Topics in Algebraic Geometry: Smooth, étale and unramified morphisms

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Throughout, A will always denote a commutative ring, and k a field.

1 Smooth morphisms

Definition 1.

Let $r, n \geq 0$ be integers, let $g_{r+1}, \dots, g_n \in A[x_1, \dots, x_n]$, and write $B = A[x_{r+1}, \dots, x_n]/(g_{r+1}, \dots, g_n)$. Then B is called standard smooth over A if for all $x \in \operatorname{Spec} B$ the matrix

$$\left(\frac{\partial g_i}{\partial x_j}(x) \right) \in \operatorname{Mat}_{(n-r)\times n}(\kappa(x))$$

is of rank $n - r$.

Definition 2. *A map of schemes $\varphi : X \to Y$ is called smooth if for all $x \in X$ there are opens $U \subset X$ and $V \subset Y$ such that $x \in U$, $\varphi(U) \subset V$, and $U \to V$ is isomorphic to* $\operatorname{Spec} B \to \operatorname{Spec} A$*, with B standard smooth over A.*

Remark 3.

- The integer r in the definition of standard smoothness is the relative dimension of the map $\operatorname{Spec} B \to \operatorname{Spec} A$.
- An A-algebra $A \to B = A[x_1, \dots, x_n]/(g_{r+1}, \dots, g_n)$ is standard smooth if and only if $\operatorname{Spec} B \to \operatorname{Spec} A$ is smooth.

Remark 4. A map of schemes $\varphi : X \to Y$ is smooth if and only if φ is formally smooth and locally of finite presentation. See [Sta23, Tag 02H6].

Example 5.

(i) The maps $\mathbb{A}^n_k \to \operatorname{Spec} k$ and $\mathbb{P}^n_k \to \operatorname{Spec} k$ are smooth.

(ii) Consider the affine patch of the blow-up of the plane in the origin

$$\pi : \operatorname{Spec} k[u, v] \to \operatorname{Spec} k[x, y],$$

defined by $x \mapsto u, y \mapsto uv$. Then π is not smooth. Indeed, we have a commutative diagram

$$\begin{array}{ccc} \operatorname{Spec} k[u,v] & \xrightarrow{\pi} & \operatorname{Spec} k[x,y] \\ \downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle =} \\ \operatorname{Spec} k[x,y][v]/(xv - y) & \longrightarrow & \operatorname{Spec} k[x,y] \end{array}$$

and it's easy to see that the lower horizontal map is not standard smooth, because $\frac{\partial}{\partial v}(xv - y) = x$ is not a unit. By Remark 3, we are done.

In the second example above it should be noted that π does have smooth fibers. This shows that it is in general not enough to check smoothness on the fibers. This turns out te be enough if we additionally assume that our map is flat. This is summarized in the theorem below, which is often taken as the definition of smoothness (including on the website).

Theorem 6. *A map of schemes $\varphi : X \to Y$ is smooth if and only if φ is flat, locally of finite presentation, and its fibers are smooth.*

Proof. The fact that any morphism which is flat, locally of finite presentation, with smooth fibers, is smooth, is the content of [Sta23, Tag 01V8]. Conversely, a smooth morphism is clearly locally of finite presentation, has smooth fibers by the fact that smoothness is stable under base change, and is flat by [Sta23, Tag 01VF]. ■

Example 7.
(i) The map $\mathrm{Bl}_O \mathbb{A}^2_k \to \mathbb{A}^2_k$ is not smooth, because it is not flat (see the notes of my last talk).
(ii) The family of conics $\mathbb{A}^2_k = \operatorname{Spec} k[x, y] \to \operatorname{Spec} k[t]$ defined by $t \mapsto xy$ is not smooth, because the fiber over 0 is not smooth.

2 Étale and unramified morphisms

Definition 8. *A map of schemes $\varphi : X \to Y$ is called unramified if it is locally of finite type, and for all $x \in X$ we have*

- $\varphi^\#_x \mathfrak{m}_{\varphi(x)} = \mathfrak{m}_x$*;*
- $\kappa(x)/\kappa(\varphi(x))$ *is a finite separable extension.*

Remark 9. A map $\varphi : X \to Y$ is unramified if and only if it is formally unramified and locally of finite type. See [Sta23, Tag 02HE].

We start with two number-theoretic examples before we move onto geometric examples.

Example 10.
(i) Consider the extension of local fields $\mathbb{Q}_3 \subset \mathbb{Q}_3(\sqrt{2})$. Then the induced map
$$\operatorname{Spec} \mathbb{Z}_3[\sqrt{2}] \to \operatorname{Spec} \mathbb{Z}_3$$
of the spectra of the valuation rings is unramified. Indeed, it is of finite type, the prime 3 is inert, and the extension of residue fields $\mathbb{F}_3 \subset \mathbb{F}_3(\sqrt{2})$ is separable.
(ii) Consider the extension of local fields $\mathbb{Q}_3 \subset \mathbb{Q}_3(\zeta_3)$. Then the induced map
$$\operatorname{Spec} \mathbb{Z}_3[\zeta_3] \to \operatorname{Spec} \mathbb{Z}_3$$
is ramified. Indeed, the prime 3 now splits as
$$3\mathbb{Z}_3[\zeta_3] = (1 - \zeta_3)^2 \mathbb{Z}_3[\zeta_3].$$

Example 11. The map $\varphi : \mathbb{A}^1_k \to \mathbb{A}^1_k$ defined by $t \mapsto t^2$ is ramified. Indeed, we have

$$\varphi^\#_{(t)}((t)) = t^2 \mathscr{O}_{\mathbb{A}^1_k,(t)} \neq t\mathscr{O}_{\mathbb{A}^1_k,(t)}.$$

Lemma 12. *Suppose $\varphi : X \to Y$ is locally of finite type. Then φ is unramified if and only if its fibers are.*

Proof. Let $x \in X$ and set $y = \varphi(x)$. Then we have

$$\mathscr{O}_{X_y,x} = \mathscr{O}_{X,x} \otimes_{\mathscr{O}_{Y,y}} \kappa(y) = \mathscr{O}_{X,x}/\mathfrak{m}_y\mathscr{O}_{X,x}.$$

The map of local rings $\kappa(y) \to \mathscr{O}_{X,x}/\mathfrak{m}_y\mathscr{O}_{X,x}$ is unramified if and only if $\mathfrak{m}_x/\mathfrak{m}_y\mathscr{O}_{X,x} = 0$ and the extension $\kappa(x)/\kappa(y)$ is finite separable; i.e. if and only if $\varphi^\#_x : \mathscr{O}_{Y,y} \to \mathscr{O}_{X,x}$ is unramified. The result follows. ■

Proposition 13. *A map $\varphi : X \to Y$, locally of finite type, is unramified if and only if for all $x \in X$ the fiber of φ over X is a disjoint union of spectra of finite separable extensions of k.*

Proof. By Lemma 12, we only need to show that a finite type map $X \to \operatorname{Spec} k$ is unramified if and only if X is a disjoint union of spectra of finite separable extensions of k. Suppose $X \to \operatorname{Spec} k$ is unramified. Let $U = \operatorname{Spec} A$ be an affine open of X. Then for all $x \in U$, the ring $\mathscr{O}_{X,x}$ is a finite separable extension of k. In particular, A is Artinian, and hence a finite product $A = \prod_{i \in I} A_i$ of local Artinian rings. Each of the A_i is also a finite separable extension of k.

Conversely, a disjoint union of spectra of finite separable extensions of k is clearly unramified over $\operatorname{Spec} k$. ■

Example 14. The map $\operatorname{Spec} k \to \operatorname{Spec} k[\varepsilon]/(\varepsilon^2)$ is unramified.

Definition 15. *A map $\varphi : X \to Y$, locally of finite presentation, is called étale*[1] *if φ is flat and unramified.*

Remark 16. A map $\varphi : X \to Y$ is étale if and only if it is formally étale and locally of finite presentation. See [Sta23, Tag 02HM].

Example 17. The map $\operatorname{Spec} k \to \operatorname{Spec} k[\varepsilon]/(\varepsilon^2)$ is not étale.

Corollary 18. *If $\varphi : X \to Y$ is étale, then φ is smooth of relative dimension zero.*

Proof. If $\varphi : X \to Y$ is étale, then it is locally of finite presentation, flat, and its fibers are smooth of dimension zero by Proposition 13. ■

Remark 19. The converse of the Corollary above is also true.

[1]The word étale means something along the lines of 'calm', or 'immobile'. It is not to be confused with the word étalé, which means 'spread out'.

Étale maps should be thought of as maps which are local isomorphisms. This intuition can be made more precise for maps of $\mathbb{C}$-varieties: a map of $\mathbb{C}$-varieties $\varphi : X \to Y$ is étale if and only if the induced map $\varphi^{\mathrm{an}} : X^{\mathrm{an}} \to Y^{\mathrm{an}}$ is a local isomorphism of analytic spaces.

A given analytic space X is smooth if and only if it is locally isomorphic to an open of $\mathbb{C}^n$. The algebraic converse to this statement is wrong: A smooth algebraic $\mathbb{C}$-variety is not always locally isomorphic to an open of $\mathbb{A}^n_{\mathbb{C}}$. The Zariski topology is simply too coarse for this. We do, however, get a true statement if we take 'locally' to mean 'étale locally', instead of 'Zariski locally'. The following proposition makes this precise.

Proposition 20. *A map $\varphi : X \to Y$ is smooth if and only if for all $x \in X$ there are opens $U \subset X$, $V \subset V$ such that $x \in U$, $\varphi(U) \subset V$, and we have a commutative diagram*

$$\begin{tikzcd} X \arrow[d, "\varphi"] & U \arrow[l] \arrow[r] \arrow[d] & \mathbb{A}^r_V \arrow[dl] \\ Y & V, \arrow[l] & \end{tikzcd}$$

with $U \to \mathbb{A}^r_V$ étale.

Proof. Suppose we have a standard smooth map

$$A \to A[x_1, \dots, x_n]/(g_{r+1}, \dots, g_n) = B.$$

Without loss of generality, we can assume that

$$\det \begin{pmatrix} \frac{\partial g_{r+1}}{\partial x_{r+1}} & \cdots & \frac{\partial g_{r+1}}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial g_n}{\partial x_{r+1}} & \cdots & \frac{\partial g_n}{\partial x_n} \end{pmatrix} \neq 0.$$

Then the map

$$A[x_1, \dots, x_r] \to B$$

is also standard smooth, and the induced map $\operatorname{Spec} B \to \mathbb{A}^r_A$ is smooth of relative dimension zero; hence, by Remark 19, this map is étale. We now have a commutative triangle

$$\begin{tikzcd} \operatorname{Spec} B \arrow[r] \arrow[d] & \mathbb{A}^n_A \arrow[dl] \\ \operatorname{Spec} A & \end{tikzcd}$$

with the top arrow étale. It follows that any smooth map $\varphi : X \to Y$ locally factors through affine space via an étale map.

For the converse, simply note that $\mathbb{A}^r_V \to V$ is smooth, and that a composition of smooth maps is smooth. ∎

3 Smoothness in terms of the sheaf of relative differentials

We recall the following fact about the sheaf of relative differentials.

Proposition 21. *For a map of schemes $\varphi : X \to Y$ over a base scheme S, we have an exact sequence*

$$\varphi^* \Omega^1_{Y/S} \to \Omega^1_{X/S} \to \Omega^1_{X/Y} \to 0.$$

Theorem 22. *Let $\varphi : X \to Y$ be flat and locally of finite presentation. Then φ is smooth if and only if the sheaf of relative differentials $\Omega^1_{X/Y}$ is locally free of rank $\dim_x \varphi$ for all $x \in X$.*

Proof. See [Sta23, Tag 02G1] and [Sta23, Tag 01V9]. ■

Example 23. Set $A = k[x_1, \ldots, x_n]$. Then $\operatorname{Spec} A = \mathbb{A}^n_k \to \operatorname{Spec} k$ is smooth. Indeed, the sheaf of differentials

$$\Omega^1_{\mathbb{A}^n_k/k} = A dx_1 \oplus \ldots \oplus A dx_n$$

is free of rank n, which is precisely the dimension of $\mathbb{A}^n_k$.

Example 24. Suppose $2 \neq 0$ in k. Consider the map

$$\varphi : \mathbb{A}^1_k(t) \to \mathbb{A}^1_k(s)$$

defined by $s = t^2$. Notice that it is of relative dimension zero. We set out to compute the sheaf of relative differentials Ω^1_φ. By Proposition 21 we have

$$\Omega^1_\varphi = k[t]dt/\langle d(t^2)\rangle = k[t]dt/\langle 2t dt\rangle,$$

which is nonzero. Indeed, it is supported only at the origin. We conclude that φ is not smooth.

The reader should stop to ponder what happens if we *do* assume k to be of characteristic 2.

Example 25. Set $B = k[x, y]$ and $A = k[t]$. Consider the family of conics

$$\varphi : \mathbb{A}^2_k = \operatorname{Spec} B \to \operatorname{Spec} A = \mathbb{A}^1_k,$$

defined by $t \mapsto xy$. Notice that φ is of relative dimension 1. We compute the sheaf of relative differentials to be

$$\Omega^1_{\mathbb{A}^2/\mathbb{A}^1} = B dx \oplus B dy/\langle d(xy)\rangle = B dx \oplus B dy/\langle x dy + y dx\rangle,$$

which is not locally free at the origin: it cannot be generated by one element.

References

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[Sta23] The Stacks project authors. *The Stacks project.* `https://stacks.math.columbia.edu`. 2023.