# Smooth base change I.

**Recall** $f: X \to Y$ is **smooth** at $x \in X$ if $\exists x \in U \subseteq X$, $\underset{\ni f(x)}{V \subseteq Y}$ affine opens, ~~$f(U) \subseteq V$~~, $\pi: U \to \mathbb{A}^n_V$ étale s.t.

$$\begin{tikzcd}
X \arrow[d, "f"] & U \arrow[l, hook'] \arrow[r, "\pi"] \arrow[d, "f"] & \mathbb{A}^n_V \arrow[dl] \\
Y & V \arrow[l, hook']
\end{tikzcd}$$

commutes.

**E.g.** $C/_{k = k^{perf}}$ a curve, $x \in C$ smooth

$\iff \exists \pi \in \mathcal{O}_{C,x}$ unif.

$\rightsquigarrow k[x]_{(x)} \to \mathcal{O}_{C,x}$, $x \mapsto \pi$ étale $\rightsquigarrow$ spread out to an étale map $x \in U \xrightarrow{\pi} \mathbb{A}^1_k$.

**Thm.** Let

$$\begin{tikzcd}
X' \arrow[r, "g'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\
S' \arrow[r, "g"] & S
\end{tikzcd}$$

cartesian w. $f$ qcqs, $g$ smooth.

Then, for $F \in Ab(X_{et})$, ~~prime~~ ~~torsion~~ torsion w. orders inv. on $S$, ~~$\varinjlim_{n \in \mathcal{O}_S^\times \cap \mathbb{Z}} n.F = 0$~~, the natural map

$$g^* R^i f_* F \xrightarrow{\simeq} R^i f'_* g'^* F \quad (i \geq 0)$$

is an iso.

**appl. 1** $X \to \operatorname{Spec} k$ qcqs, $k = \bar{k}$. Let $k \subset L$ w. $L = \bar{L}$ (e.g. $\bar{\mathbb{Q}} \subset \mathbb{C}$). Then

$$H^i(X; \mathbb{F}_\ell) \xrightarrow{\simeq} H^i(X_L; \mathbb{F}_\ell) \quad (i \geq 0)$$

for $\ell \in k^\times$.

**Proof.** $L = \varinjlim R$ over $k \overset{\text{smooth}}{\subset} R \subset L$. Then

$$H^i(X_L; \mathbb{F}_\ell) = \varinjlim H^i(X_R; \mathbb{F}_\ell) \underset{(SBC)}{=} H^i(X; \mathbb{F}_\ell). \qquad \square$$

appl. 2 (Ehresmann's Lemma) For $f: X \to Y_1$ smooth proper, $\underbrace{R^i f_* \mathbb{F}_\ell \text{ locally constant constructible}}_{(lcc)}$ $\ell \in \mathcal{O}_Y^\times$

(also for $\mathbb{Q}_\ell$)

Motto: $\ell$-adic betti #'s are loc. constant in sm. proj. families.

Lcc sheaves

$X$ a scheme, $\bar{x} \to X$ geom. pt. Set

$$X_{(\bar{x})} := \operatorname{Spec} \mathcal{O}^{hs}_{X,\bar{x}}$$

$$:= \varinjlim_{\substack{\bar{x} \to U \\ \downarrow \text{ étale} \\ X}} \mathcal{O}(U)$$

For $\bar{\eta} \to X \leftarrow \bar{x}$ a specialization $\bar{\eta} \rightsquigarrow \bar{x}$ is a morph. $\bar{\eta} \to X_{(\bar{x})}$ s.t.

$$\begin{tikzcd} \bar{\eta} \arrow[r] \arrow[dr] \arrow[r, phantom, shift right=4ex, "\circlearrowleft"] & X_{(\bar{x})} \arrow[d] \\ & X \end{tikzcd}$$

For $F$ ~~ab($X_{et}$)~~ on $X_{ét}$ and $\bar{\eta} \rightsquigarrow \bar{x}$ we define

$$sp : F_{\bar{x}} = F(X_{(\bar{x})}) \longrightarrow F_{\bar{\eta}}.$$

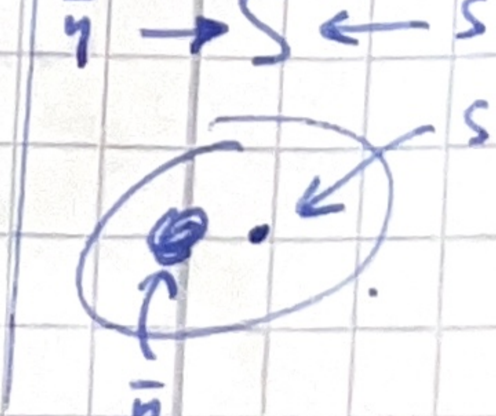
Def. $F$ on $X_{ét}$ is lcc if

(i) for all $\bar{x} \to X$, $F_{\bar{x}}$ is finite.

(ii) For all $\bar{\eta} \rightsquigarrow \bar{x}$, $sp : F_{\bar{x}} \xrightarrow{\simeq} F_{\bar{\eta}}$ is iso.

Locally acyclic morphisms

Motivation: $\mathcal{X} \xrightarrow{f} S$ proper, $S = \operatorname{Spec} R$ (str. henselian dvr). $\bar{\eta} \to S \leftarrow s$




Q.: "How do $H^i(\mathcal{X}_{\bar{\eta}}; \mathbb{F}_\ell)$ and $H^i(\mathcal{X}_s; \mathbb{F}_\ell)$ compare?" for $F = R^i f_* \mathbb{F}_\ell$.

Have $\exists$ $sp : H^i(\mathcal{X}_s; \mathbb{F}_\ell) \longrightarrow H^i(\mathcal{X}_{\bar{\eta}}; \mathbb{F}_\ell)$.

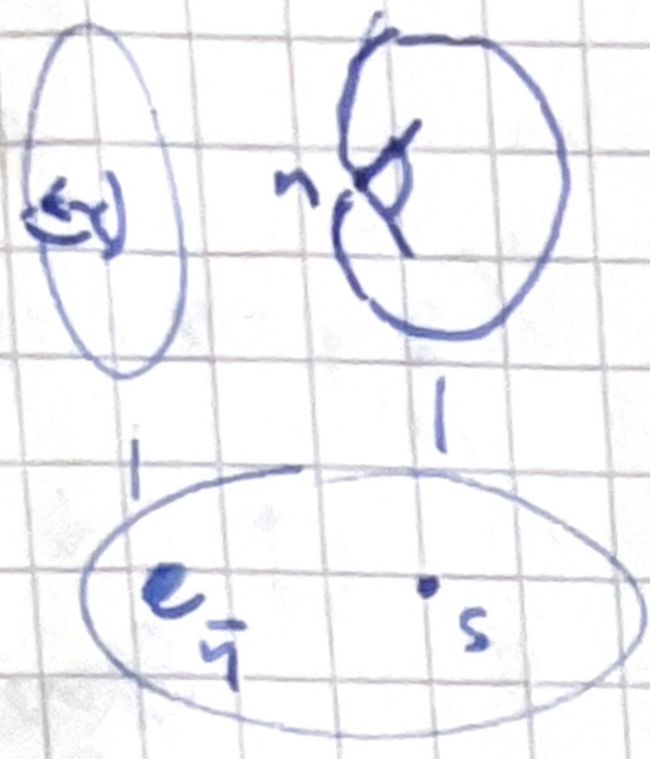
$\exists R\Phi \mathbb{F}_\ell \in D^b_c(\mathcal{X}_{\bar{s},ét}; \mathbb{F}_\ell)$ s.t. we have a les:

$$\cdots \to H^i(\mathcal{X}_{\bar{s}}; \mathbb{F}_\ell) \xrightarrow{sp} H^i(\mathcal{X}_{\bar{\eta}}; \mathbb{F}_\ell) \to H^i(\mathcal{X}_{\bar{s}}; R\Phi\mathbb{F}_\ell)$$
$$\to H^{i+1}(\mathcal{X}_s; \mathbb{F}_\ell) \xrightarrow{sp} \cdots$$

"compl. of vanishing cycles"

Exp. $\mathcal{X} \xrightarrow{f} S$ ell. fibration, $\mathcal{X}_s$ rational w. 1 node, $n \in \mathcal{X}_s$ : ~~e.g. $y^2 = x^2(x-1)$~~

$$\mathcal{O}^{hs}_{\mathcal{X}_s,n} \simeq \left(k(s)[x,y]/(xy)\right)^{hs}_{(x,y)}$$



Then

$$\begin{tikzcd}
0 \arrow[r] & H^1(\mathcal{X}_s;\mathbb{F}_\ell) \arrow[r, "sp"] \arrow[d, phantom, "\rotatebox{90}{$\simeq$}"] & H^1(\mathcal{X}_{\bar\eta};\mathbb{F}_\ell) \arrow[r] \arrow[d, phantom, "\rotatebox{90}{$\simeq$}"] & H^1(\mathcal{X}_s;R\Phi\mathbb{F}_\ell) \arrow[d, phantom, "\rotatebox{90}{$\simeq$}"] \\
& \mathbb{F}_\ell & \mathbb{F}_\ell^{\oplus 2} \arrow[r] & 0. \quad \mathbb{F}_\ell
\end{tikzcd}$$

In fact : $R\Phi\mathbb{F}_\ell \simeq \mathbb{F}_{\ell,n}[-1]$.

skyscraper sheaf.

Nearby and vanishing cycles

$X \xrightarrow{f} S$, $S$ str. henselian local. $\bar\eta \to S$ some geom. pt., $s \to S$ closed point.

Def. For $K \in D^+(X_{ét};\mathbb{F}_\ell)$, we set

$$R\Psi K := i^* Rj_* j^* K \in D^+(X_s;\mathbb{F}_\ell).$$

complex of nearby cycles.

$$\begin{tikzcd}
X_s \arrow[r, "i"] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \arrow[dr, phantom, "\square"] & X_{\bar\eta} \arrow[l, "j"'] \arrow[d] \\
s \arrow[r] & S & \bar\eta \arrow[l]
\end{tikzcd}$$

The point : natural map

$$i^*K \longrightarrow R\Psi K$$

induces for $f$ proper :

$$\begin{tikzcd}
H^i(X_s;i^*K) \arrow[r] \arrow[ddr, "sp"'] & H^i(X_s;R\Psi K) \\
& H^i(X;Rj_*j^*K) \arrow[u, "\simeq"', "(PBC)"] \arrow[d, leftrightarrow, "\simeq"] \\
& H^i(X_{\bar\eta};j^*K)
\end{tikzcd}$$

Define $R\Phi K$ as a cone

$$K \longrightarrow R\Psi K \longrightarrow R\Phi K \longrightarrow K[1].$$

complex of ~~nearby~~ vanishing cycles

## Local computation of $R\Psi K$.

**Prop.** For $\bar{x} \to X$ a geom. pt., $\bar{x} \mapsto s$, we have

$$\begin{tikzcd} K_{\bar x} \arrow[r] \arrow[dr, "\cong"'] & R\Gamma((X_{(\bar x)})_{\bar\eta}; K) \arrow[d, phantom, "\wr\!\!\|"] \\ & (R\Psi K)_{\bar x}. \end{tikzcd}$$

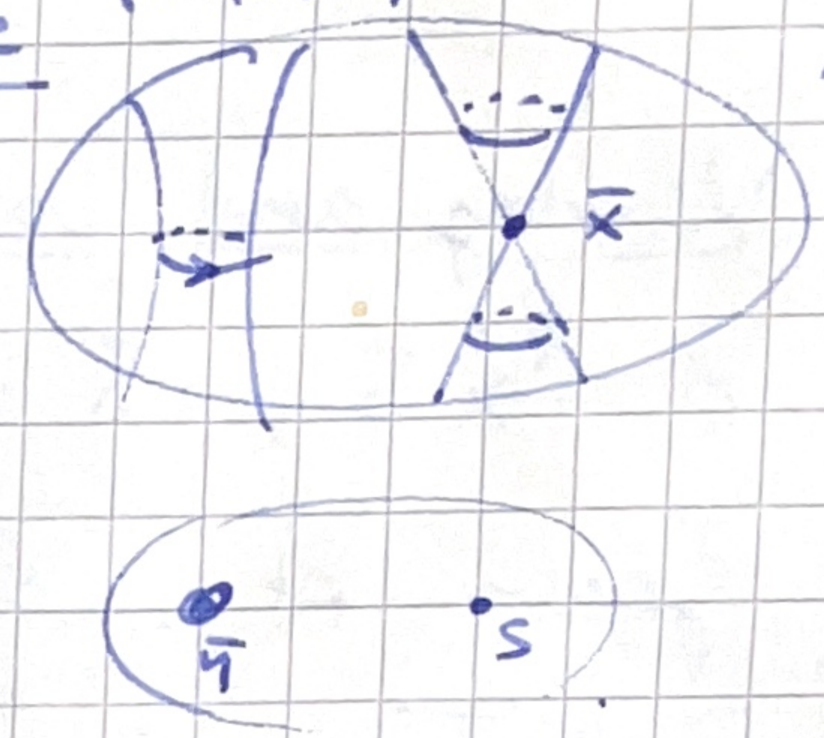
$$\begin{tikzcd} (X_{(\bar x)})_{\bar\eta} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X_{(\bar x)} \arrow[d] \\ \bar\eta \arrow[r] & S \end{tikzcd}$$

$\Leftarrow$ "Milnor fiber"

**Proof**

$$\begin{tikzcd} \bar x \arrow[r] \arrow[d] & X_{(\bar x)} \arrow[d] & (X_{(\bar x)})_{\bar\eta} \arrow[l, "j'"'] \arrow[d] \\ X_{\bar s} \arrow[r] \arrow[d] & X \arrow[d] & X_{\bar\eta} \arrow[l, "j"'] \arrow[d] \\ \bar s \arrow[r] & S & \bar\eta \arrow[l] \end{tikzcd}$$

**Picture**




$$\begin{aligned} (R\Psi K)_{\bar x} &= (Rj_* j^* K)_{\bar x} \\ &= R\Gamma R j'_*(K|(X_{(\bar x)})_{\bar\eta}) \\ &= R\Gamma((X_{(\bar x)})_{\bar\eta}; K). \end{aligned}$$ $\square$

~~$K \in D^+(X_{et}; \mathbb{F}_\ell)$~~

**Def.** $f: X \to S$ a morphism of schemes is locally acyclic ~~for $K$~~ if for all $\bar\eta \rightsquigarrow \bar s$ on $S$,

$\mathbb{F}_\ell \xrightarrow{\sim} R\Psi\mathbb{F}_\ell$ is an iso $\Leftrightarrow$ $R\Phi\mathbb{F}_\ell = 0$.

$R\Psi$ computed w.r.t. $X \times_S S_{(\bar s)} \to S_{(\bar s)}$, with $\bar\eta \to S_{(\bar s)}$.

~~"f is locally acyclic" if loc. ac. for $\mathbb{F}_\ell$.~~

$\Leftrightarrow$ $\forall \bar x \to X$ over $\bar s$: $\mathbb{F}_\ell \xrightarrow{\sim} R\Gamma((X_{(\bar x)})_{\bar\eta}; \mathbb{F}_\ell)$.

**Prop.** $f: X \to S$ smooth $\Rightarrow$ $f$ loc. acyclic. (See last page)

~~Proof. Can assume $S = S_{(\bar s)}$.~~

$$\begin{tikzcd} X_{\bar s} \arrow[r, "i"] \arrow[d] & X \arrow[d] \arrow[dr, phantom, "\square"] & X_{\bar\eta} \arrow[l, "j"'] \arrow[d] \\ \bar s \arrow[r] & S & \bar\eta \arrow[l] \end{tikzcd}$$

Suff. to show: $\mathbb{F}_\ell \xrightarrow{\sim} Rj_*\mathbb{F}_\ell$ on $X_{et}$.

By SBC only need: ~~$\mathbb{F}_\ell \xrightarrow{\sim} R\bar\eta_*\mathbb{F}_\ell$.~~

(1) $R^*\bar\eta_*\mathbb{F}_\ell = 0$, $* > 0$;

(2) $(\bar\eta_*\mathbb{F}_\ell)(S) = \mathbb{F}_\ell$.

This is clear: $S$ str. henselian. $\square$

# Bonus

— $C$ proper rational curve w. $C^{\mathrm{sing}} = \{n\}$, $n \in C$ a nodal point. Claim:

$$H^*(C;\mathbb{F}_\ell) = \begin{cases} \mathbb{F}_\ell, & * = 0,1,2; \\ 0, & * > 2. \end{cases}$$

Let $\nu : \mathbb{P}^1 \to C$ normalization: 

On $C$, have an exact sequence

$$\circledast \quad 0 \longrightarrow \mathbb{F}_\ell \longrightarrow \nu_* \mathbb{F}_\ell \longrightarrow \mathbb{F}_{\ell,n} \longrightarrow 0.$$

Easy to show: $\nu_* \mathbb{F}_\ell \xrightarrow{\simeq} R\nu_* \mathbb{F}_\ell$.

$\Rightarrow H^*(C;\nu_*\mathbb{F}_\ell) = H^*(\mathbb{P}^1;\mathbb{F}_\ell)$.

LES of $\circledast$:

$$0 \to H^0(C;\mathbb{F}_\ell) \xrightarrow{\simeq} \underbrace{H^0(\mathbb{P}^1;\mathbb{F}_\ell)}_{\mathbb{F}_\ell} \to \underbrace{H^0(C;\mathbb{F}_{\ell,n})}_{\mathbb{F}_\ell}$$

$$\xrightarrow{\simeq} H^1(C;\mathbb{F}_\ell) \to H^1(\mathbb{P}^1;\mathbb{F}_\ell) = 0 \to H^1(C;\mathbb{F}_{\ell,n}) = 0$$

$$\to H^2(C;\mathbb{F}_\ell) \xrightarrow{\simeq} \underbrace{H^2(\mathbb{P}^1;\mathbb{F}_\ell)}_{\mathbb{F}_\ell} \to H^2(C;\mathbb{F}_{\ell,n}) = 0.$$

— Rmk. $X \to \operatorname{spec} k$ qcqs, $\underset{p=0 \text{ in } k}{k} = k^{\mathrm{alg}}$, $k \subset L = L^{\mathrm{alg}}$

$\not\Rightarrow H^*(X;\mathbb{F}_p) \xrightarrow{\simeq} H^*(X_L;\mathbb{F}_p)$.

E.g. $X = \mathbb{A}^1_k$. Artin–Schreier: $0 \to \mathbb{F}_p \to \mathbb{G}_a \xrightarrow{F-1} \mathbb{G}_a \to 0$

$\rightsquigarrow H^1(X;\mathbb{F}_p) \simeq k/\langle t^p - t : t \in k\rangle$

$\not\simeq L/\langle t^p - t : t \in L\rangle = H^1(X_L;\mathbb{F}_p)$.

— Rmk. $f : \mathfrak{X} \to S$ smooth proper, $p \notin \mathcal{O}_S^\times$

$\not\Rightarrow R^i f_* \mathbb{F}_p$ lcc.

E.g. $\mathfrak{X}$ curve of genus $g$ over $\mathbb{Z}_p^{\mathrm{nr}}$

$\Rightarrow \dim_{\mathbb{F}_p} H^1(\mathfrak{X}_{\bar{\mathbb{Q}}_p};\mathbb{F}_p) = 2g$;

$0 \leq \dim_{\mathbb{F}_p} H^1(\mathfrak{X}_{\bar{\mathbb{F}}_p};\mathbb{F}_p) \leq g$

Artin–Schreier

<u>Proof</u> Can assume $S = S_{(\bar{s})}$.

$$\begin{tikzcd} X_{\bar{s}} \arrow[r, "i"] \arrow[d] & X \arrow[d, "f"] & X_{\bar{\eta}} \arrow[l, "j"'] \arrow[d] \\ \bar{s} \arrow[r] & S & \bar{\eta} \arrow[l] \end{tikzcd}$$

<u>To show</u>:

$$\mathbb{F}_\ell \xrightarrow{\sim} i^* Rj_* \mathbb{F}_\ell = R\Psi \mathbb{F}_\ell$$

iso on $X_{\bar{s}}$.

Have ~~Suff. to show~~ :

(1) $R^v \bar{\eta}_* \mathbb{F}_\ell = 0 \quad (v > 0)$;

(2) $\bar{\eta}_* \mathbb{F}_\ell (S) = \mathbb{F}_\ell$.

(1) + SBC $\Rightarrow i^* R^v j_* \mathbb{F}_\ell = i^* f^* R^v \bar{\eta}_* \mathbb{F}_\ell = 0 \quad (v>0)$;

(2) $\Rightarrow (i^* j_* \mathbb{F}_\ell)_{\bar{x}} = (j_* \mathbb{F}_\ell)_{\bar{x}} = (f^* \bar{\eta}_* \mathbb{F}_\ell)_{\bar{x}}$

$$= (\bar{\eta}_* \mathbb{F}_\ell)_{\bar{s}}$$

$$= (\bar{\eta}_* \mathbb{F}_\ell)(S) = \mathbb{F}_\ell.$$

~~(1) + (2) clear [illegible]~~ $\square$

<u>Rmk.</u> SBC for $\bar{\eta} \to S$ is all we need

$\to$ crucial case in proof of SBC.