A familiar short exact sequence

We take the following two facts for granted.

- If κ is a finite field, then $\mathrm{cd}(\kappa) \leq 1$; meaning that $H^i(\Gamma_\kappa, M) = 0$ for any torsion module M over $\Gamma_\kappa = \mathrm{Gal}(\overline{\kappa}/\kappa)$ and any integer $i \geq 2$. From this it follows that $\mathrm{scd}(\kappa) \leq 2$: we have $H^i(\Gamma_\kappa, M) = 0$ for *any* module M over Γ_κ and any integer $i \geq 3$. You can find a proof of this in any book on Galois cohomology.
- Let X be a a nice curve over a finite field κ. Let $\overline{X} = X \times_\kappa \overline{\kappa}$ denote its base extension to a curve over $\overline{\kappa}$. Then the group $\mathrm{Pic}_0 \overline{X}$ is divisible, and $H^1(\kappa, \mathrm{Pic}_0 \overline{X}) = 0$. Indeed, $\mathrm{Pic}_0 \overline{X}$ is the group $J(\overline{\kappa})$, where J denotes the jacobian of X; since multiplication by n defines an isogeny, the map $n : J(\overline{\kappa}) \to J(\overline{\kappa})$ is surjective. The assertion about $H^1(\kappa, \mathrm{Pic}_0 \overline{X})$ is a theorem of Lang; in fact, we have $H^1(\kappa, A(\overline{\kappa})) = 0$ for any abelian variety A over κ (see [Ste10, Theorem 2.1]).

Throughout this exercise, κ will denote a finite field, X a nice curve over κ, and $k = \kappa(X)$ the field of functions of X. We denote the set of its closed points (or prime divisors) by $X^{(1)}$. The field k is a so called "global field of positive characteristic".

(a) Establish an isomorphism $\mathrm{Br}\, k \simeq H^2(\kappa, \overline{\kappa}(X)^\times)$, where $\overline{\kappa}(X)$ denotes the field of functions of $\overline{X}$. *Hint: notice that* $\mathrm{Gal}(\overline{\kappa}(X)/\kappa(X)) = \mathrm{Gal}(\overline{\kappa}/\kappa)$.

(b) Establish an isomorphism $\mathrm{Br}\, k \simeq H^2(\kappa, \mathrm{Princ}\, \overline{X})$, where $\mathrm{Princ}\, \overline{X} \subset \mathrm{Div}\, \overline{X}$ denotes the group of principal divisors of $\overline{X}$.

(c) Derive an exact sequence

$$0 \to \mathrm{Br}\, k \to H^2(\kappa, \mathrm{Div}\, \overline{X}) \to H^2(\kappa, \mathrm{Pic}\, \overline{X}) \to 0$$

from the exact sequence

$$0 \to \mathrm{Princ}\, \overline{X} \to \mathrm{Div}\, \overline{X} \to \mathrm{Pic}\, \overline{X} \to 0.$$

(d) Use the fact that ([BLT, Exercise 14.7.8]) $\mathrm{Div}\, \overline{X} = \bigoplus_{v \in X^{(1)}} \mathrm{Ind}_{\Gamma_{\kappa(v)}}^{\Gamma_\kappa}(\mathbb{Z})$ to construct an isomorphism

$$H^2(\kappa, \mathrm{Div}\, \overline{X}) \simeq \bigoplus_{v \in X^{(1)}} \mathrm{Br}\, k_v.$$

Here $\kappa(v)$ denotes the residue field of X at v, and k_v is the "completion of k at the place v". You can use that we have an isomorphism $\mathrm{inv}_v : \operatorname{Br} k_v \xrightarrow{\simeq} \mathbb{Q}/\mathbb{Z}$ as for p-adic fields.

(e) Show that $H^2(\kappa, \operatorname{Pic} \overline{X}) \simeq \mathbb{Q}/\mathbb{Z}$. *Hint: use divisibility of* $\operatorname{Pic}^0 \overline{X}$ *to prove that, not just the first, but all of its higher cohomology groups vanish as a module over* Γ_κ.

(f) Deduce that we have a short exact sequence

$$0 \to \operatorname{Br} k \to \bigoplus_{v \in X^{(1)}} \operatorname{Br} k_v \to \mathbb{Q}/\mathbb{Z} \to 0.$$

Compare this with the Brauer-Hasse-Noether sequence for the Brauer group of a number field from [BLT, Theorem 10.4.5].

Solutions

(a) Fix a separable closure $\overline{k}$ of k, so that we have inclusions

$$k = \kappa(X) \subset \overline{\kappa}(X) \subset \overline{k}.$$

By Hilbert 90 we have $H^1(\overline{\kappa}(X), \overline{k}^\times) = 0$, and hence we obtain an inflation-restriction sequence

$$0 \to H^2(\kappa, \overline{\kappa}(X)^\times) \to H^2(k, \overline{k}^\times) = \operatorname{Br} k \to H^2(\overline{\kappa}(X), \overline{k}^\times) = \operatorname{Br} \overline{\kappa}(X).$$

Here the groups $H^2(\kappa, \overline{\kappa}(X)^\times)$ and $H^2(\overline{\kappa}(X)/k, \overline{\kappa}(X)^\times)$ have been identified: the Galois groups $\operatorname{Gal}(\overline{\kappa}/\kappa)$ and $\operatorname{Gal}(\overline{\kappa}(X)/k)$ are identified via the natural restriction map and this identification is compatible with the action of these groups on $\overline{\kappa}(X)^\times$.

We have $\operatorname{Br} \overline{\kappa}(X) = 0$, since $\overline{\kappa}(X)$ is C_1 by [BLT, Theorem 10.4.8] and the Brauer group of a C_1-field vanishes by [BLT, Theorem 10.4.9]. The result follows.

(b) Consider the exact sequence

$$0 \to \overline{\kappa}^\times \to \overline{\kappa}(X)^\times \xrightarrow{\operatorname{Div}} \operatorname{Princ} \overline{X} \to 0.$$

The long exact sequence in combination with $\operatorname{Br} \kappa = H^2(\kappa, \overline{\kappa}^\times) = H^3(\kappa, \overline{\kappa}^\times) = 0$ (finite fields are C_1 by [BLT, p. 10.4.7] and $\operatorname{scd}(\kappa) \le 2$) shows that

$$H^2(\kappa, \overline{\kappa}(X)^\times) \simeq H^2(\kappa, \operatorname{Princ} \overline{X}).$$

By part (a): $\operatorname{Br} k \simeq H^2(\kappa, \operatorname{Princ} \overline{X})$.

(c) The exact sequence

$$0 \to \operatorname{Pic}_0 \overline{X} \to \operatorname{Pic} \overline{X} \xrightarrow{\deg} \mathbb{Z} \to 0$$

and the fact that $H^1(\kappa, \operatorname{Pic}_0 \overline{X}) = H^1(\kappa, \mathbb{Z}) = 0$ shows $H^1(\kappa, \operatorname{Pic} \overline{X}) = 0$. We have $H^3(\kappa, \operatorname{Princ} \overline{X}) = 0$ by $\operatorname{scd}(\kappa) \le 2$. The result now follows from the long exact sequence.

(d) We apply the fact that Galois cohomology commutes with direct sums and Shapiro's lemma to find

$$\begin{aligned} H^2(\kappa, \operatorname{Div} \overline{X}) &\simeq \bigoplus_{v \in X^{(1)}} H^2(\kappa, \operatorname{Ind}_{\Gamma_{\kappa(v)}}^{\Gamma_\kappa}(\mathbb{Z})) \\ &\simeq \bigoplus_{v \in X^{(1)}} H^2(\kappa(v), \mathbb{Z}) \\ &\simeq \bigoplus_{v \in X^{(1)}} H^1(\kappa(v), \mathbb{Q}/\mathbb{Z}). \end{aligned}$$

Furthermore, we have an isomorphism

$$H^1(\kappa(v), \mathbb{Q}/\mathbb{Z}) \xrightarrow{\simeq} \mathbb{Q}/\mathbb{Z},$$

for every place v, by evaluating at the Frobenius. Composing with inv_v^{-1}, we get what we're after.

(e) We prove that $H^i(\kappa, \mathrm{Pic}^0 \overline{X}) = 0$ for $i \geq 2$. For every $n > 0$, consider the short exact sequence

$$0 \to (\mathrm{Pic}_0 \overline{X})[n] \to \mathrm{Pic}_0 \overline{X} \xrightarrow{\cdot n} \mathrm{Pic}_0 \overline{X} \to 0.$$

From the associated long exact sequence, we obtain for all i a surjection

$$H^i(\kappa, (\mathrm{Pic}_0 \overline{X})[n]) \twoheadrightarrow H^i(\kappa, \mathrm{Pic}_0 \overline{X})[n].$$

For $i \geq 2$, the source of this surjection is trivial: $\mathrm{cd}(\kappa) \leq 1$. Since Galois cohomology groups are torsion, we conclude that $H^i(\kappa, \mathrm{Pic}_0 \overline{X}) = 0$ for $i \geq 2$. Now we use the short exact sequence

$$0 \to \mathrm{Pic}^0 \overline{X} \to \mathrm{Pic}\,\overline{X} \xrightarrow{\deg} \mathbb{Z} \to 0$$

to deduce that

$$H^2(\kappa, \mathrm{Pic}\,\overline{X}) \simeq H^2(\kappa, \mathbb{Z}) \simeq \mathbb{Q}/\mathbb{Z}.$$

(f) This is now obvious.

References

[BLT] Martin Bright, Ronald van Luijk, and Damiano Testa. *Geometry and Arithmetic of Surfaces (DRAFT)*. URL: `http://www.boojum.org.uk/maths/book.html`.

[Har17] David Harari. *Galois Cohomology and Class Field Theory*. Universitext. Springer Cham, 2017. DOI: `https://doi.org/10.1007/978-3-030-43901-9`.

[Ste10] William Stein. *Lecture 14: Galois Cohomology of Abelian Varieties over Finite Fields*. 2010. URL: `https://wstein.org/edu/2010/582e/lectures/582e-2010-02-12/582e-2010-02-12.pdf`.