Almost purity

10.06.25

k perf'd field, k° val'n ring, $\varpi \in k^\circ$ p. unif.,
$k^{\flat\circ} = \varprojlim_{\varphi} k^\circ/(\varpi)$, $k^\flat = \mathrm{Frac}\, k^{\circ\flat}$. R perf'd k-algebra,
$A = R^{\circ a}$, $\bar{A} = R^{\circ a}/\varpi$.

harder! (later) — talk 3 — Goal today

$$\begin{tikzcd} R_{\text{fét}} \arrow[d] & A_{\text{fét}} \arrow[l, "\simeq"'] \arrow[r, "\simeq"] \arrow[d] & \bar{A}_{\text{fét}} \arrow[d] & A^\flat_{\text{fét}} \arrow[l, "\simeq"'] \arrow[r, "\simeq (*)"] \arrow[d] & R^\flat_{\text{fét}} \arrow[d] \\ k\text{-perf} & k^{\circ a}\text{-perf} \arrow[l, "\simeq"'] \arrow[r, "\simeq"] & k^{\circ a}/\varpi\text{-perf} & k^{\flat\circ a}\text{-perf} \arrow[l, "\simeq"'] \arrow[r, "\simeq"] & k^\flat\text{-perf} \end{tikzcd}$$

- Tilting equivalence (Talk 5)
- Theorem 4.17 (Talk 3) + Gabber-Ramero, Thm 3.5.13

Thm. There is a fully faithful functor
$$R^\flat_{\text{fét}} \longrightarrow R_{\text{fét}}$$
$$S \mapsto S^\#$$
that is inverse to the tilting functor. Furthermore, it preserves degrees.

Proof. "invert $\simeq$ where necessary". For degrees: see notes.
"suffices to do Galois covers."

Skip this

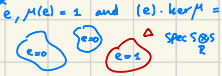
$S/R^\flat$ Galois $\Rightarrow S^\#/R$ Galois:
- connectedness ✓
- $\deg(S/R^\flat) = \deg(S \otimes_{R^\flat} S/S) = \deg(S^\# \otimes_R S^\#/S^\#) = \deg(S^\#/R)$ (S^d)
  $+ \#\mathrm{Aut}(S/R^\flat) = \#\mathrm{Aut}(S^\#/R)$

For general $R^\flat \to S$ conn. $\rightsquigarrow$ take Galois closure $R^\flat \to S \to \tilde{S}$
$\rightsquigarrow \deg(S/R^\flat) = \deg(\tilde{S}/R^\flat)/\deg(\tilde{S}/S) = \deg(\tilde{S}^\#/R)/\deg(\tilde{S}^\#/S) = \deg(S^\#/R)$.

Rmk. Actually $R^\flat_{\text{fét}} \xrightarrow{\sim} R_{\text{fét}}$.

(Almost) finite étale algebras

Def. $R \to S$ an R-alg. is finite étale if
(i) S is fin. projective as R-module;
(ii) $R \to S$ is unramified.
$\rightsquigarrow$ $\mathrm{Spec}\, S \xrightarrow{\Delta} \mathrm{Spec}\, S \times_R \mathrm{Spec}\, S$ is an open immersion.
$\rightsquigarrow$ $\exists e \in S \otimes_R S$ s.t. $e^2 = e$, $\mu(e) = 1$ and $(e) \cdot \ker \mu = 0$.



Def B an A-algebra is almost finite étale if
(i) B is alm. fin. gen'd (Talk 3);
(ii) B is almost projective: for any R-module M and any $i > 0$,
$$\mathrm{Ext}^i_{R^\circ}(B_*, M)^a = 0 \quad \text{in} \quad A\text{-Mod}.$$
(iii) A is almost unramified: $\exists e \in (B \otimes_A B)_*$ s.t. $e^2 = e$
$\mu_*(e) = 1$, and $(e) \cdot \ker \mu_* = 0$.

Rmk. in (ii): equivalently $\mathrm{alHom}(B, \_) : R^a\text{-Mod} \to \mathrm{Ab}$ is exact
adjoint to $\_ \otimes \_$, not $= \mathrm{Hom}_{R^a}(\_, \_)$!!!

Rmk. fin. étale over $\bar{A}$ defined analogously.

Exp. (Talk 3) $k = \mathbb{Q}_p(p^{1/p^\infty})^\wedge$, $L = k(\sqrt{p})$. $p \neq 2$
Then L°/k° is almost finite étale.

Prop $R^\flat \longrightarrow S$ fin. étale $\Rightarrow$ S perfectoid $k^\flat$-alg.

Proof (sketch) Seminorm on S: let $S_0 \subset S$ fin. gen'd $R^{\flat\circ}$-submodule s.t. $S = S_0[\varpi^{-1}]$. Set
$$|f| = \min\{|\varpi|^n : n \in \mathbb{Z}, \varpi^n f \in S_0\}, \quad f \in S.$$
$\rightsquigarrow$ makes S Banach alg over $k^\flat$.
- S is perfect by claim below;
- S° bounded uses perfect trace pairing
  $\mathrm{tr}_{S/R} : S \otimes_R S \to R$. $\boxtimes$

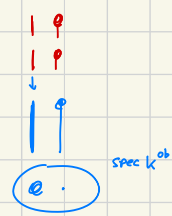
Claim: $A \to B$ (weakly) étale map of $\mathbb{F}_p$-alg., then
$$\begin{tikzcd} A \arrow[r, "\Phi"] \arrow[d] & A \arrow[d] \\ B \arrow[r, "\Phi"] & B \end{tikzcd}$$
cocartesian. $\rightsquigarrow$ See GR Thm. 3.5.13. $\boxtimes$

Have functor
$$A_{\text{fét}} \longrightarrow R_{\text{fét}}$$
$$B \mapsto B_*[\varpi^{-1}].$$

- for finite proj.: commute Ext w. localization;
- for unramified: $e \in (B \otimes_A B)_* = ((B \otimes_A B)^a)_*$
  $\hookrightarrow ((B_* \otimes_{R^\circ} B_*)^a)_*[\varpi^{-1}] = B_*[\varpi^{-1}] \otimes_R B_*[\varpi^{-1}]$

Thm (almost purity, char. p) The functor
$$A^\flat_{\text{fét}} \xrightarrow{\sim} R^\flat_{\text{fét}}$$
is an equivalence with inverse $S \mapsto S^{\circ a}$.

$\rightsquigarrow$ almost purity?
Faltings originally proved similar statement by showing almost unramifiedness at generic points in char p
that this implies global unramifiedness is kind of like Zar.-Nag purity.



Proof

Let $R^\flat \to S$ be fin. étale. We're to show: $R^{\flat\circ} \to S^\circ$ is alm. fin. ét

- finite proj. Let $\varepsilon \in \mathfrak{m}^\flat \trianglelefteq K^{\flat\circ}$. We will find maps

$$S^\circ \xrightarrow{\alpha} (R^{\flat\circ})^n \xrightarrow{\beta} S^\circ$$

s.t comp. is mult. by ε. Then

– clearly: S° is alm. fin. gen'd over $R^{\flat\circ}$.

– S° is alm. proj. over $R^{\flat\circ}$:

$$\operatorname{Ext}^{>0}_{R^{\flat\circ}}(S^\circ, M) \xrightarrow{\beta^*} \overbrace{\operatorname{Ext}^{>0}_{R^{\flat\circ}}(R^{\flat\circ n}, M)}^{0} \xrightarrow{\alpha^*} \operatorname{Ext}^{>0}_{R^{\flat\circ}}(S^\circ, M)$$

is mult. by ε $\Rightarrow$ $\operatorname{Ext}^{>0}_{R^{\flat\circ}}(S^\circ, M)^a = 0$.

To find α, β:

Let $e \in S \otimes_{R^\flat} S$ idempotent w. supp. the diag.

For $N \gg 0$:

$$\varpi^N e \in S^\circ \otimes_{R^{\flat\circ}} S^\circ$$

Prop. 5.9 perfectoid in char. p

Write $\varpi^N e = \sum_{i=1}^n x_i \otimes y_i$. Using perfectness: $\forall m \geq 1$

we have $\varpi^{N/p^m} e = \sum_{i=1}^n x_i^{1/p^m} \otimes y_i^{1/p^m} \in S^\circ \otimes_{R^{\flat\circ}} S^\circ$

Now for any $\varepsilon \in \mathfrak{m}^\flat$ can write:

$$\varepsilon e = \sum_{i=1}^n a_i \otimes b_i \in S^\circ \otimes_{R^{\flat\circ}} S^\circ$$

Now

$$\begin{array}{ccccc} S^\circ & \xrightarrow{\alpha} & (R^{\flat\circ})^n & \xrightarrow{\beta} & S^\circ \\ s & \mapsto & (\mathrm{tr}_{S/R^\flat}(a_i \cdot s))_i & & \\ & & (r_i) & \mapsto & \sum_{i=1}^n r_i b_i \end{array}$$

is mult. by ε.

- S° is alm. unram. over $R^{\flat\circ}$: "almost element"

$\varepsilon e \in S^\circ \otimes_{R^{\flat\circ}} S^\circ$ for all $\varepsilon \in \mathfrak{m}^\flat$, $e \in (S^\circ \otimes_{R^\flat} S^\circ)_* = \mathrm{Hom}_{R^{\flat\circ}}(\mathfrak{m}^\flat, S^\circ \otimes_{R^{\flat\circ}} S^\circ)$, $\varepsilon \mapsto \varepsilon \cdot e$

is idempotent supported on the diagonal. $\boxtimes$

Case of fields

Thm. (almost purity for char. 0 perf'd fields)

$$\mathrm{Fét}_{K^\flat} \xrightarrow{\sim} \mathrm{Fét}_K$$
$$L/K^\flat \mapsto L^\#/K$$

is an equivalence.

Proof.

To show: for $K \subset F$ finite: $\exists K^\flat \subset L$ s.t $F \simeq L^\#$.

Claim: suff. to find $K^\flat \subset L$ fin. Gal. s.t $F \subset L^\#$.

Proof.

$\leadsto$ $K \subset L^\#$ Galois $\left((L^\# : K) = (L : K^\flat)\right.$ and $\left.\mathrm{Gal}(L^\#/K) \simeq \mathrm{Gal}(L/K^\flat)\right)$.

Set $H = \mathrm{Gal}(L^\#/F)$. Then $F' = (L^\#)^H$ is s.t

$$(F')^\# \hookrightarrow (L^\#)^H = F$$

$+$ $(F:K) = ((F')^\# : K) \Rightarrow F = (F')^\#$. $\boxtimes$

Set $M = \widehat{K^\flat}$. Then M alg. clsd perf'd $\Rightarrow M^\#$ alg. clsd perf'd. Set $N = \bigcup_{K^\flat \subset L \text{ Gal.}} L^\# \subset M^\#$.

Then $N \subset M^\#$ dense (see notes)

Krasner's lemma $\Rightarrow$ N alg. closed. Now can embed $F \hookrightarrow N \Rightarrow F \subset L^\#$ for some $L/K^\flat$ Galois. $\boxtimes$

Cor. (Fontaine–Wintenberger if $K = \mathbb{Q}_p(p^{1/p^\infty})^\wedge$)

K perf'd field. Then

$$\mathrm{Gal}(\overline{K}/K) \simeq \mathrm{Gal}(\overline{K^\flat}/K^\flat).$$ $\boxtimes$

Proof

$$\begin{tikzcd} L & \mathrm{Fét}_{K^\flat} \arrow[rr, "\sim"] \arrow[dr, "\mathrm{Hom}_{K^\flat}(L,M)"'] & & \mathrm{Fét}_K \arrow[dl, "\mathrm{Hom}_K(L', M^\#)"] & L' \\ & & \mathrm{Sets} & & \end{tikzcd}$$

is an equivalence of Galois cats. $\boxtimes$

$S \xrightarrow{\alpha} R^n \xrightarrow{\beta} S$, $e = \sum a_i \otimes b_i$

$s \mapsto tr(sa_i)$
$(r_i) \mapsto \sum r_i b_i$.

finite proj. modules as direct summands.

$\beta \circ \alpha(s) = \beta(tr(sa_i)) = \sum b_i tr(sa_i)$

$S \to S \otimes_R S \simeq S \times S'$
$s \mapsto 1 \otimes s$

$\leadsto Tr_{S \otimes_R S/S}(e) = Tr_{S/S}(e) = 1$

$1 = Tr_{S \otimes_R S/S} = Tr_{S \otimes_R S/S}(\sum a_i \otimes b_i)$

$= \sum b_i Tr_{S/R}(a_i)$.

In general: $(s \otimes 1) \cdot e = (1 \otimes s) \cdot e$ $\quad (e \cdot \ker(\mu) = 0)$

$s = Tr_{S \otimes S/S}((1 \otimes s) \cdot e)$

$= Tr_{S \otimes S/S}((s \otimes 1) \cdot e)$

$= Tr_{S \otimes S/S}(\sum sa_i \otimes b_i)$

$= \sum b_i Tr(sa_i)$.

$M^\# = (\operatorname{colim}^{Ban}_{k^b \subset L, \text{Galois}} L)^\# = \operatorname{colim}^{Ban}_{k^b \subset L} L^\#$

density of $N \subset M^\#$.

$\leadsto (\operatorname{colim}^{Ban}_{k^b \subset L} L^\#)^\circ \simeq (\operatorname{colim}_{k^b \subset L} L^{\#\circ})^\wedge$

$\Rightarrow N \subset M^\#$.

Thm (4.17, Talk 3) The functor

$$A_{fét} \xrightarrow{\simeq} \bar{A}_{fét}$$
$$B \mapsto B \otimes_A \bar{A}$$

is an equivalence. Furthermore, $B \in A_{fét}$ is ϖ-adically complete.

Prop.(5.22) B as above is a perfectoid k^{oa}-algebra.

Proof (sketch) B is ϖ-adically compl. by Thm. 4.17. To prove $\Phi: B/\varpi^{1/p} \xrightarrow{\simeq} B/\varpi$, use almost analog of Claim.

Interlude Realizing finite étale algebras as direct summands.

Let $R \to S$ be finite étale. Have

$$tr_{S/R}: S \longrightarrow End_R(S,S) \simeq S \otimes_R S^\vee \xrightarrow{ev} R$$
$$s \mapsto (x \mapsto s \cdot x) \qquad s \otimes \varphi \mapsto \varphi(s)$$

Let $e \in S \otimes_R S$ a diagonal idempotent:

$$e^2 = e, \mu(e) = 1, (e) \cdot \ker\mu = (0).$$

write $e = \sum_{i=1}^n a_i \otimes b_i$.

Fact: $S \xrightarrow{\alpha} R^n \xrightarrow{\beta} S$

$s \mapsto (tr(sa_i))_{i=1}^n$

$(r_i) \mapsto \sum_{i=1}^n r_i b_i$.

is ids. $\leadsto$ S is a direct summand of R^n.