1. 27 LINES ON A CUBIC SURFACE

Let k be an algebraically closed field. By a *cubic surface over k* we mean the zero locus $X \subset \mathbb{P}^3_k$ of a degree 3 homogeneous polynomial over k, which we always assume to be nonsingular. By a *line on X* we mean a subset $L \subset X$ which is the zero locus of a homogenous polynomial of degree 1.

Theorem 1.1. *Let $X \subset \mathbb{P}^3$ be a cubic surface. There are exactly 27 lines on X.*

There are a few strategies to prove this Theorem. We suggest only one here. The key idea is to realize that any cubic surface is isomorphic to the *blow-up* of the projective plane $\mathbb{P}^2$ in 6 points in general position. Then the general theory of blow-ups can be used to "count all the lines". If you google for blowups on the internet, you will find many nice pictures. They are the most important tool in the field of *birational geometry*. For a more elementary introduction to blowups, you can have a look at [BTL, Chapter 7], but beware that that text is still very incomplete.

Definition 1.2. A sequence of points $P_1, \dots, P_6 \in \mathbb{P}^2$ is said to be in general position if

(i) no 3 of them lie on a line, and

(ii) they do not all lie on a single conic (the zero locus of a degree 2 homogeneous equation).

Theorem 1.3. *Let $X \subset \mathbb{P}^3$ be a cubic surface. There exist points $P_1, \dots, P_6 \in \mathbb{P}^2$ in general position such that X is isomorphic to the blow-up of $\mathbb{P}^2$ in the points $P_1, \dots, P_6$.*

Proof. This is [Poo17, Theorem 9.4.4]. Indeed, the anticanonical divisor of X is just a hyperplane section of the emedding $X \hookrightarrow \mathbb{P}^3$, which is very ample by definition. So X is a del Pezzo surface of degree 3 as in [Poo17, Section 9.4]. ∎

Proof of Theorem 1.1. This is done in [Har77, Chapter 5, Theorem 4.9]. The 27 lines correspond to

- the 6 exceptional divisors, one for each point;
- the $\binom{6}{2} = 15$ lines going through 2 of the 6 points;
- the $\binom{6}{5} = 6$ conics going through 5 of the 6 points.

Summing these all up we get

$$6+15+6=27.$$

∎

A detailed proof of Theorem 1.3 is a little out of reach for a report. However, it is feasible to prove in a report that the blow-up of $\mathbb{P}^2$ in $P_1,\dots,P_6 \in \mathbb{P}^2$ in general position, is a cubic surface. This is done in [Har77, Section 5.4].

Denote this blowup by $\widetilde{X} \to \mathbb{P}^2$. We will sketch a little more context here to motivate Hartshorne's construction. It was mentioned in the proof above already that for a cubic surface $X \hookrightarrow \mathbb{P}^3$, an anticanonical divisor is precisely minus a hyperplane section. So to realize $\widetilde{X}$ as a cubic surface, we should use its anticanonical divisor to embed it in projective space. The anticanonical divisor on $\mathbb{P}^2$ is well known to be $3L$, where L is a line. Now, [Har77, Chapter 5, Proposition 3.3] tells us that the anticanonical divisor of $\widetilde{X}$ is

$$-K_{\widetilde{X}} = 3\pi^* L - E_1 - \ldots - E_6,$$

where $E_1,\dots,E_6$ denote the special fibers of $\widetilde{X} \to \mathbb{P}^2$: E_i is the fiber of $\widetilde{X} \to \mathbb{P}^2$ over P_i. The same proposition also tells us that

$$(-K_{\widetilde{X}})^2 = K_{\widetilde{X}}^2 = (3L)^2 - 6 = 9 - 6 = 3. \tag{1.1}$$

Here the symbol $(-K_{\widetilde{X}})^2$ is an integer that denotes the *intersection pairing* of $-K_{\widetilde{X}}$ with itself on $\widetilde{X}$ (see [Har77, Chapter 5, Section 1]). This is precisely what we want to see, because once we embed $\widetilde{X}$ into projective space using $|-K_{\widetilde{X}}|$, we will want its degree to be that of a cubic surface: 3. The tricky part in proving that we will get the embedding we are after is now summed up by the following two facts.

(i) the divisor $-K_{\widetilde{X}}$ is very ample, i.e., we obtain an embedding

$$\varphi_{|-K_{\widetilde{X}}|} \colon \widetilde{X} \hookrightarrow \mathbb{P}^N$$

such that $-K_{\widetilde{X}}$ is precisely a hyperplane section under this embedding.

(ii) The dimension of $|-K_{\widetilde{X}}|$ is 3, i.e., the integer N in the embedding above is exactly 3.

The two facts above are proved in [Har77, Chapter 5, Corollary 4.7].

The converse direction of Theorem 1.3 is much harder, but Harshorne argues that, at least, there cannot be many more cubic surfaces. See [Har77, Chapter 5, Remark 4.7.2].

2. POSSIBLE GOALS FOR A REPORT

A report on this topic could dive into the proof of Theorem 1.1. As already indicated above, a few facts should be taken for granted. The existence of an intersection pairing on surfaces, as shown in [Har77, Chapter 5, Section 1] should be made intuitive, but it would be excessive to prove this in detail. The theory in [Har77, Chapter 5, Section 3] about blow-ups of surfaces and how this plays with divisors can also be taken for granted. A description of how a linear system gives a rational map to projective space and what it means to be very ample should be included in the report. The conditions Hartshorne uses to prove that the linear system he constructs is very ample should probably be taken for granted, or should only very briefly be motivated.

If the student ends up finding the approach described here too technical, she can also have a look at the final chapter in [Edi]. The approach there is more elementary. The advantage of the method we describe here is that it can easily be generalized to count curves on *Del Pezzo surfaces*, which could also be done in a report.

2.1. **Prerequisites.** You should be comfortable with a few things from algebraic geometry, in particular the notion of very ample divisors and the intersection pairing on algebraic surfaces. This material can all be found in [Har77]. This is a hard (but also very beautiful) project and should not be underestimated!

REFERENCES

[BTL] Martin Bright, Damiano Testa, and Ronald van Luijk. *Geometry and Arithmetic of Surfaces*. URL: `http://www.boojum.org.uk/maths/book.html`.

[Edi] Bas Edixhoven. *Algebraic Geometry*. URL: `https://webspace.science.uu.nl/~kool0009/AG_notes.pdf`.

[Har77] Robin. Hartshorne. *Algebraic geometry*. Vol. 52. Graduate texts in mathematics. New York: Springer, 1977.

[Poo17] Bjorn Poonen. *Rational points on varieties*. Vol. 186. Graduate Studies in Mathematics. Providence, Rhode Island: American Mathematical Society, 2017.