Strong approximation for ℓ-adic local systems (joint w. M. Kerz)

§0 Prelims

Thm ⊛ $\varphi : Y \to X$ morph. of smooth (q.proj.) $\mathbb{C}$-var's, $\rho : \pi_1(X(\mathbb{C})) \to GL_r(\mathbb{C})$ a s.simple repr. Then also

$$\varphi^*\rho : \pi_1(Y(\mathbb{C})) \xrightarrow{\varphi_*} \pi_1(X(\mathbb{C})) \to GL_r(\mathbb{C})$$

is s.simple.

Conj. ⊛ K sep. closed field, ℓ prime $\neq 0$ in K, $\varphi : Y \to X$ morph. of smooth K-var's, $\rho : \pi_1^{\acute{e}t}(X) \to GL_r(\bar{\mathbb{Q}}_\ell)$ s.simple. Then $\varphi^*\rho$ is s.simple.

First consider $K = \bar{\mathbb{F}}_p$. Say $X = X_0 \otimes_{\mathbb{F}_q} \bar{\mathbb{F}}_p$, $X_0 \to \operatorname{Spec} \mathbb{F}_q$. Have s.e.s.

$$1 \to \pi_1(X) \to \pi_1(X_0) \to Gal(\bar{\mathbb{F}}_p/\mathbb{F}_q) \to 1.$$

Get outer action

$$\underset{\text{Frob.}}{\Phi} \in Gal(\bar{\mathbb{F}}_p/\mathbb{F}_q) \curvearrowright \pi_1(X)$$

$\rho : \pi_1(X) \to GL_r(\bar{\mathbb{Q}}_\ell)$ is arithmetic if $\rho^{\Phi^e} \simeq \rho$ for some $e \geq 1$.

Prop. (DL) Conj. ⊛ holds for ρ simple arithmetic.

Proof ρ ar. $\Rightarrow \exists\, \rho_0 : \pi_1(X_0) \to GL_r(\bar{\mathbb{Q}}_\ell)$ s.t. $\rho_0|X \simeq \rho$ and ρ_0 pure of weight 0 (Langlands for GL_r/f.f.) $\Rightarrow \rho_0|Y_0$ pure of weight 0 $\Rightarrow \rho|Y$ s.simple (Weil II) ⊠

§1. Sketch of proof à la Drinfeld

Fix $y \in Y(\mathbb{C})$, find $R \subset \mathbb{C}$ fin. type$/\mathbb{Z}$ s.t. φ, $y \to Y$ spread:

$$\begin{tikzcd}
Y \arrow[r] \arrow[d] \arrow[rrr, bend left, "\varphi"] & Y_S \arrow[r, "\varphi_S"] \arrow[d] & X_S \arrow[d] & X \arrow[l] \arrow[d] \\
\operatorname{Spec} \mathbb{C} \arrow[r, "\bar\eta"] \arrow[u, bend left, "y"] & S \arrow[u, bend right, "y_S"'] \arrow[r, equal] & S & \operatorname{Spec} \mathbb{C} \arrow[l, "\bar\eta"'] \\
 & \operatorname{Spec} R \arrow[u, equal] & &
\end{tikzcd}$$

For $\bar s \to S$ a closed geom. pt., pick $\bar\eta \rightsquigarrow \bar s$ spec., get

$$sp : \pi_1(X(\mathbb{C})) \to \hat{\pi}_1(X(\mathbb{C})) \twoheadrightarrow \pi_1^t(X_{\bar s}).$$

For $\ell \in k(s)^\times$ prime #, $\tau : \bar{\mathbb{Q}}_\ell \simeq \mathbb{C}$, $\rho_{\bar s} : \pi_1^t(X_{\bar s}) \to GL_r(\bar{\mathbb{Q}}_\ell)$, get $\rho := (sp^*\rho)^\tau : \pi_1(X(\mathbb{C})) \to GL_r(\mathbb{C})$.

Def. If $\rho_{\bar s}$ arithmetic, we say ρ is weakly arithmetic.

Prop. Thm. ⊛ holds for ρ weakly arithmetic simple.

Proof

$\varphi^*\rho$ simple $\Longleftarrow$ $\varphi_{\bar s}^*\rho_{\bar s}$ simple

$$\begin{tikzcd}
\pi_1(Y(\mathbb{C})) \arrow[r, "sp"] \arrow[d, "\varphi_*"'] \arrow[dr, phantom, "\circlearrowleft"] & \pi_1^{\acute{e}t}(Y_{\bar s}) \arrow[d, "\varphi_{\bar s *}"] \\
\pi_1(X(\mathbb{C})) \arrow[r, "sp"] & \pi_1^{\acute{e}t}(X_{\bar s})
\end{tikzcd}$$

Prop. DL

ρ simple $\Longrightarrow$ $\rho_{\bar s}$ simple

⊠

Write $M_B^{\square}(X,r)^{irr} = M_B^{\square} \to \operatorname{Spec} \mathbb{Z}$ for "framed Betti-moduli space of π_1":

$$M_B^{\square}(\mathbb{C}) = \{\rho : \pi_1(X(\mathbb{C})) \to GL_r(\mathbb{C}) \text{ simple}\}$$

Note $PGL_r \curvearrowright M_B^{\square}$. Set

$$\begin{array}{c} M_B^{\square} \\ \downarrow \pi \\ M_B(X,r) = M_B := M_B^{\square}/PGL_r. \end{array}$$

Proof of Thm. ⊛

Set $Z^{\square} = \{\rho \in M_B^{\square}(\mathbb{C}) \mid \varphi^*\rho \text{ not s.simple}\}$. Set $Z = \pi(Z^{\square})$.

Goal: $Z = \emptyset$. Assume $Z \neq \emptyset$.

Claim: $Z^{\square}$ is constructible in Zar. top. of $M_B^{\square}$.

Proof uses univ. repr. on $M_B^{\square}$. ⊠

$\Rightarrow Z$ constr. (Chevalley's thm.)

Claim: Weakly arithmetic points of $\bar Z$ are Zar. dense.

Proof De Jong's conjecture. ⊠

Z constr. $\Rightarrow \exists\, U \subset Z \subset \bar Z$ w. U dense open in $\bar Z$. U contains no w. ar. points ↯ ⊠

§2. Strategy in pos. char.

Prop. Conj. (*) over $\overline{\mathbb{F}}_p$ $\Rightarrow$ Conj. over any K.

Proof Alterations + specialization argument. $\boxtimes$

Problem: No Betti moduli for X.

After reframing ρ:

$$\begin{tikzcd} \pi_1^{ét}(X) \arrow[rr, "\rho"] \arrow[dr] & \circleddash & GL_r(\overline{\mathbb{Q}}_\ell) \\ & GL_r(\mathcal{O}_E) \arrow[ur] & \end{tikzcd}$$

$\mathbb{Q}_\ell \overset{f}{\subset} E \subset \overline{\mathbb{Q}}_\ell$

$\mathbb{F} = \mathcal{O}_E/\mathfrak{m}_E$.

ρ is a deformation of $\bar\rho = \rho \otimes_{\mathcal{O}_E} \mathbb{F}$.

Problem $\bar\rho$ is not well-defined if $\bar\rho$ not abs. simple.

Exp. $\hat{\mathbb{Z}} \to GL_2(\overline{\mathbb{Q}}_\ell)$, $1 \mapsto \begin{pmatrix}1&1\\0&1\end{pmatrix}$ $\simeq$ $\hat{\mathbb{Z}} \to GL_2(\overline{\mathbb{Q}}_\ell)$, $1 \mapsto \begin{pmatrix}1&\ell\\0&1\end{pmatrix}$

However, $\mathrm{Det}\,\bar\rho$ is well-defined.

Def A a ring, G a group. An A-valued *determinant* on G of rk. r is a compatible family of multiplicative maps

$$D_B : B[G] \to B \quad \forall A \to B \text{ comm. } A\text{-alg.}$$

s.t. $D_B(b \cdot x) = b^r D_B(x)$ for all $b \in B$, $x \in D_B[G]$.

Exp. $\rho : A[G] \to M_r(A) \xrightarrow{\det} A$ repr.

$\rightsquigarrow$ $\mathrm{Det}\,\rho$ is a determinant.

$G = \pi_1^{ét}(X)$, $\bar D = \mathrm{Det}\,\bar\rho$. e.g. $\mathcal{O}, \mathcal{O}[[t]], \mathcal{O}\langle t\rangle$

Let $\mathcal{C}$ = cat. of admissible $\mathcal{O} \overset{=}{} \mathcal{O}_E$-algebras

Consider

$$\mathcal{P}sR_{\bar D} : \mathcal{C} \to \text{Sets}$$
$$A \mapsto \left\{ \begin{array}{l} \text{cont. } D : A[G] \to A : \\ D \text{ res. const. equal } \bar D \end{array} \right\}$$

Prop. $\mathcal{P}sR_{\bar D} = \mathrm{Spf}\, R_{\bar D}$ w. $R_{\bar D} \overset{\in \mathcal{C}}{\twoheadleftarrow} \mathcal{O}[[t_1, \dots, t_n]]$

Define $\mathcal{S}_{\bar\rho} = \left\{ \begin{array}{l} \rho : G \to GL_r(\overline{\mathbb{Q}}_\ell) \text{ s. simple}: \\ \mathrm{Det}\,\rho \otimes_{\mathbb{Z}_\ell} \overline{\mathbb{F}}_\ell = \bar D \otimes_{\mathbb{F}} \overline{\mathbb{F}}_\ell \end{array} \right\} / \simeq$

Then

$$\mathcal{S}_{\bar\rho} \underset{\mathrm{Det}}{\xrightarrow{\simeq}} \varinjlim_{E \subset E'} \mathrm{Hom}^{cont}_{\mathcal{O}_E}(R_{\bar D}, \mathcal{O}_{E'})$$

(Gabber-Loeser) $\xrightarrow{\simeq} \mathrm{Hom}_{\overline{\mathbb{Q}}_\ell}(R_{\bar D} \otimes_{\mathcal{O}_E} \overline{\mathbb{Q}}_\ell, \overline{\mathbb{Q}}_\ell)$

$\xrightarrow{\simeq} \mathrm{Spm}\, R_{\bar D} \otimes \overline{\mathbb{Q}}_\ell$

has Zar. top.

After enlarging $\mathbb{F}_q$: $\bar\rho^{\Phi} \simeq \bar\rho$

$\Rightarrow$ $\Phi : R_{\bar D} \to R_{\bar D}$

$\Rightarrow$ $\Phi : \mathcal{S}_{\bar\rho} \to \mathcal{S}_{\bar\rho}$ cont.

Set $\mathcal{A}_{\bar\rho} = \bigcup_{e \geq 1} \mathcal{S}_{\bar\rho}^{\Phi^e} \subset \mathcal{S}_{\bar\rho}$ (arithm. pts.)

Def. $Z \subset \mathcal{S}_{\bar\rho}$ is *special* if $\Phi^e(Z) = Z$ for some $e \geq 1$.

We say ρ has (SAP) if for all $\rho \in Z \subset \mathcal{S}_{\bar\rho}$ special closed, ρ is an acc. pt. of $Z \cap \mathcal{A}_{\bar\rho}$.

Conj. (Esnault-Kerz) Every ρ has (SAP).

Known cases

- ρ rank 1 and X proper (?)
- ρ tame rk 2 on $\mathbb{P}^1 \setminus \{0, 1, \infty\}$.

Define

$$\mathcal{R}ep^\square_{\bar D} : \mathcal{C} \to \text{Sets}$$
$$A \mapsto \left\{ \begin{array}{l} \rho : A[G] \to M_r(A) : \\ \mathrm{Det}\,\rho \in \mathcal{P}sR_{\bar D}(A) \end{array} \right\}$$

Have natural transf.

$$\mathcal{R}ep^\square_{\bar D} \xrightarrow{\mathrm{Det}} \mathcal{P}sR_{\bar D}$$

Prop. (1) $\mathcal{R}ep_{\bar D} = \mathrm{Spf}\, R^\square_{\bar D}$ w. $R^\square_{\bar D} \in \mathcal{C}$ noeth.

(2) $\pi : \mathrm{Spec}\, R^\square_{\bar D} \otimes \overline{\mathbb{Q}}_\ell \to \mathrm{Spec}\, R_{\bar D} \otimes \overline{\mathbb{Q}}_\ell$ is surj.

Upshot: Have univ. repr.

$$\rho^u_{\bar D} : G \to GL_r(R^\square_{\bar D}).$$

Thm. Conj. (*) holds for ρ simple w. SAP

Proof sketch Set

$$Z^\square = \left\{ \begin{array}{l} \mathfrak{p} \in \mathrm{Spec}\, R^\square_{\bar D} \otimes \overline{\mathbb{Q}}_\ell : \rho^u_{\bar D} \otimes \overline{k(\mathfrak{p})} \text{ simple,} \\ \text{but } \rho^u_{\bar D} \otimes \overline{k(\mathfrak{p})}\,|_Y \text{ not s.simple} \end{array} \right\}$$

$Z^\square$ is constr. $\Rightarrow$ $Z = \pi(Z^\square)$ is constr.

$\pi^{-1} Z = Z^\square$.

Furthermore, $Z(\overline{\mathbb{Q}}_\ell) \subset \mathcal{S}_{\bar\rho}$ special.

$\exists U \overset{\text{special}}{\subset} Z(\overline{\mathbb{Q}}_\ell) \subset \overline{Z}(\overline{\mathbb{Q}}_\ell)$ dense open

Spse $\rho \in Z(\overline{\mathbb{Q}}_\ell)$. $\rho \in U \rightsquigarrow \lightning$ by SAP.

Else: $\rho \in Z(\overline{\mathbb{Q}}_\ell) \setminus U \overset{\text{Noeth. ind.}}{\rightsquigarrow} \lightning$ $\boxtimes$