1. CONICS AND QUATERNIONS

Let k be a field of characteristic not equal to 2. Let a and b be units in k, and define the k-algebra $(a,b)_k$ generated by i and j subject to the relations

$$i^2 = a, \quad j^2 = b, \quad ij = -ji.$$

We call $(a,b)_k$ a *quaternion algebra over k*.

Example 1.1. (i) The algebra $\mathbb{H} = (-1,-1)_{\mathbb{R}}$ is commonly known as Hamilton's quaternion algebra.

(ii) We denote by $M_2(k)$ the algebra of 2×2-matrices over k. We have an isomorphism

$$(1,1)_k \xrightarrow{\simeq} M_2(k)$$

$$i \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, j \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

When a quaternion algebra $(a,b)_k$ is isomorphic to $M_2(k)$, it said to be *split*.

To the pair (a,b), we can also associate the conic $C(a,b)_k$ over k given by the homogeneous equation

$$ax^2 + by^2 = z^2.$$

The algebra $(a,b)_k$ and the curve $C(a,b)_k$ share a close connection. A first glimpse of this connection is the following theorem.

Theorem 1.2 ([GS17, Proposition 1.3.2])**.** *The quaternion algebra $(a,b)_k$ is split if and only if the conic $C(a,b)_k$ admits a k-rational point. In other words, if the equation*

$$ax^2 + by^2 = z^2$$

admits a solution (x_0, y_0, z_0) with coordinates in k, not all zero.

This connection becomes especially apparent with the next theorem.

Theorem 1.3 ([GS17, Theorem 1.4.2])**.** *Two quaternion algebras $(a,b)_k$ and $(c,d)_k$ are isomorphic if and only if the associated conics $C(a,b)_k$ and $C(c,d)_k$ are.*

Notice that the latter Theorem implies the former.

It is now reasobale to ask if there is some "conceptual explanation" for the close connection between these objects. Indeed, there is. Let $k \subset k^s$ be a separable closure of k. We have the following two characterizations of quaternion algebras and conics.

Proposition 1.4. *(i) Let A be an algebra over k, then A is isomorphic to a quaternion algebra if and only if $A \otimes_k k^s$ is isomorphic to $M_2(k^s)$ as an algebra over k^s.*

(ii) *Let C be an algebraic curve over k. Then C is isomorphic to a conic if and only $C \times_k k^s$ is isomorphic to $\mathbb{P}^1_{k^s}$ as algebraic curves over k^s.*

Proof. (i) This is [GS17, Proposition 2.2.5].

(ii) The reader can take this fact for granted, but for those interested, we briefly sketch the proof. Suppose C is a conic over k, then $C \times_k k^s$ admits a rational point by the fact that any variety over a separably closed field admits a rational point. A conic with a rational point is always isomorphic to $\mathbb{P}^1$. Conversely, if C is an algebraic curve over k such that $C \times_k k^s \simeq \mathbb{P}^1_{k^s}$, then C is a smooth projective curve of genus 0 over k. By the Theorem of Riemann-Roch, such a curve is a conic: embed it in $\mathbb{P}^2$ using the anticanonical divisor (which is very ample), and deduce by Riemann Roch that the resulting plane curve is of degree 2.

■

Remark 1.5. For all intents and purposes, the object $A \otimes_k k^s$, respectively $C \times_k k^s$, should be thought of as "A viewed as an algebra over k^s", respectively "C viewed as a curve over k^s". In particular, we have

$$(a,b)_k \otimes_k k^s = (a,b)_{k^s} \quad \text{and} \quad C(a,b)_k \times_k k^s = C(a,b)_{k^s}.$$

Morally, Proposition 1.4 should be thought of as saying: "all quaternion algebras are geometrically the same as $M_2(k^s)$", and all conics are "geometrically the same as $\mathbb{P}^1$".

The following observation is now a first hint as to why quaternions and conics are "the same".

Proposition 1.6 ([GS17, Lemma 2.4.1])**.** *Any automorphism φ of $M_2(k^s)$ as an algebra over k^s is inner: there exists some $X \in \mathrm{GL}_2(k^s)$ such that for $Y \in M_2(k^s)$ we have*

$$\varphi(Y) = XYX^{-1}.$$

The matrix X is unique with this property, up to multiplication by a scalar from k^s.

As a result of the above proposition, we find an isomorphism of groups

$$\mathrm{PGL}_2(k^s) := \mathrm{GL}_2(k^s)/(k^s)^\times \xrightarrow{\simeq} \mathrm{Aut}_{k^s}(M_2(k^s)).$$

But $\mathrm{PGL}_2(k^s)$ is also precisely the automorphism group of $\mathbb{P}^1_{k^s}$! In other words, the objects $M_2(k^s)$ and $\mathbb{P}^1_{k^s}$ "have the same symmetries". The following theorem now hits it home.

Theorem 1.7. *[Zoc22, Theorem 7.10] There are bijections of pointed sets*

$$\{\textit{quaternion algebras over } k\} \simeq H^1(\mathrm{Gal}(k^s/k), \mathrm{PGL}_2(k^s)) \simeq \{\textit{conics over } k\},$$

under which the quaternion algebra $(a,b)_k$ corresponds to the conic $C(a,b)_k$.

In particular, we obtain Theorem 1.2 as a corollary. The sets in the theorem above are really "isomorphism classes of (...)". The theorem above is a corollary of the general theory of "twists" or "twisted forms". It is described in great generality in [Zoc24]. For a quick overview, one can consult [Poo17, Section 4.5]. "Azumaya algebras of dimension 4" in Poonen's language, are just the quaternion algebras described here. "Severi-Brauer varieties of dimension 1", are just conics over k.

2. Possible goals for a report

Developing the machinery necessary to provide a proof of Theorem 1.3 and at least the implication ⇒ in Theorem 1.2 should be possible for any student. A student who has seen a little bit more algebraic geometry, can also prove the converse implication in Theorem 1.2 directly, as is done in [GS17]. The approach using twists is probably the most challenging one, but also the most enlightening one. A complete proof of the isomorphisms in Theorem 1.7 is a little subtle, but can certainly be sketched. It requires a technique known as Galois descent. Using the theory of twisted forms, it is also easy to generalize the theorems to statements about so called "Central simple algebras" and "Severi-Brauer varieties". This is also done in the book of Gille and Szamuely.

2.1. Prerequisites.

You need to be familiar with a little bit of algebra, in particular, the notion of a (non-commutative) algebra over a field plays a big role in this project. It would be nice if you knew a little bit about algebraic curves. If you want to explore the direction involving Galois cohomology, then perhaps you need to know some Galois theory, but not much more than the definition of a Galois group.

References

[GS17] Philippe Gille and Tamás Szamuely. *Central Simple Algebras and Galois Cohomology*. Second edition. Vol. 165. Cambridge studies in advanced mathematics. Cambridge University Press, 2017.

[Poo17] Bjorn Poonen. *Rational points on varieties*. Vol. 186. Graduate Studies in Mathematics. Providence, Rhode Island: American Mathematical Society, 2017.

[Zoc22] Hugo Zock. *Bachelor's thesis: Twists*. 2022.

[Zoc24] Hugo Zock. *Twists*. 2024.