1. Every group is a fundamental group

Fix a group G.

Question 1.1. Does there exist a path-connected space X such that

$$G \simeq \pi_1(X)?$$

▲

It turns out that the answer to the question above is yes! To construct such a space, we use the notion of a CW complex.

First, we write down a presentation of G. This is an exact sequence of groups

$$F' \to F \to G \to 1,$$

where F and F' are free groups. You should think of the generators of F as the generators of G, and of F' as the group describing the relations between these generators. Notice that such a represenation is in no way canonical.

Example 1.2. Let G be the abelian group $\mathbb{Z} \oplus \mathbb{Z}$. A representation of G is given by the exact sequence

$$\mathbb{Z} \to \langle a, b \rangle \to G \to 0.$$

Here $\langle a, b\rangle$ is the free group on the generators a and b. The arrow $\langle a, b\rangle \to G$ sends a to 1 in the first summand, and b to 1 in the second summand. The arrow $\mathbb{Z} \to \langle a, b\rangle$ sends 1 to $aba^{-1}b^{-1}$. We will usually write this simply as

$$G = \langle a, b \mid ab = ba \rangle.$$

This should be read as "G is generated by elements a and b, and there is a single relation between them: $ab = ba$." ▲

Based on this datum of a presentation we construct a topological space X in two steps.

(i) We start with a wedge sum of circles

$$X^{(1)} = \bigvee_{\alpha} S^1$$

running over the generators α of F. In other words, for every generator of F we have a circle, and they are all glued together in a single point. The space $X^{(1)}$ has a canonical basepoint x: the point where all the circles are glued together. Every generator α of F is identified with a loop in $X^{(1)}$ with basepoint x.

(ii) For every generator r of F', its image in F is a word in the generators of F which we view as a composition of loops in $X^{(1)}$. Along this loop we attach a 2-cell D^2. More precisely, we form the pushout

$$\begin{tikzcd} \bigsqcup_r S^1 \arrow[r] \arrow[d] & X^{(1)} \arrow[d] \\ \bigsqcup_r D^2 \arrow[r] & X \end{tikzcd}$$

to construct our space X. In this pushout diagram, the above horizontal map sends the copy of S^1 corresponding to r to the path in $X^{(1)}$ corresponding to the image of r. For the definition of a pushout, see [Hat01, Page 461].

Example 1.3. We return to our previous example. Constructing the space X corresponding to the presentation in that example yields the torus $S^1 \times S^1$.

Theorem 1.4. *The fundamental group $\pi_1(X, x)$ of X at x is isomorphic to G.*

Proof. This is [Hat01, Corollary 1.28]. ■

2. Goals for a report

The proof of Theorem 1.4 is effectively an application of the Van Kampen Theorem. This would therefore be a good start for a report. This is treated in [Hat01, Section 1.2]. A proof can either be sketched or given in full depending on which route the student takes to a proof. The second main ingredient in this project is the notion of a CW complex. Notions taken for granted in this description, such as a pushout, should be recalled in detail. Of course, the proof of Theorem 1.4 itself should be given, and I believe this can be done in quite some detail. Finally, make sure to include lots of example computations to demonstrate the usefulness of your theorems!

2.1. **Prerequisites.** Basic algebraic topology: ideally you have already been introduced to the fundamental group at some point. Basic group theory: group presentations and pushouts of groups.

References

[Hat01] Allen Hatcher. *Algebraic Topology*. 2001. URL: `https://pi.math.cornell.edu/~hatcher/AT/ATpage.html`.