Fano Scheme of Lines

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$k = \bar{k}$ a field, $\mathbb{P} := \mathbb{P}^{n+1}$

$X \hookrightarrow \mathbb{P}$ closed subscheme, $0 \leq m \leq n+1$ an int.

Def. (Fano functor of m-planes in X)

$$\underline{F}(X,m) : (sch/k)^{op} \longrightarrow Set$$

$$\begin{array}{c} T \\ \downarrow \\ Speck \end{array} \mapsto \left\{ \begin{array}{l} L \hookrightarrow X_T,\ L \to T \\ \text{flat, f. pres., } \forall t \in T: \\ L_t \hookrightarrow X_t \hookrightarrow \mathbb{P}_{\kappa(t)} \text{ an} \\ m\text{-plane} \end{array} \right\}$$

↑ copy of $\mathbb{P}^m \hookrightarrow \mathbb{P}^{n+1}$

representable!

Note. $\underline{F}(X,m) \simeq \underline{Hilb}^{P_m}(X)$, $P_m(\ell) = \binom{m+\ell}{m}$.

Exp. $\underline{F}(\mathbb{P},m) = \underline{G}(m,\mathbb{P})$ Grassmann functor.

$$\underline{G}(m,\mathbb{P})(T) = \left\{ \begin{array}{l} L \hookrightarrow \mathbb{P}_T,\ L \to T \text{ flat, f.p.,} \\ \forall t \in T,\ L_t \hookrightarrow \mathbb{P}_{\kappa(t)} \\ m\text{-plane} \end{array} \right\}$$

$$\begin{tikzcd} L \arrow[r, hook] \arrow[dr, "\varphi"'] & \mathbb{P}_T \arrow[d] \\ & T \end{tikzcd}$$

$\mathbb{P}(\mathcal{E}) \hookrightarrow \mathbb{P}_T$

↑

$\mathcal{O}_T^{\oplus n+2} \twoheadrightarrow \mathcal{E}$

↓

$\mathcal{O}_T^{\oplus n+2} \twoheadrightarrow \varphi_* \mathcal{O}_L(1)$

$\downarrow \simeq$

$$\left\{ \begin{array}{l} \mathcal{O}_T^{\oplus n+2} \twoheadrightarrow \mathcal{E} \text{ loc.} \\ \text{free of rk. } m+1/T \end{array} \right\}$$

$\left(\rightsquigarrow \underline{G}(m,\mathbb{P}) \simeq \underline{Quot}^{p:=m+1}_{k^{\oplus n+2}/Speck/Speck} \right)$ $Grass_{m+1,n+2}$

We have $\underline{G}(m,\mathbb{P}) = Hom_k(\_, G(m,\mathbb{P}))$.

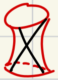
We have $\underline{F}(X,m) \hookrightarrow \underline{G}(m,\mathbb{P})$

$L \hookrightarrow X_T \mapsto L \hookrightarrow X_T \hookrightarrow \mathbb{P}_T$

closed subfunctor.

$\mathbb{P}^3$ segre embedding

Exp. $F(\mathbb{P}^1 \times \mathbb{P}^1) = \mathbb{P}^1 \amalg \mathbb{P}^1$



Relative Fano scheme

$\mathcal{X} \to |\mathcal{O}(d)|$ tautological family of deg. d hypersurf.:

$\mathcal{X} = V(\sum a_I \otimes x^I) \hookrightarrow |\mathcal{O}(d)| \times \mathbb{P}$, where

$$\sum a_I \otimes x^I \in H^0(|\mathcal{O}(d)| \times \mathbb{P}, \mathcal{O}(1) \boxtimes \mathcal{O}(d)).$$

Idea:

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{X} \arrow[d] \\ Speck \arrow[r, "X"] & {|\mathcal{O}(d)|.} \end{tikzcd}$$

Def. (relative Fano functor)

$$\underline{F}(\mathcal{X},m) : (Sch/|\mathcal{O}(d)|)^{opp} \longrightarrow Set.$$

Fact $\underline{F}(\mathcal{X},m)$ is representable by a proj. scheme $F(\mathcal{X},m) \to |\mathcal{O}(d)|$. (e.g. relative Hilb. scheme)

Local theory $X \hookrightarrow \mathbb{P}$ proj. scheme,

$Hilb = Hilb_{X/k}$ the Hilbert scheme.

$q = [Y \hookrightarrow X] \in Hilb(k)$.

Q.1: "Can we compute T_q Hilb?"

Q.2: "When is Hilb smooth at q?"

Let $\mathcal{I} := \ker(\mathcal{O}_X \to \mathcal{O}_Y)$.

Def. The normal sheaf of Y in X is

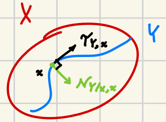
$$N_{Y/X} := \mathcal{H}om_X(\mathcal{I}, \mathcal{O}_Y) = \mathcal{H}om_Y(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_Y) = (\mathcal{I}/\mathcal{I}^2)^\vee.$$

Note: For $Y \hookrightarrow X$ a regular embedding:

$$0 \longrightarrow \mathcal{I}/\mathcal{I}^2 \longrightarrow \Omega^1_{X/k} \otimes \mathcal{O}_Y \longrightarrow \Omega^1_{Y/k} \to 0$$

$$\rightsquigarrow 0 \to \mathcal{T}_{Y/k} \longrightarrow \mathcal{T}_{X/k} \otimes \mathcal{O}_Y \to N_{Y/X} \longrightarrow 0$$

v.b. of rk codim Y in X.



Let $D = k[\varepsilon]/\varepsilon^2$. We have

$$T_q Hilb = \left\{ \varphi : Spec D \to Hilb \;\middle|\; q = Speck \to Spec D \to Hilb \right\}$$

$$= \left\{ \begin{tikzcd} Y' \arrow[r, hook] \arrow[dr, "flat"'] & X' = X \otimes_k D \arrow[d] \\ & Spec D \end{tikzcd} \;\middle|\; Y = Y' \otimes_D k \right\}.$$

↑ deformation of Y.

Q.1.

Prop. $T_q Hilb \simeq Hom_{\mathcal{O}_X}(\mathcal{I}, \mathcal{O}_Y) = H^0(Y, N_{Y/X})$.

Proof (sketch) See also diagram

Consider split s.e.s.

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O}_Y \arrow[r, "\cdot \varepsilon"] & (\mathcal{I} + \varepsilon \mathcal{O}_X)/\varepsilon \mathcal{I} \arrow[r] & \mathcal{I} \arrow[r] \arrow[l, dashed, bend right] & 0 \end{tikzcd} \quad (*)$$

$\subset \mathcal{O}_{X'}/\varepsilon\mathcal{I}$

$$\left\{ \begin{array}{c} \text{def's} \\ Y' \hookrightarrow X' \text{ of} \\ Y \hookrightarrow X \end{array} \right\} \simeq \left\{ \begin{array}{l} \text{Sections} \\ \text{of } (*) \end{array} \right\} = Hom_{\mathcal{O}_X}(\mathcal{I}, \mathcal{O}_Y)$$

☒

Q.2.

A local Artin k-algebra, $A' \to A'/I = A$

w. $A/\mathfrak{m}_A = k$

"small extension", i.e., $I \cdot \mathfrak{m}_{A'} = 0$.

Hilb (formally) smooth at q $\Leftrightarrow$ Every def. $\begin{tikzcd} Y_A \arrow[r, hook] \arrow[dr] & X_A \arrow[d] \\ & SpecA \end{tikzcd}$ is liftable to A':

$$\begin{tikzcd} X \arrow[r] & X_A \arrow[r] & X_{A'} \\ Y \arrow[u, hook] \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & Y_A \arrow[u, hook] \arrow[r, dashed] \arrow[d] \arrow[dr, phantom, "\square"] & Y_{A'} \arrow[u, dashed, hook] \arrow[d, dashed] \\ Speck \arrow[r] & SpecA \arrow[r] & SpecA' \end{tikzcd}$$

generalize ans. to Q1.

Fact: "Obstruction to liftability" lives in

$Ext^1_X(\mathcal{I}, \mathcal{O}_Y \otimes_k I)$.

Suppose now $Y \hookrightarrow X$ regular embedding. We have

$$E_2^{pq} = H^p(X, \mathcal{E}xt_X^q(\mathcal{I}, \mathcal{O}_Y \otimes_k I)) \Rightarrow \mathrm{Ext}_X^{p+q}(\mathcal{I}, \mathcal{O}_Y)$$

$$\rightsquigarrow 0 \to H^1(Y, N_{Y/X} \otimes_k I) \to \mathrm{Ext}^1(\mathcal{I}, \mathcal{O}_Y \otimes_k I)$$

$$\to H^0(Y, \underset{\substack{\uparrow \\ \text{local obstr.} \\ \text{vanish!}}}{\mathcal{E}xt^1}(\mathcal{I}, \mathcal{O}_Y \otimes_k I))$$

$\Rightarrow$ obstr. come from $H^1(Y, N_{Y/X}) \otimes_k I$

Prop. For $[L] \overset{X \text{ regular along } L}{\in} F(X,m)(k)$ we have $T_{[L]} F(X,m) = H^0(L, N_{L/X})$ and $H^1(L, N_{L/X}) = 0 \Rightarrow F(X,m)$ is smooth at $[L]$. $\boxtimes$

Exp. $L = \mathbb{P}(W) \hookrightarrow \mathbb{P}(V)$ a point of $\mathbb{G}(m,\mathbb{P})$.

Let $\mu : GL(V) \to \mathbb{G}(m,\mathbb{P})$. $\rightsquigarrow$ submersion

$A \mapsto AW$

Differentiating at 1:

$$\begin{tikzcd}
\mathfrak{gl}(V) \arrow[r] \arrow[dr] \arrow[dr, phantom, shift left=3, "\oslash"] & T_L \mathbb{G}(m,\mathbb{P}) \\
& \mathrm{Hom}(W, V/W) \arrow[u, dashed, "\simeq"']
\end{tikzcd}$$

From our method we also have

$$T_L \mathbb{G}(m,\mathbb{P}) = H^0(L, N_{L/\mathbb{P}})$$

Smoothness for the relative Fano scheme

Prop. $F(\mathcal{X}, m) \to |\mathcal{O}(d)|$ is smooth at $[L] \in \underbrace{F(X,m) \hookrightarrow}_{X \text{ smooth along } L} F(\mathcal{X}, m)$ if $H^1(X, N_{L/X}) = 0$. $\boxtimes$

Normal bundles of lines

$L \overset{\simeq \mathbb{P}^m}{\subset} X$ an m-plane, X smooth along L.

Consider s.e.s.

$$0 \to N_{L/X} \to N_{L/\mathbb{P}} \to N_{X/\mathbb{P}} \otimes \mathcal{O}_L \to 0$$

- $N_{L/\mathbb{P}}$:

$$\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{O}_L \arrow[r, "="] \arrow[d] & \mathcal{O}_L \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}_L(1)^{\oplus m+1} \arrow[r] \arrow[d] & \mathcal{O}_L(1)^{\oplus n+2} \arrow[r] \arrow[d] & \mathcal{O}_L^{\oplus n+1-m} \arrow[r] \arrow[d, "\simeq"'] & 0 \\
0 \arrow[r] & T_L \arrow[r] \arrow[d] & T_{\mathbb{P}} \otimes \mathcal{O}_L \arrow[r] \arrow[d] & N_{L/\mathbb{P}} \arrow[r] \arrow[d] & 0 \\
& 0 & 0 & 0 &
\end{tikzcd}$$

- $\overset{\text{smooth}}{X} = V(F) \hookrightarrow \mathbb{P}$, $F \in \Gamma(\mathbb{P}, \mathcal{O}(d))$.

$$N_{X/\mathbb{P}} = (\mathcal{I}_X / \mathcal{I}_X^2)^\vee = \mathcal{O}_X(d)$$

$$\rightsquigarrow N_{X/\mathbb{P}} \otimes \mathcal{O}_L = \mathcal{O}_L(d).$$

We get

$$(*) \quad 0 \to N_{L/X} \to \mathcal{O}_L(1)^{\oplus n+1-m} \to \mathcal{O}_L(d) \to 0$$

We compute

$$\det(N_{L/X}) = \det(\mathcal{O}_L(1)^{\oplus n+1-m}) \otimes \mathcal{O}_L(d)^\vee$$
$$= \mathcal{O}_L(n+1-m-d),$$

and

$$\chi(N_{L/X}) = \chi(\mathcal{O}_L(1)) \cdot (n+1-m) - \chi(\mathcal{O}(d))$$
$$= (m+1)(n+1-m) - \binom{m+d}{d}.$$

Specializing to $m=1$, $d=3$:

Prop. $L \subseteq X$ a line in X, sm. cubic hypersurf. Then $N_{L/X} \simeq \mathcal{O}_L(a_1) \oplus \cdots \oplus \mathcal{O}_L(a_{n-1})$, with

$$(a_1, \dots, a_{n-1}) = \begin{cases} (1, \dots, 1, 0, 0); & \text{type I} \\ (1, \dots, 1, 1, -1). & \text{type II} \end{cases}$$

Proof $\mathrm{rk}\, N_{L/X} = n-1$ + v.b on $\mathbb{P}^1$ is $\oplus$ of $\mathcal{O}(a)$'s. $\det(N_{L/X}) = \mathcal{O}(\sum a_i) = \mathcal{O}(n-3)$

$\Rightarrow \sum a_i = n-3$. Finally, $N_{L/X} \hookrightarrow \mathcal{O}(1)^{\oplus n}$

$\Rightarrow a_i \leq 1 \ \forall i$. $\boxtimes$

Exp.

- $n=2 \Rightarrow$ all lines are of type II.
- $n=3$: $X \overset{\subseteq \mathbb{P}^3_{\mathbb{C}}}{:} x_0^2 x_2 + x_0 x_1 x_3 + x_1^2 x_4 + x_2^3 + x_3^3 + x_4^3 = 0$,

  $L : x_2 = x_3 = x_4$ line on X.

  Then L is of type I.

Cor. (i) For $X = V(F) \overset{\subseteq \mathbb{P}}{}$ sm. cubic, $F(X)$ smooth. Furthermore, if $F(X) \neq 0$,

$$\dim F(X) = 2n-4.$$

(ii) $F(\mathcal{X}) \longrightarrow |\mathcal{O}_{\mathbb{P}}(3)|$ is smooth over dense open of smooth hyp. surf's. $|\mathcal{O}(3)|_{sm}$

Proof. For $L \subseteq X$ a line

$$H^1(L, N_{L/X}) = 0 \quad + \quad h^0(L, N_{L/X}) = \chi(N_{L/X})$$
$$= (1+1) \cdot (n+1-1) - \binom{1+3}{3}$$
$$= 2n-4.$$

Prop. $X \subseteq \mathbb{P}^{n+1}$ cubic hyp. surf., $n \geq 2$.
Then $F(X) \neq \emptyset$.

Proof. $(3 \neq 0)$ Consider $X_0 : x_0^3 + \ldots + x_{n+1}^3 = 0$.

X_0 contains $L_0 : x_0 + x_1 = x_2 + x_3 = x_4 = \ldots = x_{n+1} = 0$.

so $F(X_0) \neq 0$ + $F(X) \to |\mathcal{O}(3)|$ smooth at L_0 $\Rightarrow$ $\operatorname{im}(F(X) \to |\mathcal{O}(3)|)$ is closed + contains dense open $\Rightarrow$ surj.! $\boxtimes$

Cor. $X \subseteq \mathbb{P}^{n+1}$ sm. cubic hyp. surf., $n > 1$.
There exists

$$\mathbb{P}^n \dashrightarrow\!\!\!\!\!\rightarrow X$$

dominant rational of deg. 2.

Proof. (Sketch) $L \subseteq X$ a line. Consider

$\mathbb{P}(T_X \otimes \mathcal{O}_L) \longrightarrow L$. **Note:** this thing is rational. For $0 \neq v \in T_\ell X$ $(\ell \in L)$ a point of $\mathbb{P}(T_X \otimes \mathcal{O}_L)$, let L_v line through ℓ with $T_\ell L_v$ spanned by v.

Generically: $L_v \cap X = 2\ell + y_v$.

Define

$$\begin{aligned} \mathbb{P}(T_X \otimes \mathcal{O}_L) &\dashrightarrow X \\ v &\mapsto y_v. \end{aligned}$$

$(\boxtimes)$