Cyclic Homology

k comm unital ring, A/k assoc. alg.

Recap

- For $n \geq 1$: $b : A^{\otimes n+1} \longrightarrow A^{\otimes n}$

  gave "Hochschild complex":

$$HH(A): \quad \cdots \xleftarrow{0} \underset{\text{deg. } 0}{A} \xleftarrow{b} A^{\otimes 2} \xleftarrow{b} A^{\otimes 3} \xleftarrow{b} \cdots$$

$$\begin{tikzcd} A^{\otimes n+1} \arrow[r, shift left=3, "d_0"] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=3, "d_n"'] & A^{\otimes n} \arrow[l, shift left=1, "s_0"'] \end{tikzcd} \quad \overset{s_i}{\leftarrow} \text{ if } A \text{ unital}$$

$$d_i(a_0 \otimes \dots \otimes a_n) = \begin{cases} a_0 \otimes \dots \otimes a_i a_{i+1} \otimes \dots \otimes a_n, & i = 0, \dots, n-1 \\ a_n a_0 \otimes a_1 \otimes \dots \otimes a_{n-1}, & i = n. \end{cases}$$

$\rightsquigarrow b = \sum_{i=0}^{n} (-1)^i d_i$

- $n = 1$: $b(x \otimes y) = x \cdot y - y \cdot x$;
- $n = 2$: $b(x \otimes y \otimes z) = x \cdot y \otimes z - x \otimes y \cdot z + z \cdot x \otimes y$.
- $n = 0$: $HH_0(A) = A/[A,A]$
- $n = 1$, A comm. unital

$$HH_1(A) \xrightarrow{\sim} \Omega^1_{A/k}$$
$$x \otimes y \mapsto x \cdot dy.$$

- For $n \geq 2$: $b' : A^{\otimes n+1} \longrightarrow A^{\otimes n}$,

$$b' = \sum_{i=0}^{n-1} (-1)^i d_i$$

gave "bar resolution"

$$C^{bar}_\bullet : \quad \cdots \xleftarrow{0} \underset{\text{deg. } 0}{A^{\otimes 2}} \xleftarrow{b'} A^{\otimes 3} \xleftarrow{b'} \cdots$$

Then $C^{bar}_\bullet \overset{\mu\,(=b')}{\underset{\sim}{\longrightarrow}} A[0]$
$x \otimes y \mapsto x \cdot y$

resol'n of A as $A \otimes_k A^{op}$-module for A unital:

constr. contraction

$$s = s_0 : A^{\otimes n} \longrightarrow A^{\otimes n+1}$$
$$a_1 \otimes \dots \otimes a_n \mapsto 1 \otimes a_1 \otimes \dots \otimes a_n.$$

Cyclic homology

For $n \geq 0$: $\mathbb{Z}/(n+1)\mathbb{Z} \circlearrowright A^{\otimes n+1}$ by (t_n generator)

$$t_n(a_0 \otimes \dots \otimes a_n) = (-1)^n (a_n \otimes a_0 \otimes \dots \otimes a_{n-1}).$$

Write $N = 1 + t + \dots + t^n : A^{\otimes n+1} \longrightarrow A^{\otimes n+1}$ for the "norm".

Lemma $(1-t)b' = b(1-t)$, $Nb = b'N$.

Sketch $d_i t_n = -t_{n-1} d_{i-1}$, $d_0 t_n = (-1)^n d_n$

$\rightsquigarrow ((-1)^i d_i)t = t((-1)^{i-1} d_{i-1})$

$\rightsquigarrow bt = \sum_{i=0}^{n} (-1)^i d_i t = t \sum_{i=1}^{n} (-1)^{i-1} d_{i-1} + (-1)^n d_n$

$= tb' + b - b' \rightsquigarrow (1-t)b' = b(1-t)$. $\boxtimes$

Define bicomplex $CC(A)$ "anti-commuting"

$$\begin{tikzcd} \vdots \arrow[d, "b"] & \vdots \arrow[d, "-b'"'] & \vdots \arrow[d, "b"] & \vdots \arrow[d, "-b'"'] & \\ A^{\otimes 2} \arrow[d, "b"] & A^{\otimes 2} \arrow[l, "1-t"'] \arrow[d, "-b'"'] \arrow[u, bend right, "s"'] & A^{\otimes 2} \arrow[l, "N"'] \arrow[d, "b"] & A^{\otimes 2} \arrow[l, "1-t"'] \arrow[d, "-b'"'] \arrow[u, bend right, "s"'] & \cdots \arrow[l, "N"'] \\ A & A \arrow[l, "1-t"'] \arrow[u, bend right, "s"'] & A \arrow[l, "N"'] & A \arrow[l, "1-t"'] \arrow[u, bend right, "s"'] & \cdots \arrow[l, "N"'] \end{tikzcd}$$

Def $HC(A) = \operatorname{Tot} CC(A)$

$\rightsquigarrow HC_n(A) = H_n(\operatorname{Tot} CC(A))$

Bicomplex $B(A)$ Assume A unital

For $n \geq 0$ get

$$B = (1-t)sN : A^{\otimes n+1} \longrightarrow A^{\otimes n+2}.$$

Then (by formal reasons)

$$bB = -Bb, \quad B^2 = 0.$$

Obtain a bicomplex $B(A)$

$$\begin{tikzcd} \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] \\ A^{\otimes 3} \arrow[d, "b"] & A^{\otimes 2} \arrow[l, "B"'] \arrow[d, "b"] & A \arrow[l, "B"'] \\ A^{\otimes 2} \arrow[d, "b"] & A \arrow[l, "B"'] & \\ A & & \end{tikzcd}$$

Exp. $B : A \to A^{\otimes 2}$

$$x \overset{N}{\longmapsto} x \overset{s}{\longmapsto} 1 \otimes x \overset{1-t}{\longmapsto} 1 \otimes x - x \otimes 1$$

Have a natural morphism of complexes

$$\operatorname{Tot} B(A) \longrightarrow HC(A) = \operatorname{Tot} CC(A)$$

$$\begin{array}{ccc} B(A)_{p,q} \ni x & \longmapsto & x \oplus sN(x) \\ \parallel & & \cap \\ CC_{2p,q-p} & & CC_{2p,q-p} \oplus CC_{2p-1,q-p+1} \\ & & \downarrow \\ & & (\operatorname{Tot} CC(A))_{p+q} \end{array}$$

Thm. The map

$$\operatorname{Tot} B(A) \xrightarrow{\sim} HC(A)$$

is a quasi-isomorphism.

Computations

$A = k$: Recall

$$HH(k): \quad \cdots \xrightarrow{1} k \xrightarrow{0} k \xrightarrow{1} k \xrightarrow{0} k$$

$$\Rightarrow HH_n(k) = \begin{cases} k, & n = 0 \\ 0, & \text{else.} \end{cases}$$

From $B(A)$ get spectral sequence

$$E^2_{pq} = H^{hor}_p H^{ver}_q B(A) \Rightarrow HC_{p+q}(A).$$

$E^2 = E^\infty$

$$\begin{array}{ccc} & & \iddots \\ & & k \\ & k & \\ k & & \end{array} \quad \rightsquigarrow HC_n(k) = \begin{cases} k, & n \in 2\mathbb{Z} \\ 0, & \text{else} \end{cases}$$

$HC_0(A)$:

$$\begin{tikzcd} x\otimes y \arrow[d, mapsto] & A^{\otimes 2} \arrow[d, "b"] & 0 \arrow[d, phantom, "\shortparallel"] \\ x\cdot y - y\cdot x & A & A \arrow[l, "1-t"'] \end{tikzcd} \rightsquigarrow HC_0(A) = A/[A,A] = HH_0(A).$$

$HC_1(A)$ for A comm. unital

Recall
- $b: A\otimes A \to A$ is zero (A comm.)
- $b: A^{\otimes 3} \to A^{\otimes 2}$, $x\otimes y\otimes z \mapsto x\cdot y\otimes z - x\otimes y\cdot z + z\cdot x\otimes y$

Have seen

$$HH_1(A) = A\otimes A/\langle x\cdot y\otimes z - x\otimes y\cdot z + z\cdot x\otimes y\rangle \xrightarrow{\simeq} \Omega^1_{A/k}$$
$$x\otimes y \mapsto xdy$$

For $HC_1(A)$:

$$\begin{tikzcd} A^{\otimes 3} \arrow[d, "b"] & \\ A^{\otimes 2} & A \arrow[l, "B"'] \\ 1\otimes x - x\otimes 1 & x \arrow[l, mapsto] \end{tikzcd}$$

$$\rightsquigarrow HC_1(A) = HH_1(A)/\langle 1\otimes x - x\otimes 1\rangle \xrightarrow{\simeq} \Omega^1_{A/k}/dA$$

Connes Periodicity

Thm. A assoc. k-algebra. There is a natural l.e.s.

$$\dots \to HH_n(A) \xrightarrow{I} HC_n(A) \xrightarrow{S} HC_{n-2}(A) \xrightarrow{B} HH_{n-1}(A) \xrightarrow{I} \dots$$

Rmk. $S^1 \to E \to B$ topological S^1-fibr.

Have Serre spectral sequence

$$E^2_{pq} = H_p(B, H_q(S_1)) \Rightarrow H_{p+q}(E)$$

E^2

$$\begin{tikzcd} H_0(B) & H_1(B) \arrow[l] & H_2(B) \arrow[l] & \dots \\ H_0(B) & H_1(B) & H_2(B) \arrow[ull, "d"] & H_3(B) \arrow[ull, "d"] \dots \end{tikzcd}$$

Get Gysin sequence

$$\dots \to H_3(B) \xrightarrow{d} H_1(B) \to H_2(E) \to H_2(B) \to H_0(B) \to$$
$$\to H_1(E) \to H_1(B) \to 0$$

periodicity map

Proof of Connes' periodicity

For simpl.: A unital.

Have s.e.s of bicomplexes

$$0 \to HH(A) \to B(A) \to B(A)[1,1] \to 0$$

"as first column"

"shift to right and up"

$\rightsquigarrow$ get ses of complexes

$$0 \to HH(A) \to HC(A) \to HC(A)[2] \to 0$$

Get les

$$\dots \to HH_n(A) \xrightarrow{I} HC_n(A) \xrightarrow{S} HC_{n-2}(A) \xrightarrow{B} HH_{n-1}(A) \to \dots$$

□

Exp. $A = k$: $S: HC_n(A) \xrightarrow{\simeq} HC_{n-2}(A)$.

Rmk. S can be made explicit also, in char. 0 but I won't do this

Bonus Connes' periodicity is a Gysin sequence.

X "cyclic set" $\rightsquigarrow k[X]$ cyclic k-module

Can form

$$HH(k[X]) = C(k[X]) = C(|X|, k).$$

E.g. G a group, $X: [n] \mapsto G^{n+1}$. Then $k[X]: [n] \mapsto k[G]^{\otimes n+1}$ and

$$C(k[X]) = HH(k[G])$$

Hochschild complex.

X cyclic set $\rightsquigarrow |X|$ inherits S^1-action.

Can form $|X|_{hS^1} = ES^1 \times_{S^1} X$ "homotopy orbits".

Thm. $HC(k[X]) \simeq C(|X|_{hS^1}, k)$

Have fibr. sequence

$$S^1 \to |X| \to |X|_{hS^1}$$

$\rightsquigarrow$ Gysin sequence = les from Connes periodicity.

cyclic set $X: [n] \mapsto G^{n+1}$

$d_i: G^{n+1} \to G^n$

$(g_0, \dots, g_n) \mapsto (g_0, \dots, g_i g_{i+1}, \dots, g_n)$, $i = 0, \dots, n-1$

" $\mapsto (g_n g_0, g_1, \dots, g_{n-1})$, $i = n$;

$s_j: G^{n+1} \to G^{n+2}$, $j = 0, \dots, n$

$(g_0, \dots, g_n) \mapsto (g_0, \dots, g_j, 1, g_{j+1}, \dots, g_n)$.

Cyclic Simplicial Objects

Connes' cyclic cat. $C\Delta$ contains Δ but w. additional arrows $c_n : [n] \to [n]$ subject to

- $c_n d_i = d_{i-1} c_{n-1}$, $c_n d_0 = d_n$;
- $c_n s_i = s_{i-1} c_{n+1}$, $c_n s_0 = s_n c_{n+1}^2$;
- $c_n^{n+1} = id$.

Then cyclic object in $\mathcal{C}$ is a presheaf $C\Delta^{op} \to \mathcal{C}$. Can define $HH(X)$ and $HC(X)$ for $X : C\Delta^{op} \to k\text{-mod}$ any cyclic k-module.