COHOMOLOGY OF SHEAVES ON SCHEMES: FLAT MORPHISMS

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In algebraic geometry one often encounters families of varieties or schemes indexed by a certain parameter space.

Example 0.1. Every smooth, projective, genus one curve E over $\mathbb{C}$ is isomorphic to the zero locus of a Legendre polynomial

$$E_\lambda = \{y^2 z = x(x-z)(x-\lambda z)\} \subset \mathbb{P}^2_{\mathbb{C}},$$

for some $\lambda \in \mathbb{C} - \{0,1\}$. Set $A = \mathbb{C}[\lambda, (\lambda(\lambda-1))^{-1}]$. Consider the morphism

$$\mathscr{E} = \operatorname{Proj} \frac{A[x,y,z]}{(y^2 z - x(x-z)(x-\lambda z))} \to \operatorname{Spec} A = \mathbb{A}^1_{\mathbb{C}} - \{0,1\}.$$

It fits into a commutative triangle

$$\begin{tikzcd} \mathscr{E} \arrow[r, hook] \arrow[dr] & \mathbb{P}^2 \times (\mathbb{A}^1 - \{0,1\}) \arrow[d] \\ & \mathbb{A}^1 - \{0,1\}. \end{tikzcd}$$

The fiber of $\mathscr{E} \to \mathbb{A}^1_{\mathbb{C}} - \{0,1\}$ over λ is precisely $E_\lambda \subset \mathbb{P}^2_{\mathbb{C}}$.

Naively, one might consider the collection of fibers of *any* morphism of schemes to be a family. However, in order to justify calling a given collection a family, we at least want the objects of the family to share certain invariants or properties. The notion of flatness guarantees that the fibers of a morphism are reasonably well-behaved with respect to each other, while still being flexible enough to include practically everything we might wish to call a family.

Example 0.2. The family $\mathscr{E} \to \mathbb{A}^1 - \{0,1\}$ naturally extends to a family over $\mathbb{A}^1$:

$$\begin{tikzcd} \mathscr{E} \arrow[r, hook] \arrow[d] & \overline{\mathscr{E}} \arrow[r] \arrow[d] & \mathbb{P}^2 \times \mathbb{A}^1 \arrow[d] \\ \mathbb{A}^1 - \{0,1\} \arrow[r] & \mathbb{A}^1 \arrow[r, "="] & \mathbb{A}^1 \end{tikzcd}$$

The fibers of $\mathscr{E} \to \mathbb{A}^1 - \{0,1\}$ were all smooth, whereas $\overline{\mathscr{E}} \to \mathbb{A}^1$ has singular fibers over 0 and 1. The fibers of $\mathscr{E} \to \mathbb{A}^1 - \{0,1\}$ naturally degenerate to the singular fibers of $\overline{\mathscr{E}} \to \mathbb{A}^1$, and so we do wish to consider the fibers of

$\overline{\mathscr{E}} \to \mathbb{A}^1$ a family of varieties. we will see that this is an example of a flat family, even though it is not a smooth family.

1. Flat morphisms: generalities

1.1. **Flat modules over a ring.**

Throughout we fix a commutative ring A. We will denote a general field by k.

Definition 1.1. Let M be a module over A. We say that M is *flat* (over A) if the functor $-\otimes_A M$ from A-modules to A-modules is exact. We say that M is *faithfully flat* if, in addition to being flat, the following implication holds for all A-modules N:

$$N \otimes_A M = 0 \Rightarrow N = 0.$$

The functor $-\otimes_A M$ is always right exact, regardless of whether M is a flat module. To ensure that M is flat, it therefore suffices to check that M preserves injections. The following proposition states that it is in fact enough to check that injections of the form $I \hookrightarrow A$, with I an ideal of A, are preserved.

For $i \geq 0$ we set $\mathrm{Tor}_i^A(-,M) := L_i(-\otimes_A M)\colon A\text{-mod} \to \mathrm{Ab}$, the i-th left derived functor of $-\otimes_A M$. Since $-\otimes_A M$ is right exact, we have $\mathrm{Tor}_0^A(-,M) = -\otimes_A M$.

Proposition 1.2 (ideal criterion)**.** *Let M be a module over A. Then M is flat over A if and only if for all finitely generated ideals I of A we have*

$$\mathrm{Tor}_1^A(A/I, M) = 0$$

Sketch of proof. If M is flat, then clearly $\mathrm{Tor}_1^A(A/I,M) = 0$ for all ideals I of A.

Conversely, to show that M is flat, we must show that $\mathrm{Tor}_1^A(N,M) = 0$ for all A-modules N. Using the fact that any module is the filtered direct limit of its finitely generated submodules, and the fact that $\mathrm{Tor}_1^A(N,M)$ commutes with filtered direct limits, we reduce to the case that N is finitely generated. By induction and the long exact sequence for Tor-modules, we further reduce to the case where N is generated by a single element $a \in A$. Then $N \simeq A/\mathrm{Ann}(a)$, where $\mathrm{Ann}(a) \subset A$ denotes the anihilator of a, so that we only have to show $\mathrm{Tor}_1^A(A/I,M) = 0$ for all ideals I of A. To further reduce to the case where I is finitely generted, notice that

$$A/I = \varinjlim A/J,$$

where J ranges over all the finitely generated ideals of A contained in I. Then we again use the fact that Tor commutes with filtered direct limits. ■

The above proposition already allows us to identify the flat modules over a principal ideal domain, as shown in the example below.

Example 1.3. Suppose A is a principal ideal domain. Let M be a module over A. Then by the ideal criterion for flatness, M is flat over A if and only if the functor $-\otimes_A M$ preserves exact sequences of the form

$$0 \to A \xrightarrow{\cdot a} A,$$

with $a \in A - \{0\}$. Tensoring such a sequence with M yields the sequence:

$$0 \to M \xrightarrow{\cdot a} M,$$

which is exact for all $a \in A - \{0\}$ if and only if M is torsion-free.

Let M and N be A-modules, then a balancing argument using the spectral sequence of a double complex shows that

$$L_i(N \otimes_A -)(M) = L_i(- \otimes_A M)(N) = \mathrm{Tor}_i^A(N, M);$$

see [Wei13, Section 5.6]. We use this to provide another application of the Tor functors.

Proposition 1.4. *Let*

$$0 \to M \to M_1 \to M_2 \to \ldots \to M_n \to 0$$

be an exact sequence of A-modules and assume $M_1, \ldots, M_n$ are all flat. Then also M is flat.

Proof. Suppose the sequence is a short exact sequence, then by the balancing argument mentioned above, we find for any A-module N an exact sequence

$$\mathrm{Tor}_2^A(N, M_2) \to \mathrm{Tor}_1^A(N, M) \to \mathrm{Tor}_1^A(N, M_1)$$

from the long exact sequence associated to $L_*(N \otimes_A -)$. By flatness of M_1 and M_2 we get $\mathrm{Tor}_2^A(N, M_2) = \mathrm{Tor}_1^A(N, M_1) = 0$, and so $\mathrm{Tor}_1^A(N, M) = 0$. It follows that M is flat over A. The general case then follows by cutting the sequence up into short exact sequences. ■

Below are some standard facts from algebra about flat modules.

Proposition 1.5. *(i) A module M over A is flat if and only if for all prime ideals $\mathfrak{p}$ of A, the module $M_\mathfrak{p}$ is flat over $A_\mathfrak{p}$.*

(ii) If $A \to B$ is a flat map of rings (i.e. A is flat as a module over B), and M is a flat B-module, then M is also flat as a A-module.

(iii) If M is flat over A, and $A \to B$ is any map of rings, then the module $M \otimes_A B$ is flat over B.

(iv) Let $A \to B$ be a map of rings and let M be a B-module, flat over A. If $S \subset B$ denotes a multiplicative subset, then $S^{-1}M$ is again flat over A.

Proof. Properties (ii) and (iii) are basic exercises. For part (i) we will refer to [Sta22, Tag 00HT]. We will prove part (iv). Let T be the inverse image of S in A. We will prove that $S^{-1}M$ is flat over $T^{-1}A$. Then (iv) will follow from the fact that $T^{-1}A$ is flat over A (localization is exact) and part (ii). By Proposition 1.2, we need only show that for any $I \subset T^{-1}A$ an ideal of $T^{-1}A$, the induced map $I \otimes_{T^{-1}A} S^{-1}M \to S^{-1}M$ is injective. Write $I = T^{-1}J$ for an ideal $J \subset A$. We know $J \otimes_A M \to M$ to be injective, since M is flat over A. The map $I \otimes_{T^{-1}A} S^{-1}M \to S^{-1}M$ is just the localization of this map at S, which is again injective by the fact that localization is exact. ■

We give some more examples of flat and non-flat modules.

Example 1.6. (i) If P is a projective module over A, then P is flat over A. This can be seen by writing P as a direct summand of a free module and using the fact that free modules are flat. if P happens to be finitely presented over A, then this can also be seen by the fact that P is locally free and Proposition 1.5(i).

(ii) The map of rings $k[\varepsilon]/(\varepsilon^2) \to k$ is not flat. This can be seen as follows. The inclusion $(\varepsilon) \hookrightarrow k[\varepsilon]/(\varepsilon^2)$ when tensored with k gives the zero map, whereas the tensor product $(\varepsilon) \otimes k$ is nonzero.

(iii) If $A \hookrightarrow B$ is an inclusion of integral domains such that the field of fractions of A and B coincide, then B is faithfully flat over A if and only if $A = B$. See [Mat86, Exercise 7.2], which can be done using Theorem 7.5 in the same reference. Take for example the map of rings
$$k[x, y]/(y^2 - x^3) \to k[t]$$
sending x to t^2 and y to t^3. If this map were flat, it would be faithfully flat by the Going-down Theorem for integral extensions (as we will see in Proposition 1.8 (ii)). The field of fractions of both domains is $k(t)$, and so this would imply that the rings are equal. They clearly are not, and so we conclude that this map of rings is not flat.

(iv) We provide a partial converse to (i). We show that if M is finitely presented over A, then M is flat if and only if M is projective. If M is projective, then we know M to be flat by part (i). Conversely, if we assume that M is flat, we need only show that M is locally free. So assume that A is a local ring, and write $\kappa = A/\mathfrak{m}$ for its residue field. Let
$$0 \to F' \to F \to M \to 0$$
be a finite presentation of M, where F sends the standard basis to a minimal set of generators $m_1, \ldots, m_n$ of M. Since M is flat, the sequence
$$0 \to F' \otimes k \to F \otimes k \to M \otimes k \to 0$$

is exact. The elements $m_1 \otimes 1, \ldots, m_n \otimes 1$ form a basis of $M \otimes k$ by Nakayama's lemma, and so $F \otimes k \to M \otimes k$ is an isomorphism of k-vector spaces. Hence, $F' \otimes k = 0$. By applying Nakayama's lemma again we find $F' = 0$. So $F = M$ and we win.

1.2. **Flat morphisms.** We turn to the geometric notion of flatness.

Definition 1.7. Let $f : X \to Y$ be a map of schemes, and let $\mathscr{F}$ be a sheaf of $\mathscr{O}_X$-modules. For all $x \in X$, $\mathscr{F}_x$ is a module over $\mathscr{O}_{X,x}$, and so it is a module over $\mathscr{O}_{Y,f(x)}$ via the map $f_x^\# : \mathscr{O}_{Y,f(x)} \to \mathscr{O}_{X,x}$. We say that $\mathscr{F}$ is flat over Y if for all $x \in X$ the module $\mathscr{F}_x$ is flat over $\mathscr{O}_{Y,f(x)}$. We say that f is flat if $\mathscr{F} = \mathscr{O}_X$ is flat over Y. We say that f is faithfully flat if, in addition to being flat, it is surjective.

Proposition 1.8. *(i) Let $A \to B$ be a map of rings and let M be a B-module. Then M is flat over A if and only if $\widetilde{M}$ is flat over* Spec A.
(ii) A map of rings $A \to B$ is faithfully flat if and only if the induced map on spectra $\operatorname{Spec} B \to \operatorname{Spec} A$ *is faithfully flat.*

Proof. See [Sta22, Tag 00HT] for (i) and [Sta22, Tag 00HQ] for (ii). ■

Using the above example we can translate all our algebraic examples to geometric ones.

Example 1.9. (i) The map $\operatorname{Spec} k \to \operatorname{Spec} k[\varepsilon]/(\varepsilon^2)$ from a point to a fuzzy point is not flat.
(ii) The projection map $\pi : \mathbb{A}_k^2 \to \mathbb{A}_k^1$ is flat. This follows from the fact that the ring $k[x, y]$ is free over $k[x]$, and hence flat.
(iii) Let $\mathbb{A}_k^1(t) = \operatorname{Spec} k[t]$ be the affine line and let $X = \operatorname{Spec} k[x, y]/(y^2 - x^3)$ be the cuspidal curve. The map of rings from Example 1.6(iii) induces a map of schemes $\mathbb{A}_k^1 \to X$ which is not flat. This example can be generalized by showing that the normalization of a scheme is flat if and only if it is an isomorphism.

Example 1.10 (Relative effective Cartier divisors)**.** Let $f : X \to Y$ be a morphism of schemes, and let $D \subset X$ be an effective Cartier divisor on X. Then D is said to be an effective Cartier divisor *relative to* Y if $D \rightarrowtail X \to Y$ is flat. This terminology is justified by the fact that for any morphism $Y' \to Y$, the pullback $D' = D \times_Y Y'$ is a Cartier divisor of $X' = X \times_Y Y'$ relative to Y'; see [Sta22, Tag 056Q]. In particular, the restriction of D to any fibre of f is an effective Cartier divisor of the fibre, so that we can think of D as a *family of effective Cartier divisors.* See also [Har10, Chapter III, Example 9.8.5].

Proposition 1.11. *(i) Compositions of flat morphisms are flat.*
(ii) Flatness is stable under base change.

Proof. Flatness is a local condition so that it suffices to check this for morphisms of affine schemes. Then apply Proposition 1.5 (ii) and (iii). ■

1.2.1. *Associated points.*

Definition 1.12. Let X be a scheme. A point x of X is called an associated point of X if the unique maximal ideal $\mathfrak{m}_x$ of $\mathcal{O}_{X,x}$ is an associated prime.

Remark 1.13. If X is locally Noetherian, then $x \in X$ is an associated point if and only if $\mathfrak{m}_x \subset \mathcal{O}_{X,x}$ consists entirely of zero divisors.

Example 1.14. If X is reduced, then the associated points of X correspond to the generic points of the irreducible components of X.

The most important examples of associated points to keep in mind are generic points of irreducible components[1]. If X is reduced, these are the only associated points. If X is non-reduced, there may be non-maximal associated points, called *embedded points.*

Example 1.15. Let $X = \operatorname{Spec} \mathbb{C}[x, y]/(xy, y^2)$. The scheme X should be thought of as an affine line "with a fuzzy origin". Indeed, the element $y \in \mathcal{O}_{X,(x,y)}$ is a non-trivial nilpotent element, so that X supports more functions at the origin than the ordinary affine line $\mathbb{A}^1_{\mathbb{C}}$. The associated points of X are the generic point (y) and the embedded point (x, y).

Proposition 1.16 (flat maps send associated points to associated points)**.** *Let $f : X \to Y$ be a flat map of locally Noetherian schemes and let $x \in X$ be an associated point. Then $y = f(x)$ is an associated point of Y.*

Proof. The map $f^\#_x : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is flat. Let $t \in \mathfrak{m}_y$, and suppose that t is not a zero divisor. Then by flatness, the element $f^\#_x t \in \mathfrak{m}_x$ would not be a zero divisor. This is a contradition, since x is an associated point of X. We conclude that y is an associated point of Y. ■

The above proposition will allow us to exhibit more examples of non-flat morphisms.

Example 1.17.
(i) The map of schemes $\operatorname{Spec} k[x, y]/(xy) \to \operatorname{Spec} k[x]$ is not flat. Indeed, the associated prime (x) of $k[x, y]/(xy)$ pulls back to the prime (x) of $k[x]$, which is not an associated prime. Geometrically, we're projecting the union of the x- and y-axis down to the y-axis, which sends the entire y-axis to the origin.
(ii) The map $\pi : \mathrm{Bl}_O \mathbb{A}^2_k \to \mathbb{A}^2_k$, corresponding to the blow-up of the plane in the origin, is not flat. To see this, consider the inclusion of the y-axis into the plane $\mathbb{A}^1_k \hookrightarrow \mathbb{A}^2_k$. If π were flat, then the pullback of π

[1]Algebraically, these correspond to minimal primes.

along this inclusion $\pi^{-1}\mathbb{A}^1_k \to \mathbb{A}^1_k$ would be flat by 1.11 (ii). This map is not flat, because $\pi^{-1}\mathbb{A}^1_k$ consists of two irreducible components, one of which, the exceptional fiber, gets sent to the origin of $\mathbb{A}^1_k$. The associated point correponding to this irreducible component does not get mapped to an associated point of $\mathbb{A}^1_k$, and so we conclude that $\pi^{-1}\mathbb{A}^1_k \to \mathbb{A}^1_k$ is not flat.

The following Theorem provides a partial converse to Proposition 1.16.

Theorem 1.18. *Let $f : X \to Y$ be a map of locally Noetherian schemes, with Y regular, integral, and of dimension 1. Then f is flat if and only if every associated point $x \in X$ gets mapped to the generic point of Y by f.*

Proof. If f is flat, then by Proposition 1.16 f must send associated points to associated points. Since Y is integral, the only associated point of Y is its generic point.

Conversely, suppose that every associated point of X gets sent to the generic point of Y. Let $x \in X$ be a point. We set out to prove that $f^\#_x : \mathscr{O}_{Y,y} \to \mathscr{O}_{X,x}$ is flat. If $y = f(x)$ is the generic point of Y, then certainly the map on local rings $f^\#_x : \mathscr{O}_{Y,y} \to \mathscr{O}_{X,x}$ is flat, because $\mathscr{O}_{Y,y}$ is a field. Suppose that y is a closed point of Y. Then the local ring $\mathscr{O}_{Y,y}$ is a discrete valuation ring by the regularity assumption on Y. Let π be a uniformizer. Since $\mathscr{O}_{Y,y}$ is a principal ideal domain, it suffices to prove that $f^\#_x\pi$ is not a zero divisor of $\mathscr{O}_{X,x}$ by Example 1.3. Suppose this is the case. Then $f^\#_x\pi$ is contained in an associated prime $\mathfrak{p}$ of $\mathscr{O}_{X,x}$. This prime corresponds to an associated point x' of X, and by the assertion $\mathfrak{p} \ni f^\#_x\pi$ it lies in the fiber of f over y. Since y is not the generic point of Y and x' is associated, this leads to a contradiction. ∎

Remark 1.19. In case X is also reduced in the above theorem, we see that $f : X \to Y$ is flat if and only if all irreducible components of X dominate Y. One might say that all irreducible components of X "lie flat over Y".

Example 1.20. (i) We consider a family of conics. Let $A = \mathbb{C}[t]$ and let X be the scheme $\operatorname{Proj} A[x,y,z]/(xy - tz^2)$. Then the morphism $X \to \operatorname{Spec} A = \mathbb{A}^1_{\mathbb{C}}$ is faitfhfully flat by the above remark. The fibers over the closed points of $\mathbb{A}^1_{\mathbb{C}}$ are as follows.
- Over $t \neq 0$, the fiber is a non-degenerate conic in $\mathbb{P}^2_{\mathbb{C}}$.
- Over $t = 0$, the fiber is a degenerate conic in $\mathbb{P}^2_{\mathbb{C}}$.

This example illustrates that even if the general member of a flat family of varieties is irreducible, the special member can be reducible.

(ii) The family of curves $\overline{\mathscr{E}} \to \mathbb{A}^1$ from Example 0.2 is flat.

The following example shows that neither the assumption about regularity, nor the assumption about the dimension of Y can be dropped.

Example 1.21. (i) Let $X = \operatorname{Spec} k[x,y]/(y^2 - x^3)$ be the cuspidal curve and let $\mathbb{A}^1_k \to X$ be the map from Example 1.9 (iii). The domain of this morphism is integral, and the map is surjective, but it is not flat. Indeed, the scheme X has a singular point and so Theorem 1.18 does not apply.

(ii) Suppose $2 \neq 0$ in k[2].Let $A = k[x,y]$ and $B = A[u,v]/IJ$, where $I = (x-u, y-v)$ and $J = (x+u, y+v)$ are ideals of $k[x,y,u,v]$. It can be shown that the map $X = \operatorname{Spec} B \to \operatorname{Spec} A = \mathbb{A}^2_k$ is not flat. Geometrically, X consists of two planes meeting in a point, both mapping isomorphically to a plane. The irreducible components of X do dominate $\mathbb{A}^2_k$, but $\mathbb{A}^2_k$ is not of dimension 1, and so Theorem 1.18 does not apply.

Lemma 1.22 (Extending flat families)**.** *Let Y be a regular, integral and locally noetherian scheme of dimension 1. Let $P \in Y$ be a closed point. Let $X \subset \mathbb{P}^r_{Y-\{P\}}$ be a closed subscheme such that $X \to Y - \{P\}$ is flat. Then there exists a* unique *closed subscheme $\overline{X} \subset \mathbb{P}^r_Y$, flat over Y, which fits into a cartesian diagram*

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \overline{X} \arrow[d] \\ Y - \{P\} \arrow[r] & Y. \end{tikzcd}$$

Roughly speaking, the above lemma tells us that we can "take the limit of the fibers in a flat family over a regular curve".

Sketch of proof. Let $\overline{X}$ be the scheme theoretic closure of X in $\mathbb{P}^r_Y$. It is defined to be the closed subscheme associated to the quasi-coherent sheaf of ideals $\mathscr{I} = \ker(\mathscr{O}_{\mathbb{P}^r_Y} \to i_*\mathscr{O}_X)$, where $i \colon X \to \mathbb{P}^r_Y$ denotes the inclusion map. Then X and $\overline{X}$ have the same associated points by [Vak, Exercise 8.3D], so that $\overline{X} \to Y$ is flat by Theorem 1.18. It fits into the cartesian diagram above by construction. Furthermore, $\overline{X}$ is unique, because any other extension of X to $\mathbb{P}^r_Y$ would have some associated point mapping to P. ∎

Example 1.23. Let $P = (0:0:\ldots:0:1) \in \mathbb{P}^{r+1}_k$, be a point, and consider the projection $\pi \colon \mathbb{P}^{r+1} - \{P\} \to \mathbb{P}^r$ onto the hyperplane $\{x_r = 0\}$ given by $(x_0 : \ldots : x_{r+1}) \mapsto (x_0 : \ldots : x_r)$. For each $a \in k^\times$, consider the automorphism $\sigma_a \colon \mathbb{P}^{r+1} \xrightarrow{\simeq} \mathbb{P}^{r+1}$ given by $(x_0 : \ldots : x_{r+1}) \mapsto (x_0 : \ldots : x_r : ax_{r+1})$. Let $X_1 \subset \mathbb{P}^{r+1}$ be a closed subvariety of $\mathbb{P}^{r+1}$, not containing P. For each $a \in k^\times$, let $X_a = \sigma_a(X_1)$. We argue that the X_a form the fibers of a flat family. Let

[2]The reader should stop to ponder what happens when $2 = 0$.

$I = (f_i)_{i=0}^n$ be the homogenous ideal of X_1. Set $A = k[a, a^{-1}]$. Consider

$$\begin{tikzcd} \mathscr{X} = \operatorname{Proj} \frac{A[x_0,\dots,x_{r+1}]}{(f_i(x_0,\dots,x_r,a^{-1}x_{r+1}))_{i=0}^n} \arrow[r, tail] \arrow[dr] & \mathbb{P}^{r+1} \times_k (\mathbb{A}^1 - \{0\}) \arrow[d] \\ & \mathbb{A}^1 - \{0\}. \end{tikzcd}$$

There is an isomorphism $\mathscr{X} \simeq X_1 \times_k (\mathbb{A}^1 - \{0\})$ of schemes over $\mathbb{A}^1 - \{0\}$; hence, $\mathscr{X} \to \mathbb{A}^1 - \{0\}$ is flat. By Lemma 1.22 there is a unique closed subscheme $\overline{\mathscr{X}} \subset \mathbb{P}^{r+1}_{\mathbb{A}^1}$, flat over $\mathbb{A}^1$, which fits into a cartesian diagram

$$\begin{tikzcd} \mathscr{X} \arrow[r] \arrow[d] & \overline{\mathscr{X}} \arrow[d] \\ \mathbb{A}^1 - \{0\} \arrow[r, hook] & \mathbb{A}^1. \end{tikzcd}$$

Topologically, the fiber of $\overline{\mathscr{X}} \to \mathbb{A}^1$ over 0 is just the projection $\pi(X_1)$. We will see later (Example 3.14) that this fiber generally does not have the reduced induced structure of $\pi(X_1)$.

1.2.2. *Fibre dimension.* As mentioned before, the notion of flatness for morphisms of schemes is partly motivated by the fact that they have "nicely varying" fibers. We will give this statement some meaning by first showing that dimension is a well-behaved invariant for the members of a flat family in Corollary 1.27

Proposition 1.24 (flat maps are generalizing)**.** *Let $f : X \to Y$ be a flat map of schemes, let $x \in X$ and write $y = f(x)$. If y' is a generalization of y (i.e. $y \in \overline{\{y'\}}$), then there is a generalization x' of x such that $f(x') = y'$. One also says that generalizations lift along f, or that f is generalizing.*

The proof of this proposition comes down to the following algebraic lemma, which is reminiscent of the situation for integral extensions.

Lemma 1.25 (Going-down for flat maps)**.** *Let $A \to B$ be a flat map of rings and write $f : \operatorname{Spec} B \to \operatorname{Spec} A$ for the induced map on spectra. If $\mathfrak{q}' \subset \mathfrak{q}$ are primes of A, and $\mathfrak{p}$ is a prime of B such that $f(\mathfrak{p}) = \mathfrak{q}$, then there exists a prime $\mathfrak{p}' \subset \mathfrak{p}$ of B such that $f(\mathfrak{p}') = \mathfrak{q}'$.*

Proof. The induced map $A_\mathfrak{q} \to B_\mathfrak{p}$ is a flat map of local rings and hence faithfully flat; thus, there exists a prime of $B_\mathfrak{p}$ pulling back to $\mathfrak{q}' A_\mathfrak{q}$. This prime is of the form $\mathfrak{p}' B_\mathfrak{p}$, where $\mathfrak{p}'$ is a prime of B pulling back to $\mathfrak{q}'$. ■

Proof of Proposition 1.24. Apply the Going-down lemma to the flat map of rings $f_x^\# : \mathscr{O}_{Y,y} \to \mathscr{O}_{X,x}$. ■

Remark 1.26. Another example of generalizing maps are open maps of schemes. It turns out that a flat map of schemes $f : X \to Y$ is also (universally) open, as long as it is locally of finite presentation. See [Sta22, Tag 01UA].

Corollary 1.27 (fibre dimension)**.** *Let $f : X \to Y$ be a map of locally Noetherian schemes. Let $x \in X$ and write $y = f(x)$. Then*

$$\dim \mathscr{O}_{X,x} \leq \dim \mathscr{O}_{Y,y} + \dim \mathscr{O}_{X,x} \otimes_{\mathscr{O}_{Y,y}} \kappa(y).$$

If, in addition, f is flat, then the above inequality becomes an equality.

Proof. For the inequality "≤" we refer to [Mat86, Theorem 15.1]. We prove the inequality

$$\dim \mathscr{O}_{X,x} \geq \dim \mathscr{O}_{Y,y} + \dim \mathscr{O}_{X,x} \otimes_{\mathscr{O}_{Y,y}} \kappa(y)$$

using the Going-down lemma for flat maps. Let

$$\mathfrak{r}_0 \subsetneq \mathfrak{r}_1 \subsetneq \ldots \subsetneq \mathfrak{r}_r = \mathfrak{m}_x \mod \mathfrak{m}_y \mathscr{O}_{X,x}$$

be a chain of prime ideals in $\mathscr{O}_{X,x} \otimes_{\mathscr{O}_{Y,y}} \kappa(y) = \mathscr{O}_{X,x}/\mathfrak{m}_y \mathscr{O}_{X,x}$ of length r, and let

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \ldots \subsetneq \mathfrak{p}_m = \mathfrak{m}_y$$

be a chain of prime ideals in $\mathscr{O}_{Y,y}$ of length m. The chain in $\mathscr{O}_{X,x}/\mathfrak{m}_y \mathscr{O}_{X,x}$ lifts to a chain

$$\mathfrak{m}_y \mathscr{O}_{X,x} \subset \mathfrak{q}_m \subsetneq \ldots \subsetneq \mathfrak{q}_{r+m} = \mathfrak{m}_x$$

in $\mathscr{O}_{X,x}$. Using the Going-down lemma for flat maps we now construct a chain of prime ideals

$$\mathfrak{q}_0 \subsetneq \ldots \subsetneq \mathfrak{q}_m$$

in $\mathscr{O}_{X,x}$ pulling back to the chain $\mathfrak{p}_0 \subsetneq \ldots \subsetneq \mathfrak{p}_m$. We've now found a chain of prime ideals $\mathfrak{q}_0 \subsetneq \ldots \subsetneq \mathfrak{q}_{r+m}$ in $\mathscr{O}_{X,x}$ of length $r+m$. We conclude that

$$\dim \mathscr{O}_{X,x} \geq \dim \mathscr{O}_{Y,y} + \dim \mathscr{O}_{X,x} \otimes_{\mathscr{O}_{Y,y}} \kappa(y).$$

■

Remark 1.28. The proof of the above corollary only relies on the fact that f is generalizing.

Notice that we can now prove in a different way that the morphisms of Example 1.17 are not flat.

Roughly speaking, the above corollary says that dimension is a well-behaved invariant of the fibers of a flat family. In Section 3 we will see that many invariants of fibers of reasonable flat families are well-behaved.

1.3. **Descending properties of morphisms.** Very often, when $f\colon X \to Y$ is a morphism which has some nice property, P, and $Y' \to Y$ is any morphism, then the base change of f along $Y' \to Y$, $f'\colon X \times_Y Y' \to Y$, will inherit P. It is now natural to ask when this implication can be reversed. In other words, if f' has P, can we deduce from this that f has P? Clearly, this is not the case in general. For instance, if $y \in Y$ is a point, then the fiber over y, $X_y \to \operatorname{Spec}\kappa(y)$, might have properties not shared with all other fibers of f. However, it turns out that we can turn this implication around for almost all properties P if $Y' \to Y$ is *faithfully* (!) flat. We list only a few in the theorem below. A more complete list can be found in [Poo17, Appendix C].

Theorem 1.29. *Suppose $Y' \to Y$ is faithfully flat and suppose f' is a closed immersion, respectively an open immersion, respectively of finite presentation, respectively flat, respectively proper, respectively smooth, respectively étale. Then f has the same property.*

Example 1.30. Suppose $Y' = \operatorname{Spec}\ell \to \operatorname{Spec} k = Y$ is the morphism corresponding to an extension of fields $k \subset \ell$. Then $Y' \to Y$ is faitfhully flat.

2. The flat base change theorem

Let $f\colon X \to Y$ be a separated, quasi-compact morphism, and let $\mathscr{F}$ be a quasi-coherent sheaf on X. Let $u\colon Y' \to Y$ be a flat morphism of noetherian schemes. Consider a cartesian diagram

$$\begin{array}{ccc} X' & \xrightarrow{v} & X \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{u} & Y. \end{array} \tag{2.1}$$

Theorem 2.1 (Flat base change)**.** *For all $i \geq 0$ there are natural isomorphisms*

$$u^* R^i f_*(\mathscr{F}) \simeq R^i g_*(v^*\mathscr{F}). \tag{2.2}$$

Remark 2.2. If (2.1) is *any* commutative diagram of schemes, we can construct a natural morphism

$$u^* R^i f_*(\mathscr{F}) \to R^i g_*(v^*\mathscr{F}),$$

by using some abstract nonsense. This is demonstrated in [FK13, §I.6]. We will construct the isomorphism (2.2) in a less general way in the proof of the theorem.

Proof of Theorem 2.1. The question is local on Y', and so we can assume that Y' and Y are affine; say, $Y' = \operatorname{Spec} A'$ and $Y = \operatorname{Spec} A$. Then by [Góm, Corollary 4.8], $R^i f_*\mathscr{F}$ is quasi-coherent. Hence, $R^i f_*\mathscr{F} \simeq H^i(X, \mathscr{F})^\sim$.

Similarly, $R^i g_*(v^*\mathscr{F}) \simeq H^i(X', v^*\mathscr{F})^\sim$. We're left with proving that the natural map

$$H^i(X,\mathscr{F}) \otimes_A A' \to H^i(X', v^*\mathscr{F})$$

is an isomorphism.

Let $\mathfrak{U} = \{U_i\}$ be a finite open affine cover of X. By [Har10, Exercise 4.11], the fact that X is separated and [GD63, §1.3.1], there is a natural isomorphism

$$\check{H}^i(\mathfrak{U},\mathscr{F}) = H^i(\check{C}^\bullet(\mathfrak{U},\mathscr{F})) \simeq H^i(X,\mathscr{F}),$$

where $\check{C}^\bullet(\mathfrak{U},\mathscr{F})$ denotes the Čech complex. Similarly, $H^i(\check{C}^\bullet(\mathfrak{U}',\mathscr{F})) \simeq H^i(X', v^*\mathscr{F})$, where $\mathfrak{U}' = \{U \times_Y Y' : U \in \mathfrak{U}\}$. Now notice that

$$\check{C}^\bullet(\mathfrak{U}', v^*\mathscr{F}) = \check{C}^\bullet(\mathfrak{U},\mathscr{F}) \otimes_A A'.$$

By flatness of $A \to A'$ we obtain an isomorphism

$$H^i(\check{C}^\bullet(\mathfrak{U},\mathscr{F})) \otimes_A A' \xrightarrow{\simeq} H^i(\check{C}^\bullet(\mathfrak{U}', v^*\mathscr{F})).$$

■

Remark 2.3. The above theorem is even true if we only assume that f is quasi-separated. The proof then uses the *Čech-to-cohomology spectral sequence*. See [Sta22, Tag 02KH].

Example 2.4. Suppose $Y' = \operatorname{Spec} \ell \to \operatorname{Spec} k = Y$ corresponds to an extension of fields $k \subset \ell$. Then $X' = X_\ell$ and the flat base change theorem translates into

$$H^i(X,\mathscr{F}) \otimes_k \ell \simeq H^i(X_\ell, \mathscr{F} \otimes_k \ell).$$

3. Hilbert polynomials of flat projective families

Example 3.1. Consider again the family of curves $\overline{\mathscr{E}} \to \operatorname{Spec} \mathbb{A}^1$ from Example 0.2. Notice that all its fibers are Curves of degree 3 in $\mathbb{P}^2_{\mathbb{C}}$ under the embedding given by $\overline{\mathscr{E}} \rightarrowtail \mathbb{P}^2_{\mathbb{A}^1}$. So, they all share the same dimension, degree and arithmetic (but not geometric!) genus. These are basic invariants of varieties, which intuitively should vary continuously in a family. The main theorem of this section, Theorem 3.10, guarantees that these invariants do indeed vary continuously for flat projective families.

3.1. **The Hilbert polynomial.** Throughout, we fix X a proper scheme over a field k, equipped with a very ample line bundle $\mathscr{L}$. If $\mathscr{F}$ is a coherent sheaf on X, then we define its *euler characteristic* to be

$$\chi(X,\mathscr{F}) = \sum_{i=0}^{\infty} (-1)^i h^i(X,\mathscr{F}),$$

where

$$h^i(X,\mathscr{F}) = \dim_k H^i(X,\mathscr{F}).$$

Lemma 3.2. *Let*

$$0 \to \mathscr{F}_1 \to \ldots \to \mathscr{F}_n \to 0 \tag{3.1}$$

be an exact sequence of coherent sheaves on X. Then

$$\sum_{i=1}^{n} (-1)^i \chi(X, \mathscr{F}_i) = 0.$$

Proof. If (3.1) is a short exact sequence, this is an easy exercise using the long exact sequence for cohomology. The general case is then proved by cutting (3.1) up into short exact sequences. ∎

For a coherent sheaf $\mathscr{F}$ on X, we write

$$\mathscr{F}(n) = \mathscr{F} \otimes \mathscr{L}^{\otimes n} \quad (n \in \mathbb{Z}).$$

Recall also that the support $\operatorname{Supp}\mathscr{F} = \{x \in X : \mathscr{F}_x \neq 0\}$ of $\mathscr{F}$ is a closed subset of X.

Theorem 3.3 (Existence of the Hilbert polynomial)**.** *If $\mathscr{F}$ is a coherent sheaf on X, then there exists a (unique) polynomial $P(z) = P_{\mathscr{F}}(z) \in \mathbb{Q}[z]$, called the* Hilbert polynomial *of $\mathscr{F}$, such that for all $n \in \mathbb{Z}$ we have*

$$P(n) = \chi(X, \mathscr{F}(n)).$$

Furthermore, P has degree at most equal to the dimension of the support of $\mathscr{F}$.

The Hilbert polynomial of $\mathscr{O}_X$ is also simply called the *Hilbert polynomial of X*, and is denoted by P_X.

Proof. Let $j \colon X \rightarrowtail \mathbb{P}^r_k$ be a closed immersion such that $\mathscr{L} \simeq \mathscr{O}_X(1) = j^*\mathscr{O}_{\mathbb{P}^r}(1)$. By replacing $\mathscr{F}$ by $j_*\mathscr{F}$, we only need to prove the theorem if $\mathscr{F}$ is a coherent sheaf on $\mathbb{P}^r$ and $\mathscr{L} = \mathscr{O}(1)$. Indeed, $j_*\mathscr{F}$ is again coherent, because j is a closed immersion, and for all $n \in \mathbb{Z}$ we have

$$\begin{aligned}
\chi(X, \mathscr{F}(n)) &= \chi(\mathbb{P}^r, j_*\mathscr{F}(n)) \\
&= \chi(\mathbb{P}^r, j_*(\mathscr{F} \otimes j^*\mathscr{O}_{\mathbb{P}^r}(n)) \\
&= \chi(\mathbb{P}^r, j_*\mathscr{F} \otimes \mathscr{O}_{\mathbb{P}^r}(n)),
\end{aligned}$$

where the last equality uses the projection formula. Furthermore $\operatorname{Supp} j_*\mathscr{F} = \operatorname{Supp}\mathscr{F}$.

Additionally, we can assume that k is algebraically closed, because for all $n \in \mathbb{Z}$ we have

$$\chi(\mathbb{P}^r_k, \mathscr{F}(n)) = \chi(\mathbb{P}^r_{\overline{k}}, (\mathscr{F} \otimes_k \overline{k})(n))$$

by Example 2.4.

We prove the theorem by induction on the dimension of $\operatorname{Supp}\mathscr{F}$. If $\operatorname{Supp}\mathscr{F} = \emptyset$, then we can just take $P(z) = 0$. In general, consider an exact sequence

$$0 \to \mathscr{G} \to \mathscr{F}(-1) \xrightarrow{\otimes x_r} \mathscr{F} \to \mathscr{H} \to 0. \tag{3.2}$$

We see that $\operatorname{Supp}\mathscr{G}$ and $\operatorname{Supp}\mathscr{H}$ must lie in $\{x_r = 0\} \cap \operatorname{Supp}\mathscr{F} \subset \mathbb{P}^r$. Since k is infinite (being algebraically closed), we can assume that no irreducible component of $\operatorname{Supp}\mathscr{F}$ lies in $\{x_r = 0\}$ after potentially applying an automorphism of $\mathbb{P}^r$. So $\operatorname{Supp}\mathscr{G}$ and $\operatorname{Supp}\mathscr{H}$ must have strictly lower dimensions than $\operatorname{Supp}\mathscr{F}$. Tensoring (3.2) with $\mathscr{L}(n)$ and applying Lemma 3.2, we find

$$\chi(\mathbb{P}^r, \mathscr{F}(n)) - \chi(\mathbb{P}^r, \mathscr{F}(n-1)) = \chi(\mathbb{P}^r, \mathscr{G}(n)) - \chi(\mathbb{P}^r, \mathscr{H}(n)),$$

where the right hand side is a polynomial in n by the induction hypothesis. By [Har10, Chapter I, Proposition 7.3], we find that $\chi(\mathbb{P}^r, \mathscr{F}(n))$ is a polynomial in n of degree at most $\dim \operatorname{Supp}\mathscr{F}$. ■

Remark 3.4. In general, the Hilbert polynomial depends on the fixed very ample line bundle $\mathscr{L}$. See also Remark 3.8. However, the constant coefficient

$$P_{\mathscr{F}}(0) = \chi(X, \mathscr{F})$$

does not. In particular, the Hilbert polynomial of X encodes the arithmetic genus of X in its constant coefficient:

$$p_a(X) = (-1)^{\dim X}(P_X(0) - 1).$$

Proposition 3.5. *Let $\mathscr{F}$ be a coherent sheaf on X, and let $P(z)$ be its Hilbert polynomial. Then for all $n \gg 0$ sufficiently large, we have*

$$P(n) = h^0(X, \mathscr{F}(n)).$$

Proof. This follows immediately from [Har10, Chapter III, Theorem 5.2(b)]. ■

Remark 3.6. Let $S = k[x_0, \dots, x_r]$ and let M be a graded S-module. The Hilbert polynomial of M is classically defined to be the unique polynomial $P_M(z) \in \mathbb{Q}[z]$ such that for all $n \gg 0$ we have

$$P_M(n) = \dim_k M_n,$$

where M_n denotes the n-th graded part of M. It exists by [Har10, Chapter I, Thoerem 7.5]. Furthermore, Hartshorne also proves that we have always $\deg P(z) = \dim Z(\operatorname{Ann} M)$, where $Z(\operatorname{Ann} M) \subset \mathbb{P}^r$ denotes the zero locus of the annihilator of M.

If $X \rightarrowtail \mathbb{P}^r$ is a closed subscheme with homogenous ideal $I \subset S$, then by the above proposition we see that the Hilbert polynomial $P(z)$ of X (with respect to $\mathscr{O}_X(1)$), as we have defined it, is precisely the Hilbert polynomial of the graded S-module $M = S/I$. In particular $\deg P(z) = \dim X$ always. We also observe from [Har10, Chapter I, Proposition 7.6] that

$$\deg X = (\dim X)! \times (\text{leading coefficient of } P).$$

In fact, Hartshorne defines the degree of a closed subvariety of $\mathbb{P}^r$ this way; but, from [Har10, Chapter I, Theorem 7.7], we see that this definition also coincides with the more convential one: the degree of X is the number of points of intersection of X with a generic linear subspace of codimension $\dim X$.

Example 3.7. (i) The Hilbert polynomial of $\mathbb{P}^r$ is given by

$$P_{\mathbb{P}^r} = \binom{z+r}{r} = \frac{1}{r!} z(z-1)\dots(z-r+1).$$

For any $d \in \mathbb{Z}$, the Hilbert polynomial of $\mathscr{O}_{\mathbb{P}^r}(d)$ is given by

$$P_{\mathscr{O}(d)} = \binom{z+r+d}{r}.$$

(ii) Set $S = k[x_0, \dots, x_r]$. Let $d \in \mathbb{Z}$ be an integer and let $f \in S_d$ be a homogeneous polynomial of degree d. Let $X = \operatorname{Proj} S/(f)$, and write $j\colon X \rightarrowtail \mathbb{P}^r$ for the inclusion of X into $\mathbb{P}^r$. We have an exact sequence

$$0 \to \mathscr{O}(-d) \to \mathscr{O}_{\mathbb{P}^r} \to j_*\mathscr{O}_X \to 0.$$

It follows that

$$P_X(z) = P_{\mathbb{P}^r}(z) - P_{\mathscr{O}(-d)}(z) = \binom{z+r}{r} - \binom{z-d+r}{r}.$$

Remark 3.8. To stress the obvious: the Hilbert polynomial of a coherent sheaf $\mathscr{F}$ on X depends *crucially* on $\mathscr{L}$. Indeed, consider the projective line $\mathbb{P}^1$ with very ample line bundles $\mathscr{O}(1)$ and $\mathscr{O}(2)$. With respect to $\mathscr{O}(1)$, the Hilbert polynomial of $\mathbb{P}^1$ is just $P(z) = z+1$, whereas with respect to $\mathscr{O}(2)$ it is $2z+1$. The constant coefficient does of course not depend on $\mathscr{L}$; see Remark 3.4.

3.2. **The Hilbert polynomial of a flat projective family.**

Example 3.9. Consider again the flat family $\overline{\mathscr{E}} \to \mathbb{A}^1$ from Example 0.2 with the embedding $\overline{\mathscr{E}} \rightarrowtail \mathbb{P}^2_{\mathbb{A}^1}$. Each of the fibers is a degree 3 curve in $\mathbb{P}^2_{\mathbb{C}}$ and hence they all share the same Hilbert polynomial: $P(z) = 3z$.

The Example above demonstrates a more general phenomenon. Let $f\colon X \to T$ be a proper morphism with T integral and locally noetherian, let $\mathscr{L}$ be an f-relatively very ample line bundle, and let $\mathscr{F}$ be a coherent sheaf on X. For $t \in T$, denote by $P_t(z) \in \mathbb{Q}[z]$ the Hilbert polynomial of $\mathscr{F}_t = \mathscr{F}|_{X_t}$ with respect to the very ample line bundle $\mathscr{L}_t$.

Theorem 3.10. *The coherent sheaf $\mathscr{F}$ is flat over T if and only if P_t is independent of t.*

With notation as in the theorem above, if $\mathscr{F}$ is flat over T, we will call $P = P_t$ simply *the Hilbert polynomial of $\mathscr{F}$*. If $\mathscr{F} = \mathscr{O}_X$, then we will call $P = P_t$ *the Hilbert polynomial of $X \to T$*.

Proof of Theorem 3.10. Let $j\colon X \rightarrowtail \mathbb{P}^r_T$ be a T-immersion such that $\mathscr{L} \simeq j^*\mathscr{O}(1)$[3]. Similarly to the proof of Theorem 3.3, we can replace $\mathscr{F}$ by $j_*\mathscr{F}$ and immediately reduce to the case $X = \mathbb{P}^r_T$ and $\mathscr{L} = \mathscr{O}(1)$. We need to prove that $\mathscr{F}$ is flat over T if and only if $P_t = P_\eta$ for all $t \in T$, where $\eta \in T$ denotes the generic point. So, it suffices to prove the theorem after base-changing along $\operatorname{Spec}\mathscr{O}_{T,t} \to T$ for all $t \in T$. We have reduced to proving the theorem for $T = \operatorname{Spec} A$ a noetherian local integral domain.

We show that the statements

(i) $\mathscr{F}$ is flat over T;
(ii) $H^0(\mathbb{P}^r_T, \mathscr{F}(m))$ is a free A-module of finite rank, for all $m \gg 0$,

are equivalent. Suppose $\mathscr{F}$ is flat over T. Let $\mathfrak{U}$ be the standard affine open cover of $\mathbb{P}^r_T$. By [Har10, Chapter III, Theorem 4.5], we have

$$H^i(\mathbb{P}^r_T, \mathscr{F}(m)) = H^i(C^\bullet(\mathfrak{U}, \mathscr{F}(m))$$

for all i. Since $\mathscr{F}$ is flat over T, so is $\mathscr{F}(m) = \mathscr{F} \otimes \mathscr{O}(m)$ (indeed, $\mathscr{O}(m)$ is a line bundle and flatness is a local condition). It follows that each term of the Čech complex $C^\bullet(\mathfrak{U}, \mathscr{F}(m))$ is flat over A. By [Har10, Chapter III, Theorem 5.2](b), for all $m \gg 0$ large enough, we have $H^i(\mathbb{P}^r_T, \mathscr{F}(m)) = 0$ for all $i > 0$. Hence, we obtain an exact sequence

$$0 \to H^0(\mathbb{P}^r_T, \mathscr{F}(m)) \to C^0(\mathfrak{U}, \mathscr{F}(m)) \to \ldots \to C^r(\mathfrak{U}, \mathscr{F}(m)) \to 0.$$

Flatness of the $C^i(\mathfrak{U}, \mathscr{F}(m))$ and Proposition 1.4 now implies that $H^0(\mathbb{P}^r_T, \mathscr{F}(m))$ is flat over A. By Example 1.6(iv), and the fact that $H^0(\mathbb{P}^r_T, \mathscr{F}(m))$ is finitely generated over A ([Har10, Chapter III, Theorem 5.2](a)), it follows that $H^0(\mathbb{P}^r_T, \mathscr{F}(m))$ is free of finite rank over A. This proves (i) $\Rightarrow$ (ii).

Conversely, assume that $H^0(\mathbb{P}^r_T, \mathscr{F}(m))$ is free of finite rank over A for all $m \geq m_0$ for some $m_0 \in \mathbb{Z}$. Let $S = A[x_0, \ldots, x_r]$, and let M be the graded S-module

$$M = \bigoplus_{m \geq m_0} H^0(\mathbb{P}^r_T, \mathscr{F}(m)).$$

[3]With few adjustments, the proof also goes through if we use the more general definition of a very ample sheaf: there is a T-immersion $j\colon X \rightarrowtail \mathbb{P}(\mathscr{E})$ for some coherent sheaf $\mathscr{E}$ on T such that $j^*\mathscr{O}(1) \simeq \mathscr{L}$.

Then M is flat over A, because each graded piece is free over A. By [Har10, Chapter III, Exercise 5.9], we have $\mathscr{F} = \widetilde{M}$. By Lemma 3.11 below it follows that $\mathscr{F}$ is flat over A. This proves (ii) $\Rightarrow$ (i).

Next, we prove that the statements

(ii) $H^0(\mathbb{P}^r_T, \mathscr{F}(m))$ is a free A-module of finite rank, for all $m \gg 0$;
(iii) The Hilbert polynomial P_t of $\mathscr{F}_t$ on $\mathbb{P}^r_{\kappa(t)}$ is independent of t,

are equivalent. This will follow after we prove the equality

$$H^i(\mathbb{P}^r_{\kappa(t)}, \mathscr{F}_t(m)) = H^i(\mathbb{P}^r_T, \mathscr{F}(m)) \otimes_A \kappa(t) \tag{3.3}$$

for $m \gg 0$ sufficiently large. Indeed, if $H^0(\mathbb{P}^r_T, \mathscr{F}(m))$ is free of finite rank for $m \gg 0$ large enough, then for $t \in T$ we find that

$$P_t(m) = \dim_{\kappa(t)} H^0(\mathbb{P}^r_{\kappa(t)}, \mathscr{F}_t(m)) = \mathrm{rk}_A\ H^0(\mathbb{P}^r_T, \mathscr{F}(m))$$

is independent of t.

On the other hand, if $P_t = P_\eta$, where $t \in T$ is the closed point, and $\eta \in T$ the generic point, then for $m \gg 0$ large enough we find that

$$\dim_{\kappa(\eta)} H^0(\mathbb{P}^r_{\kappa(\eta)}, \mathscr{F}_\eta(m)) = P_\eta(m) = P_t(m) = \dim_{\kappa(t)} H^0(\mathbb{P}^r_{\kappa(t)}, \mathscr{F}_t(m)).$$

From (3.3), and Lemma 3.12 below, it follows that $H^i(\mathbb{P}^r_T, \mathscr{F}(m))$ is a free A-module.

We now prove the equality (3.3). We can assume that t is the closed point of T by base changing along $\operatorname{Spec}\mathscr{O}_{T,t} \to T$ and by Theorem 2.1. Consider an exact sequence

$$A^{\oplus n} \to A \to \kappa(t) \to 0.$$

It exists since A is assumed to be noetherian. Pulling it back to a sequence of sheaves on $\mathbb{P}^r_T$ and tensoring with $\mathscr{F}$, we obtain the exact sequence

$$\mathscr{F}^{\oplus n} \to \mathscr{F} \to \mathscr{F}_t \to 0.$$

By Lemma 3.13 below, for all $m \gg 0$ large enough we obtain an exact sequence

$$H^0(\mathbb{P}^r_T, \mathscr{F}(m)^{\oplus n}) \to H^0(\mathbb{P}^r_T, \mathscr{F}(m)) \to H^0(\mathbb{P}^r_T, \mathscr{F}_t(m)) \to 0.$$

After pulling out the direct sum, we see that the right hand term coincides with $H^0(\mathbb{P}^r_T, \mathscr{F}(m)) \otimes_A \kappa(t)$.

■

Lemma 3.11. *Let A be a ring and write $S = A[x_0, \ldots, x_r]$. If M is a flat graded S-module, then $\mathscr{F} = \widetilde{M}$ is an $\mathscr{O}_{\mathbb{P}^r}$-module which is flat over* $\operatorname{Spec} A$.

Proof. Let $U_i = \operatorname{Spec} S_{(x_i)}$ where $S_{(x_i)}$ is the degree zero part of S_{x_i}. We know that M_{x_i} is flat over A by Proposition 1.5. We have $M_{x_i} = \bigoplus_{m\in\mathbb{Z}} x_i^m \cdot M_{(x_i)}$, and so $M_{(x_i)}$ is flat over A. It follows that $\mathscr{F}|_{U_i} = \widetilde{M_{(x_i)}}$ is flat over A. ■

Lemma 3.12. *Let A be a local integral domain. Write $K = \operatorname{Frac} A$ for the fraction field and κ for the residue field. Let M be a finitely generated A-module such that*

$$\dim_K M \otimes_A K = \dim_\kappa M \otimes_A \kappa,$$

then M is a free A-module.

Proof. Denote by $m_1, \ldots, m_n$ a minimal set of generators for M as an A-module. Then $n = \dim_\kappa M \otimes_A \kappa$ by Nakayama's lemma. The $m_1, \ldots, m_n$ are furthermore A-linearly independent, because $m_1 \otimes 1, \ldots, m_n \otimes 1$ forms a K-basis of $M \otimes K$ by the fact that $n = \dim_K M \otimes_A K$. ∎

Lemma 3.13. *Let X be a projective scheme over a noetherian ring A, and let*

$$\mathscr{F}_1 \to \ldots \to \mathscr{F}_n$$

be an exact sequence of coherent $\mathscr{O}_X$-modules. Then for all $m \gg 0$ large enough, the sequence

$$H^0(X, \mathscr{F}_1(m)) \to \ldots \to H^0(X, \mathscr{F}_n(m))$$

is exact.

Proof. This is an easy exercise using [Har10, Chapter III, Theorem 5.2(b)]. ∎

Example 3.14. Suppose $k = \overline{k}$. We let $X_1 \subset \mathbb{P}^3_k$ be a smooth, connected curve which is not contained in any hyperplane. One such example is the twisted cubic curve: it is obtained by embedding $\mathbb{P}^1$ into $\mathbb{P}^3$ using the very ample line bundle $\mathscr{O}(3)$. Explicitly, this embedding is given by

$$\begin{aligned} \mathbb{P}^1 &\to \mathbb{P}^3 \\ (s:t) &\mapsto (s^3 : s^2 t : t s^2 : t^3). \end{aligned}$$

Let $P \in \mathbb{P}^3$ be a point not in X_1, and write $\pi : \mathbb{P}^3 - \{P\} \to \mathbb{P}^2$ for the projection from P onto a hyperplane in $\mathbb{P}^3$ not containing P.

Recall that by Example 1.23 this gives rise to a flat family $X \to \mathbb{A}^1$ whose fiber over 1 is X_1. As already mentioned in that example, the fiber X_0 over 0 has support $\pi(X_1)$. We argue that X_0 must always have nilpotent elements by using the fact that the arithmetic genus of the fibers must all be the same by Theorem 3.10 and Remark 3.4.

Claim 3.15. *The curve $\pi(X_1) \subset \mathbb{P}^2$ with the induced reduced structure must be singular.*

Proof. Indeed, if it is not, then π induces an isomorphism $X_1 \simeq \pi_1(X_1)$, because it is generically 1-to-1. Since X_1 is not contained in any hyperplane, we obtain an injective map

$$H^0(\mathbb{P}^3, \mathscr{O}(1)) \rightarrowtail H^0(X_1, \mathscr{O}_{X_1}(1)),$$

and hence $h^0(X_1, \mathscr{O}_{X_1}(1)) \geq 4$. On the other hand, we have an exact sequence of sheaves of $\mathscr{O}_{\mathbb{P}^2}$-modules

$$0 \to \mathscr{I}_{\pi(X_1)} \to \mathscr{O}_{\mathbb{P}^2} \to i_* \mathscr{O}_{\pi(X_1)} \to 0.$$

Here i denotes the inclusion $\pi(X_1) \rightarrowtail \mathbb{P}^2$. Tensoring with $\mathscr{O}(1)$ and taking the long exact sequence gives rise to an exact sequence

$$H^0(\mathbb{P}^2, \mathscr{O}(1)) \to H^0(\pi(X_1), \mathscr{O}_{\pi(X_1)}(1)) \to H^1(\mathbb{P}^2, \mathscr{I}_{\pi(X_1)}(1)) = 0,$$

where the vanishing of the right term follows from the fact that $\mathscr{I}_{\pi(X_1)}(1)$ is a line bundle on $\mathbb{P}^2$ and [Har10, Chapter III, Theorem 5.1]. So,

$$h^0(\pi(X_1), \mathscr{O}_{\pi(X_1)}(1)) \leq h^0(\mathbb{P}^2, \mathscr{O}(1)) = 3.$$

However, $H^0(\pi(X_1), \mathscr{O}_{\pi(X_1)}(1)) = H^0(X_1, \mathscr{O}_{X_1}(1))$ had dimension greater than 3, so this is a contradiction. ∎

Claim 3.16. *Denote by g the genus of X_1. Then $g < p_a(\pi(X_1))$, where $p_a(\pi(X_1))$ denotes the arithmetic genus of $\pi(X_1)$.*

Proof. Since $X_1 \to \pi(X_1)$ is the normalization of $\pi(X_1)$, we have an exact sequence of sheaves of $\mathscr{O}_{\pi(X_1)}$-modules

$$0 \to \mathscr{O}_{\pi(X_1)} \to \pi_* \mathscr{O}_{X_1} \to \mathscr{Q} \to 0,$$

where $\mathscr{Q} = \sum_{x \in \pi(X_1)} \widetilde{\mathscr{O}}_{\pi(X_1),x} / \mathscr{O}_{\pi(X_1),x}$ is the skyscraper sheaf with stalk $\widetilde{\mathscr{O}}_{\pi(X_1),x} / \mathscr{O}_{\pi(X_1),x}$ at x, the normalization of $\mathscr{O}_{\pi(X_1),x}$ modulo $\mathscr{O}_{\pi(X_1),x}$. See [Har10, Chapter IV, Exercise 1.8]. In particular, it is not zero by the previous claim. This gives rise to a long exact sequence

$$\begin{aligned} k = H^0(X_1, \mathscr{O}_{X_1}) \to H^0(\pi(X_1), \mathscr{Q}) \to H^1(\pi(X_1), \mathscr{O}_{\pi(X_1)}) \\ \to H^1(X_1, \mathscr{O}_{X_1}) \to H^1(\pi(X_1), \mathscr{Q}) = 0. \end{aligned}$$

The map $H^0(X_1, \mathscr{O}_{X_1}) \to H^0(\pi(X_1), \mathscr{Q})$ is zero, and so we find

$$g = h^1(X_1, \mathscr{O}_{X_1}) < h^1(\pi(X_1), \mathscr{O}_{\pi(X_1)}) = p_a(\pi(X_1)).$$

∎

Now, from the above claim it is clear that the fiber of $X \to \mathbb{A}^1$ must have nilpotents. For if it didn't, $p_a(X_1) \neq p_a(X_0)$ would contradict Theorem 3.10.

If X_1 is the twisted cubic curve described above, and if P lies on a secant line of X_1 (a line going through two distinct points of X_1), then $\pi(X_1)$ is a rational curve with a single node. Hartshorne shows that in this case, the fiber X_0 has nilpotent elements only at the node, where it has an embedded point; see [Har10, Chapter III, Example 9.8.4].

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