Bloch-Kato ordinary bielliptic surfaces

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§1. Preliminaries

$k = \bar{k}$ of char. $p > 0$. Varieties are smooth proper conn./k.

X/k surface w./ $\mathrm{kod}(X) = 0$. Then X can be

K3 surface ($b_2 = 22$)	Abelian surface ($b_2 = 6$)
Enriques surface ($b_2 = 10$)	Bielliptic surface ($b_2 = 2$)

Prop. X bielliptic. $p_a = \chi(\mathcal{O}_X) - 1 = -1$, $q := h^{01} = \begin{cases} 1, \text{ or} \\ 2. \end{cases}$ $\boxtimes$

Thm. X biell. Then $X \cong (E \times C)/G$, where E ell. curve, $G \subset E$ subgrp. scheme, C smooth genus 1 curve or $C = \mathrm{Spec}\, k[t^2, t^3] \cup \mathrm{Spec}\, k[t^{-1}]$ w. $G \curvearrowright C$ faithfully.

cusp: $y^2 = x^3$

Fact $f: X \to E/G$ (ell. curve) albanese. If C cuspidal $\Rightarrow$ f not smooth. (not even generically!)

We then also say: X is quasi-bielliptic. (only happens in char 2 or 3.)

$\gamma = |G| := h^0(G)$ is an invariant of X.

Def. If $\gamma = 0$ in k (char 2 or 3.), then X lives in wild characteristic. (this is where stuff gets interesting!)

Exp. • E, C ell. curves, $2 = 0$ in k, $\mathbb{Z}/2\mathbb{Z} \hookrightarrow E$ (i.e., E is ordinary), $\mathbb{Z}/2\mathbb{Z} \curvearrowright C$ by $x \mapsto -x$. Then $X = (E \times C)/\tfrac{\mathbb{Z}}{2\mathbb{Z}}$ is an Igusa surface.

• Spse C s.singular, and let $\mathbb{Z}/4\mathbb{Z} \curvearrowright C$ by auto of order 4. Then $X = (E \times C)/\tfrac{\mathbb{Z}}{4\mathbb{Z}}$. ← said to be of type (c1)

Rmk. Set $Y = E \times C$. In both cases: $G \curvearrowright \Omega^1_Y \cong \mathcal{O}_Y^{\oplus 2}$ trivially $\Rightarrow \Omega^1_X \cong \mathcal{O}_X^{\oplus 2}$.

In particular: $h^2(\mathcal{O}_X) = h^0(\Omega^2_X) = 1 \Rightarrow q = 2$. ($p_a = -1$ always)

Note: Pic^0_X is not reduced: $\dim T_e \mathrm{Pic}^0_X = h^1(\mathcal{O}_X) = 2 > 1 = \dim \mathrm{Pic}^0_X$. (albanese was an ell. curve!)

Exp. • Igusa surface:

$$0 \to \mathrm{Pic}^0_{X,\mathrm{red}} \to \mathrm{Pic}^\tau_X \to C[2] \to 0.$$

$C[2] \cong \begin{cases} \mathbb{Z}/2\mathbb{Z} \times \mu_2, \text{ or} \\ 0 \to \alpha_2 \to C[2] \to \alpha_2 \to 0 \end{cases}$ (← s.singular)

• Type (c1): $0 \to \mathrm{Pic}^0_{X,\mathrm{red}} \to \mathrm{Pic}^\tau_X \to \alpha_2 \to 0$, where $\alpha_2 = \ker(\mathbb{G}_a \xrightarrow{\mathrm{Frob}} \mathbb{G}_a)$

Rmk. In char. 0: $\Theta_{X/k} := (\Omega^1_{X/k})^\vee \cong \mathcal{O}_X^{\oplus \dim X} \Rightarrow X$ abelian. Not here!!

Joshi: this is explained by torsion in $H^2_{\mathrm{cris}}(X/W)$.

§2. Hodge + de Rham #'s for these examples

Hodge is easy:

$$\begin{array}{ccccc} & & 1 & & \\ & 2 & & 2 & \\ 1 & & 4 & & 1 \\ & 2 & & 2 & \\ & & 1 & & \end{array}$$

de Rham: Set $\Gamma = \mathrm{Pic}^\tau_X / \mathrm{Pic}^0_{X,\mathrm{red}}$. Then

• Igusa: $h^1_{dR} = 2 + \log_2 |\Gamma| = 2 + 2 = 4$, $h^2_{dR} = 2 + 2\log_2 |\Gamma| = 2 + 2.2 = 6$ (formulae by Suwa)

• type (c1): $h^1_{dR} = 3$, $h^2_{dR} = 4$

Note: for Igusa: $E_1^{ij} = H^j(X, \Omega^i_X) \Rightarrow H^{i+j}_{dR}(X/k)$ degenerate

for type (c1): not degenerate. ← obstructed deformations by Deligne-Illusie

In both cases: $H^2_{\mathrm{cris}}(X/W)$ has torsion:

$$0 \to H^1_{\mathrm{cris}}(X/W) \otimes_W k \to H^1_{\mathrm{cris}}(X/W_1) \to \mathrm{Tor}_1^W(H^2_{\mathrm{cris}}(X/W), k) \to 0$$

$\dim_k = \mathrm{rk}_W H^1_{\mathrm{cris}} = b_1 = 2$ ← ℓ-adic betti #

$\dim = h^1_{dR} > 2 = b_1$

$\mathrm{Tor}_1^W(H^2_{\mathrm{cris}}(X/W), k) \neq 0 \Rightarrow H^2_{\mathrm{cris}}$ has torsion!

Prop. $X \cong (E \times C)/\tfrac{\mathbb{Z}}{2\mathbb{Z}}$ w. C ordinary or X (c1): $H^2(W\mathcal{O}_X) \cong k$. (← torsion W-module!)

Proof Consider s.e.s $0 \to \mathcal{O}_X \to W_2\mathcal{O}_X \to \mathcal{O}_X \to 0$. Then connecting map

$$\beta_1 : H^1(\mathcal{O}_X) \to H^2(\mathcal{O}_X).$$

is surjective: so-called first Bockstein operator.
(follows from structure of Γ + relations between Bockstein and Pic^0.)

[skip if no time

We get $\cdots \to H^1(\mathcal{O}_X) \twoheadrightarrow H^2(\mathcal{O}_X) \to H^2(W_2\mathcal{O}_X) \xrightarrow{\cong} H^2(\mathcal{O}_X) \to 0$.

Now consider for $n \geq 2$ the s.e.s

$$0 \to W_{n-1}\mathcal{O}_X \to W_n\mathcal{O}_X \to \mathcal{O}_X \to 0.$$

We get $H^1(\mathcal{O}_X)/H^1(W_n\mathcal{O}_X) \to H^2(W_{n-1}\mathcal{O}_X) \to H^2(W_n\mathcal{O}_X) \to H^2(\mathcal{O}_X) \to 0$

$H^1(\mathcal{O}_X)/H^1(W_n\mathcal{O}_X) = H^1(\mathcal{O}_X)/H^1(W_2\mathcal{O}_X)$ ("higher Bockstein operations vanish.")

$\cong$ for $n = 2$ and for higher n by induction

Now $H^2(W\mathcal{O}_X) = \varprojlim H^2(W_n\mathcal{O}_X) \cong k$. $\boxtimes$]

Rmk. $X \cong (E \times C)/\tfrac{\mathbb{Z}}{2\mathbb{Z}}$ w. C s.singular, then β_1 is zero!

Can still compute $H^2(W\mathcal{O}_X) \cong k \oplus k$: β_2 now surjective!

§3. Bloch-Kato ordinarity

X variety. $F: \underset{f^p \leftarrow\!\shortmid\, f}{X \to X}$ absolute Frob. For $j \geq 1$, write

$$B\Omega^j_X = \mathrm{im}\left(F_*\Omega^{j-1}_X \xrightarrow{d} F_*\Omega^j_X\right).$$

<u>Def.</u> (Bloch-Kato) X is (Bloch-Kato) ordinary if $H^i(B\Omega^j_X) = 0$ for all $i,j \geq 0$.

<u>Rmk.</u> Newt. poly of $(H^n_{cris}(X/W)/\text{tors.}, F)$ = geom. Hodge polygon (defined by $h^{q,n-q}$) $\Rightarrow$ Bk-ordinary.

We have exact sequences

$$0 \to \mathcal{O}_X \to F_*\mathcal{O}_X \xrightarrow{d} B\Omega^1_X \to 0$$

$$0 \to B\Omega^n_X \to F_*\Omega^n_X \xrightarrow{C} \Omega^n_X \to 0. \qquad n = \dim X$$

Cartier operator

<u>Grothendieck duality</u> $\Rightarrow$ $F_*\mathcal{O}_X \otimes_{\mathcal{O}_X} F_*\Omega^n_X \to \Omega^n_X$, $f \otimes w \mapsto C(fw)$ is a perfect pairing.

Then we obtain a perfect pairing

$$B\Omega^1_X \otimes B\Omega^n_X \to \Omega^n_X \quad (*)$$

<u>Cor.</u> X a surface. Then X Bk ordinary $\Leftrightarrow$ $H^*(B\Omega^1_X) = 0$.

<u>Proof.</u> $(*)$ + Serre-duality. ☒

<u>Cor.</u> X a surface. Then X Bk ordinary $\Leftrightarrow$ $H^1(B\Omega^1_X) = 0$.

<u>Proof.</u> Take long exact sequence of

$$0 \to \mathcal{O}_X \to F_*\mathcal{O}_X \to B\Omega^1_X \to 0.$$ ☒

<u>Def.</u> Variety X is <u>Frobenius split</u> if

$$0 \to \mathcal{O}_X \to F_*\mathcal{O}_X \to B\Omega^1_X \to 0$$

is a split s.e.s of $\mathcal{O}_X$-modules.

<u>Prop.</u> X a surface

X Frob. split $\Rightarrow$ X B.k ordinary.

$(\Leftarrow)$ if $\Omega^2_X \simeq \mathcal{O}_X$. Then also X ordinary $\Leftrightarrow$ $\exists w \in \Omega^2_X(X)$ non-zero s.t $Cw = w$.

<u>Proof.</u> $(\Rightarrow)$ have s.e.s

$$0 \to \underbrace{H^i(\mathcal{O}_X) \to H^i(F_*\mathcal{O}_X)}_{\text{same dim'n}} \to \underbrace{H^i(B\Omega^1_X)}_{=0} \to 0$$

$(\Leftarrow)$ Grothendieck duality: $0 \to \mathcal{O}_X \to F_*\mathcal{O}_X \to B\Omega^1_X \to 0$ split

$\Leftrightarrow$ $0 \to B\Omega^2_X \to F_*\Omega^2_X \xrightarrow{C} \Omega^2_X \simeq \mathcal{O}_X \to 0$ split $\Leftrightarrow$ $\exists w \in \Omega^2_X(X)$ non-zero w. $Cw = w$.

$\rightsquigarrow$ follows from $\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{O}_X, B\Omega^2_X) = H^1(B\Omega^2_X) = 0$. ☒

<u>Cor.</u> X Bk-ordinary surface w. $\Omega^2_X \simeq \mathcal{O}_X$. Then $Y \xrightarrow{\pi} X$ fin. étale cover $\Rightarrow$ Y ordinary. <u>Rmk.</u> Will see: false if $\Omega^2_X \not\simeq \mathcal{O}_X$!!!

<u>Proof.</u> $w \in \Omega^2_X(X)$ non-zero s.t $Cw = w$. Then $\Omega^2_Y \simeq \mathcal{O}_Y$ and

$$C(\pi^* w) = \pi^* w.$$ ☒

<u>Exp.</u>
- $X \simeq (E \times C)/\mathbb{Z}/2\mathbb{Z}$ w. C s.sing. or X of type (c1) is <u>not</u> Bk ordinary.
- $X \simeq (E \times C)/\mathbb{Z}/2\mathbb{Z}$ w. C ordinary: X is B.k ordinary: $w \in \Omega^2_{E\times C}(E \times C)$ s.t $Cw = w$ $\rightsquigarrow$ w descends to $\Omega^2_X(X)$.

<u>Thm.</u> X biell. surf., $f: X \to \mathrm{Alb}(X) = A$ albanese (ell. curve). Then A s.sing.

$\Rightarrow$ X <u>not</u> Bk-ordinary. $(\Leftarrow)$ if $h^{01} = 1$.

<u>Proof.</u> $\pi_*\mathcal{O}_X = \mathcal{O}_A \Rightarrow H^1(A, \mathcal{O}_A) \xrightarrow{f^*} H^1(X, \mathcal{O}_X)$. (Leray-Serre spectral sequence)

We get

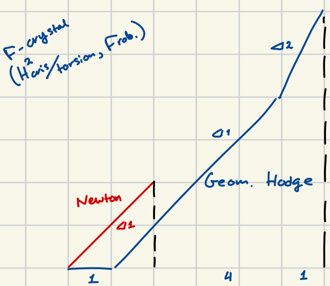
$$\begin{array}{ccc} H^1(\mathcal{O}_X) & \xrightarrow{F} & H^1(\mathcal{O}_X) \\ \uparrow & \circlearrowleft & \uparrow \\ H^1(\mathcal{O}_A) & \xrightarrow{F} & H^1(\mathcal{O}_A). \end{array}$$

A s.singular $\Rightarrow$ $H^1(\mathcal{O}_A) \xrightarrow{F} H^1(\mathcal{O}_A)$ zero $\Rightarrow$ $F: H^1(\mathcal{O}_X) \to H^1(\mathcal{O}_X)$ not bijective

$\Rightarrow$ X not Bk ordinary from $0 \to \mathcal{O}_X \to F_*\mathcal{O}_X \to B\Omega^1_X \to 0$.

$h^{01} = 1 \Rightarrow F: H^1(\mathcal{O}_X) \to H^1(\mathcal{O}_X)$ bijective $\Leftrightarrow$ A ordinary. ☒

<u>Exp.</u> X ordinary Igusa.



Hodge #'s: $h^{20} = h^{02} = 1$, $h^{11} = 4$.

For Newton polygon:

$$H^2(X/k)_{[0,1)} = H^2(W\mathcal{O}_X)[1/p] \quad \text{slope sp. seq.!}$$

$$\Rightarrow H^2(X/k)_{(1,2]} = 0 \quad \text{by Poincaré duality}$$

$\Rightarrow$ only slope 1 part remains!

I collect here a few more interesting computations for our ordinary Igusa surface and some of kay's remarks from after the lecture.

The fact that $H^2(X, W\mathcal{O}_X)$ is finitely generated implies, by Illusie's DRW paper I.3.7 that the slope spectral sequence

$$E_1^{ij} = H^i(X, W\Omega_X^j) \Rightarrow H^{i+j}_{crys}(X/W)$$

degenerates at E_1. We get an exact sequence

$$0 \to P^1H^2_{crys}(X/W) \to H^2_{crys}(X/W) \to H^2(W\mathcal{O}_X) \to 0.$$

$H^0(X, W\Omega_X^2)$ $=0$ as it has no torsion and the slope 2-part of $H^2(X/W)$ is zero!

$$\overset{\text{(1)}}{H^1(X, W\Omega_X^1)}$$

This sequence is actually split: "Finiteness, duality, ..." 3.4.1 — Illusie.

Furthermore $P^1H^2_{cris}(X/W)_{tors} = NS(X)_{p\text{-tors}} \otimes_{\mathbb{Z}_p} W$ is p-torsion, ← pour X Igusa et ordinaire, nous ont $NS(X)_{tors} \simeq \Gamma_{ét} = (\mathbb{Z}/2 \times \mu_2)_{ét} = \mathbb{Z}/2.$

by I.6.8.1 in Illusie's DRW paper.

So we get

$$\begin{aligned} H^2_{crys}(X/W) &\simeq W^{b_2} \oplus k^{\oplus h^2_{dR} - b_2} \\ &\simeq W^{\oplus 2} \oplus k^{\oplus 2} \end{aligned}$$

$$\Rightarrow H^1(X, W\Omega_X^1) = W^{\oplus 2} \oplus k.$$

kay also gave an argument for the Tate conjecture for bielliptic surfaces!

Below is kay's argument.

$X/\mathbb{F}_q$ a bielliptic surface

By 3.2.7 + 3.4.6.1 in Andrés "Introduction aux motifs" we can realize

this is a direct calculation

$NS(X) \otimes \mathbb{Q}_p \xhookrightarrow{cl} H^2(X/k)$. Now cl lands in $H^2(X/k)(1)^F$, which must have dim'n 2 as a $\mathbb{Q}_p$-vector space: $H^2(X/k)(1)^F \otimes k \xhookrightarrow{\sim} H^2(X/k)$.

So $\rho(X) = \dim_{\mathbb{Q}_p} H^2(X/k)(1)^F$ and we can conclude the Tate conjecture for X

2" by Thm 4.1 in Milne's "On a conjecture of Artin-Tate." ← the condition $2 \neq 0$ in that Thm. is likely redundant according to kay.

This maybe uses some non-obvious facts about bielliptic surfaces that you find in Ivo kroon's thesis.