(\mathcal{C}) \to \mathcal{C}$ that is full and faithful.

Let us construct an inverse functor G to the inclusion functor F, and then check the isomorphisms (0.1). This will prove the equivalence between $\mathsf{sk}(\mathcal{C})$ and $\mathcal{C}$.

Indeed, since objects of $\mathsf{sk}(\mathcal{C})$ are isomorphism classes of objects of $\mathcal{C}$, every object $X \in \mathcal{C}$ is isomorphic to some object $\mathsf{F}(x)$ for $x \in \mathsf{sk}(\mathcal{C})$. Therefore, given such an object X, we can choose an isomorphism i_x between $\mathsf{F}(x)$ and X:

$$i_x : \mathsf{F}(x) \to X.$$

Define the functor $\mathbf{G} : \mathcal{C} \to \mathsf{sk}(\mathcal{C})$ that sends an object $X \in \mathcal{C}$ to $x \in \mathsf{sk}(\mathcal{C})$ as above. We must define how $\mathbf{G}$ acts on morphisms in $\mathcal{C}$. Let $g : X \to Y$ be an arrow in $\mathcal{C}$. Consider the square:

$$
\begin{tikzcd}
\mathsf{F}\circ\mathsf{G}(X) \arrow[r, "i_x"] \arrow[d] & X \arrow[d, "g"] \\
\mathsf{F}\circ\mathsf{G}(Y) \arrow[r, "i_y"] & Y
\end{tikzcd}
$$

We obtain an arrow $i_y^{-1}\circ g\circ i_x \in \mathrm{Mor}_{\mathcal{C}}(\mathsf{F}\circ\mathsf{G}(X), \mathsf{F}\circ\mathsf{G}(Y))$ (recall that i_x and i_y are isomorphisms). Since the functor F is full and faithful, there exists a unique $f \in \mathrm{Mor}_{\mathsf{sk}(\mathcal{C})}(\mathsf{G}(X), \mathsf{G}(Y))$ that is sent by F to the composed arrow $i_y^{-1}\circ g\circ i_x$. Define $\mathsf{G}(g) := f$ for $g \in \mathrm{Mor}_{\mathcal{C}}(X, Y)$. We have defined G on objects and morphisms of $\mathcal{C}$.

Let us verify that G is a **functor**, in other words it preserves identity morphisms and repsects compositions of morphisms. If $\mathrm{id}_X : X \to X$ is the identity morphism for an object $X \in \mathcal{C}$, then $\mathsf{G}(\mathrm{id}_X)$ is a unique arrow from $\mathsf{G}(X)$ to $\mathsf{G}(X)$, such that $\mathsf{FG}(\mathrm{id}_X) = i_x^{-1}\circ \mathrm{id}_X \circ i_x = \mathrm{id}_X$. So, $\mathsf{G}(\mathrm{id}_X) = \mathrm{id}_{\mathsf{G}(X)}$.

To verify that G preserves composition of arrows we need to show that given two arrows $g_2 : X \to Y, g_1 : Y \to Z$ and their composition $g_1 \circ g_2 : X \to Z$ one obtains $\mathsf{G}(g_1\circ g_2) = \mathsf{G}(g_1)\circ\mathsf{G}(g_2)$. This can be seen from the diagram below:

$$
\begin{tikzcd}
\mathsf{F}\circ\mathsf{G}(X) \arrow[r, "i_x"] \arrow[d, "\mathsf{F}(f_2)"] \arrow[dd, bend right=60, "\mathsf{F}(f_1\circ f_2)"'] & X \arrow[d, "g_2"] \arrow[dd, bend left=60, "g_1\circ g_2"] \\
\mathsf{F}\circ\mathsf{G}(Y) \arrow[r, "i_y"] \arrow[d, "\mathsf{F}(f_1)"] & Y \arrow[d, "g_1"] \\
\mathsf{F}\circ\mathsf{G}(Z) \arrow[r, "i_z"] & Z
\end{tikzcd}
$$

Indeed, from the definition of morphisms $\mathsf{F}(f_i)$ for $i = 1, 2$ it follows (write FG short for the composition of functors $\mathsf{F}\circ\mathsf{G}$):

$$\mathsf{FG}(f_1\circ f_2) = i_z^{-1}\circ g_1\circ g_2\circ i_x = i_z^{-1}\circ g_1\circ i_y\circ i_y^{-1}\circ g_2\circ i_x = \mathsf{FG}(f_1)\circ\mathsf{FG}(f_2) = \mathsf{F}(\mathsf{G}(f_1)\circ\mathsf{G}(f_2)),$$

the last equality follows from F being a functor (the inclusion functor of $\mathsf{sk}(\mathcal{C})$ to $\mathcal{C}$). Since F is faithful, it follows that $\mathsf{G}(f_1\circ f_2) = \mathsf{G}(f_1)\circ\mathsf{G}(f_2)$.

We obtain that G is a functor, a morphism $\mathsf{F}\circ\mathsf{G} \to \mathrm{id}_{\mathcal{C}}$ is a natural transformation of functors, and by the construction it is an isomorphism of functors. Since F is an inclusion functor, we also have $\mathsf{G}\circ\mathsf{F} \simeq \mathrm{id}_{\mathsf{sk}(\mathcal{C})}$. Therefore, the categories $\mathsf{sk}(\mathcal{C})$ and $\mathcal{C}$ are equivalent.

$\square$