BLOCH-KATO ORDINARY BIELLIPTIC SURFACES

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ABSTRACT. We determine the Bloch-Kato ordinary (quasi-)bielliptic surfaces in every characteristic. We relate the notion of Bloch-Kato ordinarity to the notions of ordinary, classical and supersingular introduced for (quasi-)bielliptic surfaces in [Kro25].

1. INTRODUCTION

Throughout, we fix an algebraically closed field k of characteristic $p > 0$. All schemes considered are over k. By a *variety* X we mean a connected smooth proper scheme over the base. By a *surface* X we mean a variety of dimension 2.

In [Kro25] a trichotomy is suggested for certain bielliptic surfaces in *wild characteristic* (which exist in characteristic 2 and 3; see Definition 2.3), subdividing them into three classes: *ordinary*, *classical* and *supersingular*. This is in analogy with Enriques surfaces in characteristic 2, for which the same terminology exists; see [CDL24]. We roughly sketch the terminology for bielliptic surfaces. A bielliptic surface X can always be written as a quotient $X \simeq (E \times C)/G$, where E is an elliptic curve, C is a smooth genus one curve or the rational cuspidal curve $\operatorname{Spec} k[t^2, t^3] \cup \operatorname{Spec} k[t^{-1}]$, and G is a finite subgroup scheme of E acting faithfully on C. If the order $|G| = h^0(\mathscr{O}_G)$ is divisible by the characteristic of k, then X lives in wild characteristic. In this case, we say

- X is Kroon ordinary if E (or equivalently, the Albanese variety of X by Remark 2.7) is;
- X is Kroon classical if C is the cuspidal curve and E is ordinary;
- X is Kroon supersingular if C is the cuspidal curve and E is supersingular.

For general varieties, a notion of ordinarity exists which is due to Illusie-Raynaud and Bloch-Kato; see [IR83], [BK86] and Definition 3.1 below. We refer to it as *Bloch-Kato ordinarity*. A notable feature is that this defintion generalizes many defintions of "ordinary" for various families of varieties, for instance abelian varieties and K3 surfaces.

For Enriques surfaces it turns out that all of them are Bloch-Kato ordinary, apart from the supersingular ones in characteristic 2; see [CDL24, Theorem 1.4.19].

Inspired by this result and the trichotomy introduced in [Kro25], we seek to determine the Bloch-Kato ordinary bielliptic surfaces.

Theorem 1.1 (Theorem 4.5)**.** *Let X be a bielliptic surface. Then X is Bloch-Kato ordinary precisely when*

(i) *X lives in tame characteristic and* $\mathrm{Alb}(X)$ *is ordinary as an elliptic curve;*
(ii) *X is Kroon classical in wild characteristic;*
(iii) *X is Kroon ordinary in wild characteristic of type (a2) or (d);*
(iv) *X is Kroon ordinary in wild characteristic of type (a1) and the factor C arising in an isomorphism $X \simeq (E \times C)/G$ as in Theorem 2.5 is an ordinary elliptic curve.*

See Theorem 2.5 for the definitions of the different types of Kroon ordinary bielliptic surfaces.

2. Preliminaries on bielliptic surfaces

Definition 2.1. A minimal surface X of Kodaira dimension $\mathrm{Kod}(X) = 0$ is said to be *bielliptic* if $b_2 = 2$.

We collect a few other relevant numerical invariants of X (see [Kro25, Theorem 3.1.23]):

$$b_1 = 2, \quad p_a = \chi(\mathscr{O}_X) - 1 = -1, \quad q = h^1(\mathscr{O}_X) = 1 \text{ or } 2. \tag{2.1}$$

Theorem 2.2. *Let X be a bielliptic surface. Then $X \simeq (E \times C)/G$, where E is an elliptic curve, C is a smooth genus-one curve or the rational cuspidal curve, and G is a finite subgroup scheme of E that acts faithfully on the curve C.*

Proof. See [BM77] and [BM76]. ■

Given a bielliptic surface $X = (E \times C)/G$ as in the theorem above, there are projection maps

$$f\colon X \to E/G =: A \quad \text{and} \quad g\colon X \to C/G.$$

Since G acts on E by translations, E/G is an elliptic curve and f turns out to be the Albanese map ([Kro25, Theorem 3.4.1]). The Albanese map is always a *fibration*: $f_*\mathscr{O}_X = \mathscr{O}_A$, which can be seen, for instance, from the argument in the proof of [Bea96, Proposition V.15]. In case C is the rational cuspidal curve, the Albanese map f is non-smooth and X is also called a *quasi-bielliptic surface*.

It also turns out that $C/G \simeq \mathbb{P}^1$ and that the map g is a fibration; see [Kro25, Theorem 3.4.1].

For a given bielliptic surface $X \simeq (E \times C)/G$, the maps constructed above are intrinsic to X by [Kro25, Theorem 3.3.9], and we define the *intersection invariant* of X (see [Kro25, Notation 3.3.17]) to be

$$\gamma = F_1 \cdot F_2, \tag{2.2}$$

where F_1 denotes a fiber of f and F_2 a fiber of g. By [Kro25, Proposition 3.4.16], we have $\gamma = |G| := h^0(\mathscr{O}_G)$.

Definition 2.3. ([Kro25, Definition 3.4.7]) A bielliptic surface X is said to *live in tame characteristic* if the number γ constructed in 2.2 is coprime to p. It is said to *live in wild characteristic* otherwise.

Definition 2.4 ([Kro25, Definition 3.4.9])**.** Let X be a bielliptic surface in wild characteristic.

- (i) X is said to be *Kroon ordinary* if the Albanese map $f\colon X \to A$ is smooth; i.e., if X is *not* quasi-bielliptic.
- (ii) X is said to be *Kroon classical* if X is quasi-bielliptic and A is ordinary as an elliptic curve over k.
- (iii) X is said to be *Kroon supersingular* if X is quasi-bielliptic and A is supersingular as an elliptic curve.

2.1. **Kroon ordinary bielliptic surfaces.** Bombieri and Mumford classify bielliptic surfaces completely in [BM77] and [BM76]. We will state here only the classification of Kroon-ordinary bielliptic surfaces in wild characteristic.

Theorem 2.5. *Let X be a Kroon-ordinary bielliptic surface in wild characteristic. Then $X \simeq (E\times C)/G$ by Theorem 2.2, where E and C are elliptic curves, G is a subgroup scheme of E, and G acts faithfully on C. For the action of G on C we have the following options.*

- *$G \simeq \mathbb{Z}/2\mathbb{Z}$ acting by $x \mapsto -x$ (type (a1)).*
- *$G \simeq \mathbb{Z}/2\mathbb{Z}\times\mu_2$ where $\mathbb{Z}/2\mathbb{Z}$ acts by $x\mapsto -x$ and μ_2 acts by translation (type (a2)).*
- *$G \simeq \mathbb{Z}/3\mathbb{Z}$ with action $x \mapsto \omega x$, where $\omega\colon C\to C$ is an automorphism of C of order 3 as an elliptic curve (type (b)).*
- *$G \simeq \mathbb{Z}/4\mathbb{Z}$ with action $x\mapsto ix$, where $i\colon C\to C$ is an automorphism of C as an elliptic curve of order 4 (type (c)).*
- *$G \simeq \mathbb{Z}/6\mathbb{Z}$ with action $x \mapsto -\omega x$ (type (d)).*

Remark 2.6. The strange naming convention of the different types of ordinary bielliptic surfaces in wild characteristic is chosen to be consistent with [Kro25].

Remark 2.7. Notice in the above theorem that E is always an ordinary elliptic curve, because of the structure of the possible subgroups schemes G. Hence also $\mathrm{Alb}(X)$ is ordinary, since it is isogenous to E.

For the first four types in the Theorem above, we can easily compute by hand that the group G acts trivially on the cotangent bundle $\Omega^1_{E\times C}\simeq \mathcal{O}_{E\times C}^{\oplus 2}$, and as a result the cotangent bundle of X is trivial. In particular, for the canonical bundle we have $\omega_X\simeq\mathcal{O}_X$. So for X a Kroon ordinary bielliptic surface of type *not* (d) we find

$$g = h^2(\mathcal{O}_X) = h^0(\mathcal{O}_X) = 1.$$

From the invariants in (2.1) we deduce that in this case $q = 2$.

The strange phenomenon where Ω^1_X is a trivial bundle also occurs for some supersingular quasi-bielliptic surfaces, but the following proposition tells us that it gets no worse than that.

Proposition 2.8. *Let X be a Kroon classical bielliptic surface or a bielliptic surface in tame characteristic, then $q = 1$.*

Proof. By [Kro25, Proposition 3.4.18] the canonical bundle ω_X is a non-trivial torsion sheaf. As a result,

$$g = h^2(\mathscr{O}_X) = h^0(\omega_X) = 0.$$

Since $\chi(\mathscr{O}_X) = 1$, we find $q = 1$. ∎

Remark 2.9. For a bielliptic surface X, the statement $q = 2$ can also be interpreted as saying that Pic^0 is a non-reduced group scheme. Indeed, the tangent space of Pic^0 at the neutral element is of dimension $h^1(\mathscr{O}_X) = q$, whereas Pic^0 itself is of dimension $b_1/2 = 1$.

3. Bloch-Kato ordinary surfaces

Let X be a smooth proper variety of dimension n. Let $F\colon X \to X$ be the absolute Frobenius morphism: it is defined by $f \mapsto f^p$ on $\mathscr{O}_X$. For $j \geqslant 1$, write

$$B\Omega^j_X = F_*(d\Omega^{j-1}_X),$$

where $d(\Omega^{j-1}_X)$ denotes the image of the exterior derivative $d\colon \Omega^{j-1}_X \to \Omega^j_X$.

Definition 3.1 ([BK86, Definition 7.2])**.** The variety X is said to be *Bloch-Kato ordinary* if for all i and j we have

$$H^i(X, B\Omega^j_X) = 0.$$

3.1. Bloch-Kato ordinary surfaces.

For surfaces we can simplify the situation quite a bit with regards to Bloch-Kato ordinarity.

We have exact sequences of locally free $\mathscr{O}_X$-modules

$$\begin{aligned} &0 \to \mathscr{O}_X \to F_*\mathscr{O}_X \xrightarrow{d} B\Omega^1_X \to 0, \\ \text{and}\quad &0 \to B\Omega^n_X \to F_*\Omega^n_X \xrightarrow{C} \Omega^n_X \to 0, \end{aligned} \tag{3.1}$$

where C denotes the Cartier operator. Grothendieck duality gives rise to a perfect pairing of $\mathscr{O}_X$-modules

$$\begin{aligned} F_*\mathscr{O}_X \otimes_{\mathscr{O}_X} F_*\Omega^n_X &\to \Omega^n_X \\ f \otimes \omega &\mapsto C(f\omega). \end{aligned}$$

This perfect pairing in combination with the exact sequences of (3.1) gives rise to a perfect pairing

$$B\Omega^1_X \otimes B\Omega^n_X \to \Omega^n_X. \tag{3.2}$$

Lemma 3.2. *Let X be a surface. Then X is Bloch-Kato ordinary if and only if for all i we have*

$$H^i(X, B\Omega^1_X) = 0.$$

Proof. This follows immediately from 3.2 and Serre duality. ■

Proposition 3.3. *Let X be a surface. Then X is Bloch-Kato ordinary if and only if*

$$H^1(X, B\Omega^1_X) = 0.$$

Proof. The long exact sequence induced by the first short exact sequence of (3.1) shows that $H^0(B\Omega^1_X) = H^2(B\Omega^1_X) = 0$ if $H^1(B\Omega^1_X) = 0$. ■

Definition 3.4. A variety X is said to be Frobenius-split if the Frobenius morphism $\mathcal{O}_X \to F_*\mathcal{O}_X$ is split as a map of $\mathcal{O}_X$-modules; i.e., when the sequence

$$0 \to \mathcal{O}_X \to F_*\mathcal{O}_X \to B\Omega^1_X \to 0$$

is a split exact sequence.

Proposition 3.5. *If X is a Frobenius-split surface, then it is Bloch-Kato ordinary. If the canonical bundle Ω^2_X is trivial, then the converse is also true, and the statement is furthermore equivalent to the existence of a non-zero global 2-form fixed by the Cartier operator.*

Proof. Suppose X is Frobenius-split. Then for all j the map $H^j(\mathcal{O}_X) \to H^j(F_*\mathcal{O}_X)$ is injective, but this is a map of finite-dimensional vector spaces of the same dimension, hence an isomorphism. The long exact sequence applied to the first sequence of (3.1) now shows that X is Bloch-Kato ordinary.

Suppose now that $\Omega^2_X \simeq \mathcal{O}_X$. If X is Bloch-Kato ordinary, then

$$\operatorname{Ext}^1_{\mathcal{O}_X}(B\Omega^1_X, \mathcal{O}_X) \simeq H^1((B\Omega^1_X)^\vee) \simeq H^1(B\Omega^1_X) = 0$$

by Serre-duality. Hence X is Frobenius-split. The last part of the statement follows from the fact that the exact sequences of (3.1) are interchanged by $\operatorname{Hom}_{\mathcal{O}_X}(-, \Omega^2_X)$ by Grothendieck-duality, and so one is split if and only if the other one is. A splitting of the second sequence is equivalent to the existence of a non-zero global 2-form fixed by the Cartier operator, because $\Omega^2_X \simeq \mathcal{O}_X$. ■

Corollary 3.6. *Let X be a Bloch-Kato ordinary surface with $\Omega^2_X \simeq \mathcal{O}_X$. Then any finite étale cover $Y \xrightarrow{f} X$ of X is again Bloch-Kato ordinary.*

Proof. Clearly $f^*\Omega^2_X \simeq \Omega^2_Y$ is trivial. Let $\omega \in \Omega^2_X(X)$ be a non-zero global 2-form fixed by the Cartier operator, then $f^*\omega \in \Omega^2_Y(Y)$ is also fixed by the Cartier operator. ■

4. Bloch-Kato ordinary bielliptic surfaces

Lemma 4.1. *Let X be a bielliptic surface and let $f\colon X \to A$ be the Albanese fibration, then the induced map*

$$f^*\colon H^1(A,\mathscr{O}_A) \to H^1(X,\mathscr{O}_X)$$

is injective.

Proof. We have $f_*\mathscr{O}_X = \mathscr{O}_A$, and so the Leray-Serre spectral sequence

$$E_2^{ij} = H^i(A, R^j f_*\mathscr{O}_X) \Rightarrow H^{i+j}(X,\mathscr{O}_X)$$

gives rise to a five term exact sequence

$$0 \to H^1(A,\mathscr{O}_A) \to H^1(X,\mathscr{O}_X) \to H^0(A, R^1 f_*\mathscr{O}_X) \to \dots,$$

where the first map is precisely f^*. ■

Proposition 4.2. *If X is a bielliptic surface with supersingular Albanese, then X is* not *Bloch-Kato ordinary. Furthermore, if $q = 1$, then the converse is true.*

Proof. Let $f\colon X \to A$ be the Albanese of X. If A is supersingular, then the map

$$F^*\colon H^1(A,\mathscr{O}_A) \to H^1(A,\mathscr{O}_A)$$

is zero and the commutative diagram

$$\begin{array}{ccc} H^1(X,\mathscr{O}_X) & \xrightarrow{F^*} & H^1(X,\mathscr{O}_X) \\ {\scriptstyle f^*}\big\uparrow & & {\scriptstyle f^*}\big\uparrow \\ H^1(A,\mathscr{O}_A) & \xrightarrow{F^*} & H^1(A,\mathscr{O}_A) \end{array}$$

combined with Lemma 4.1 shows that $F^*\colon H^1(X,\mathscr{O}_X) \to H^1(X,\mathscr{O}_X)$ has non-trivial kernel. The long exact sequence induced by

$$0 \to \mathscr{O}_X \to F_*\mathscr{O}_X \to B\Omega^1_X \to 0$$

shows that $H^0(B\Omega^1_X) \neq 0$, and so X is not Bloch-Kato ordinary.

Suppose now that $q = 1$ so that f^* is an isomorphism in the above square. Consider the exact sequence

$$H^1(\mathscr{O}_X) \to H^1(F_*\mathscr{O}_X) \to H^1(B\Omega^1_X) \to H^2(\mathscr{O}_X).$$

We have $q = 1$, whence $g = h^2(\mathscr{O}_X) = 0$. It follows that $H^1(B\Omega^1_X) = 0$ if and only if $H^1(\mathscr{O}_X) \to H^1(F^*\mathscr{O}_X)$ is surjective, which by the above commutative square happens if and only if A is ordinary. We conclude by Proposition 3.3. ■

Corollary 4.3. *Let X be a bielliptic surface.*

(i) If X lives in tame characteristic, then X is Bloch-Kato ordinary if and only if $\mathrm{Alb}(X)$ *is ordinary as an elliptic curve.*
(ii) If X is Kroon classical in wild characteristic, then X is Bloch-Kato ordinary.
(iii) If X is Kroon supsersingular in wild characteristic, then X is not *Bloch-Kato ordinary.*

Proof. Parts (i) and (ii) are clear by Propositions 2.8 and 4.2, and the fact that the Albanese of a Kroon classical quasi-bielliptic surfaces is ordinary by definition. Part (iii) is by Proposition 4.2 and the fact that the Albanese of a Kroon supersingular quasi-bielliptic surface is supersingular. ■

4.1. **Kroon ordinary bielliptic surfaces.** This leaves us with the Kroon ordinary bielliptic surfaces in wild characteristic. Unlike the name would suggest, these surfaces are usually not Bloch-Kato ordinary. Notice, however, that if X is Kroon ordinary its Albanese is always an ordinary elliptic curve by Remark 2.7. This is in contrast to the other bielliptic surfaces, for which Bloch-Kato ordinarity is governed entirely by the Albanese. Indeed, in Section 2.1 we saw that for every type of Kroon ordinary bielliptic surfaces, apart from type (d), the irregularity is 2, so that Proposition 4.2 does not apply.

Corollary 4.4. *Let X be a Kroon ordinary bielliptic suface in wild characteristic. Write $X \simeq (E \times C)/G$ as in Theorem 2.5.*

(i) If X is of type (a1), X is Bloch-Kato ordinary if and only if C is an ordinary elliptic curve.
(ii) If X is of type (a2) or (d), X is Bloch-Kato ordinary.
(iii) If X is of type (b) or (c), X is not *Bloch-Kato ordinary.*

Proof. We set $Y = E \times C$.

(i) Notice that we have an étale cover $Y \to X$. If C is supersingular, then Y is not a Bloch-Kato ordinary variety. If X were Bloch-Kato ordinary, this would contradict Corollary 3.6, since we know the canonical bundle of X to be trivial from Section 2.1.

If C is Bloch-Kato ordinary, then by Proposition 3.5 we find a non-zero global 2-form $\omega \in \Omega^2_Y(Y)$ that is fixed by the Cartier operator. Since the group G operates trivially on the sheaf of 2-forms on Y, the form ω descends to a 2-form on X that is again fixed by the Cartier operator. It follows from Proposition 3.5 again that X is Bloch-Kato ordinary.

Alternatively, the fact that X is Bloch-Kato ordinary in this case follows from ordinarity of Y, the fact that $Y \to X$ is a Galois étale cover, and the Hochschild-Serre spectral sequence for étale cohomology[1]; see [Mil16, Theorem III.2.20].

[1] Thanks to Kay Rülling for pointing this out!

(ii) Suppose X is of type (a2). From Theorem 2.5 we see that Y is an ordinary variety, because C is ordinary as an elliptic curve: it admits a non-trivial 2-torsion point. By the same argument as for part (i), we see that X must be Bloch-Kato ordinary.

Alternatively, we can argue as follows. $E \times C$ is an ordinary abelian variety; hence the isogenous abelian variety $A = (E \times C)/\mu_2$ is ordinary. Then we conclude as in part (i) that $X = A/(\mathbb{Z}/2\mathbb{Z})$ is Bloch-Kato ordinary, because X admits the ordinary variety A as a Galois étale cover.

If X is of type (d), then X is Bloch-Kato ordinary by Proposition 4.2: see the paragraph at the start of Section 4.1.

(iii) We observe from Theorem 2.5 that C is a supersingular elliptic curve in this case, so that Y is not a Bloch-Kato ordinary variety. Since Y is an étale cover of X, we conclude that X is not Bloch-Kato ordinary as before.

■

We summarize everything in the theorem below.

Theorem 4.5. *Let X be a bielliptic surface. Then X is Bloch-Kato ordinary precisely when*

(i) X lives in tame characteristic and $\mathrm{Alb}(X)$ *is ordinary as an elliptic curve;*
(ii) X is Kroon classical in wild characteristic;
(iii) X is Kroon ordinary in wild characteristic of type (a2) or (d);
(iv) X is Kroon ordinary in wild characteristic of type (a1) and the factor C arising in an isomorphism $X \simeq (E \times C)/(\mathbb{Z}/2\mathbb{Z})$ as in Theorem 2.5 is an ordinary elliptic curve.

Proof. Combine Corollaries 4.3 and 4.4. ■

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