Topics in Algebraic Geometry: Flatness

Hugo Zock

Throughout we fix a commutative ring A. We will denote a general field by k.

Definition 1. *Let M be a module over A. We say that M is flat (over A) if the functor $- \otimes_A M$ from A-modules to A-modules is exact. We say that M is faithfully flat if, in addition to being flat, the following implication holds for all A-modules N:*

$$N \otimes_A M = 0 \Rightarrow N = 0.$$

The functor $- \otimes_A M$ is always right exact, regardless of whether M is flat module. To ensure M is flat, it therefore suffices to check that M preserves injections. The following proposition states that is in fact enough to check that injections of the form $I \hookrightarrow A$, with I an ideal of A, are preserved.

Proposition 2 (ideal criterion)**.** *Let M be a module over A. Then M is flat over A if and only if for all ideals I of A we have*

$$\mathrm{Tor}_1^A(M, A/I) = 0$$

Proof. See [Vak, Theorem 24.4.1]. ■

The above proposition already allows us to identify the flat modules over a principal ideal domain, as shown below.

Example 3. Suppose A is a principal ideal domain. Let M be a module over A. Then by the ideal criterion for flatness, M is flat over A if and only if the functor $- \otimes_A M$ preserves exact sequences of the form

$$0 \to A \xrightarrow{\cdot a} A,$$

with $a \in A \setminus \{0\}$. Tensoring such a sequence with M yields the sequence:

$$0 \to M \xrightarrow{\cdot a} M,$$

which is exact for all $a \in A \setminus \{0\}$ if and only if M is torsion-free.

Proposition 4. *(i) A module M over A is flat if and only if for all prime ideals $\mathfrak{p}$ of A, the module $M_\mathfrak{p}$ is flat over $A_\mathfrak{p}$.*

(ii) If $B \to A$ is a flat map of rings (i.e. A is flat as a module over B), and M is a flat A-module, then M is also flat as a B-module.

(iii) If M is flat over A, and $A \to C$ is any map of rings, then the module $M \otimes_A C$ is flat over C.

We give some more examples of flat and non-flat modules.

Example 5. (i) If P is a projective module over A, then P is flat over A. This can be seen by writing P as a direct summand of a free module and using the fact that free modules are flat. if P happens to be finitely presented over A, then this can also be seen by the fact that P is locally free and Proposition 4(i).

(ii) The map of rings $k[\varepsilon]/(\varepsilon^2) \to k$ is not flat. This can be seen as follows. The inclusion $(\varepsilon) \hookrightarrow k[\varepsilon]/(\varepsilon^2)$ when tensored with k gives the zero map, whereas the tensor product $(\varepsilon) \otimes k$ is nonzero.

(iii) If $A \hookrightarrow B$ is an inclusion of integral domains such that the field of fractions of A and B coincide, then B is faithfully flat over A if and only if $A = B$. See [Mat86, Exercise 7.2], which can be done using Theorem 7.5 in the same reference. Take for example the map of rings

$$k[x, y]/(y^2 - x^3) \to k[t]$$

sending x to t^2 and y to t^3. If this map were flat, it would be faithfully flat by the Going-down Theorem for integral extensions (as we will see in Proposition 7 (ii)). The field of fractions of both domains is $k(t)$, and so this would imply that the rings are equal. They clearly are not, and so we conclude that this map of rings is not flat.

We turn to the geometric notion of flatness.

Definition 6. *Let $\varphi : X \to Y$ be a map of schemes. We say that φ is flat if for all $x \in X$, the map of rings $\varphi_x^\# : \mathcal{O}_{Y,\varphi(x)} \to \mathcal{O}_{X,x}$ is flat. We say that φ is faithfully flat if, in addition to being flat, it is surjective.*

Proposition 7. *(i) A map of rings $A \to B$ is flat if and only if the induced map on spectra $\operatorname{Spec} B \to \operatorname{Spec} A$ is flat.*

(ii) A map of rings $A \to B$ is faithfully flat if and only if the induced map on spectra $\operatorname{Spec} B \to \operatorname{Spec} A$ is faithfully flat.

Proof. See [Sta22, Tag 00HT] for (i) and [Sta22, Tag 00HQ] for (ii). ∎

Using the above example we can translate all our algebraic examples to geometric ones.

Example 8. (i) The map $\operatorname{Spec} k \to \operatorname{Spec} k[\varepsilon]/(\varepsilon^2)$ from a point to a fuzzy point is not flat.

(ii) The projection map $\pi : \mathbb{A}_k^2 \to \mathbb{A}_k^1$ is flat. This follows from the fact that the ring $k[x, y]$ is free over $k[x]$, and hence flat.

(iii) Let $\mathbb{A}_k^1(t) = \operatorname{Spec} k[t]$ be the affine line and let $X = \operatorname{Spec} k[x, y]/(y^2 - x^3)$ be the cuspidal curve. The map of rings from Example 5(iii) induces a map of schemes $\mathbb{A}_k^1 \to X$ which is not flat. This example can be generalized by showing that the normalization of a scheme is flat if and only if it is an isomorphism.

Proposition 9. *(i) Composition of flat morphisms are flat.*

(ii) Flatness is stable under base change.

Proof. Apply Proposition 4 (ii) and (iii). ■

Definition 10. *Let X be a scheme. A point x of X is called an associated point of X if the unique maximal ideal $\mathfrak{m}_x$ of $\mathcal{O}_{X,x}$ is an associated prime.*

Remark 11. *If X is locally Noetherian, then $x \in X$ is an associated point if and only if $\mathfrak{m}_x \subset \mathcal{O}_{X,x}$ consists entirely of zero divisors.*

Example 12. If X is reduced, then the associated points of X correspond to the generic points of the irreducible components of X.

Proposition 13 (flat maps send associated points to associated points)**.** *Let $\varphi : X \to Y$ be a map of locally Noetherian schemes and let $x \in X$ be an associated point. Then $y = \varphi(x)$ is an associated point of Y.*

Proof. The map $\varphi_x^\# : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is flat. Let $t \in \mathfrak{m}_y$, and suppose that t is not a zero divisor. Then by flatness, the element $\varphi_x^\# t \in \mathfrak{m}_x$ would not be a zero divisor. This is a contradition, since x is an associated point of X. We conclude that y is an associated point of Y. ■

The above proposition will allow us to exhibit more examples of non-flat morphisms.

Example 14. (i) The map of schemes $\operatorname{Spec} k[x,y]/(xy) \to \operatorname{Spec} k[x]$ is not flat. Indeed, the associated prime (x) of $k[x,y]/(xy)$ pulls back to the prime (x) of $k[x]$, which is not an associated prime. Geometrically, we're projecting the union of the x- and y-axis down to the y-axis, which sends the entire y-axis to the origin.

(ii) The map $\pi : \mathrm{Bl}_O \mathbb{A}_2^k \to \mathbb{A}_k^2$, corresponding to the blow-up of the plane in the origin, is not flat. To see this, consider the inclusion of the y-axis into the plane $\mathbb{A}_k^1 \hookrightarrow \mathbb{A}_k^2$. If π were flat, then the pullback of π along this inclusion $\pi^{-1}\mathbb{A}_k^1 \to \mathbb{A}_k^1$ would be flat by 9 (ii). This map is not flat, because $\pi^{-1}\mathbb{A}_k^1$ consists of two irreducible components, one of which, the exceptional fiber, gets sent to the origin of $\mathbb{A}_k^1$. The associated point correponding to this irreducible component does not get mapped to an associated point of $\mathbb{A}_k^1$, and so we conclude that $\pi^{-1}\mathbb{A}_k^1 \to \mathbb{A}_k^1$ is not flat.

The following Theorem provides a partial converse to Proposition 13.

Theorem 15. *Let $\varphi : X \to Y$ be a map of locally Noetherian schemes, with Y regular, integral, and of dimension 1. Then φ is flat if and only if every associated point $x \in X$ gets mapped to the generic point of Y by φ.*

Proof. If φ is flat, then by Proposition 13 φ must send associated points to associated points. Since Y is integral, the only associated point of Y is its generic point.

Conversely, suppose that every associated point of X gets sent to the generic point of Y. Let $x \in X$ be a point. We set out to prove that $\varphi_x^\# : \mathscr{O}_{Y,y} \to \mathscr{O}_{X,x}$ is flat. If $y = \varphi(x)$ is the generic point of Y, then certainly the map on local rings $\varphi_x^\# : \mathscr{O}_{Y,y} \to \mathscr{O}_{X,x}$ is flat, because $\mathscr{O}_{Y,y}$ is a field. Suppose that y is a closed point of Y. Then the local ring $\mathscr{O}_{Y,y}$ is a discrete valuation ring by the regularity assumption on Y. Let π be a uniformizer. Since $\mathscr{O}_{Y,y}$ is a principal ideal domain, it suffices to prove that $\varphi_x^\# \pi$ is not a zero divisor of $\mathscr{O}_{X,x}$ by Example 3. Suppose this is the case. Then $\varphi_x^\# \pi$ is contained in an associated prime $\mathfrak{p}$ of $\mathscr{O}_{X,x}$. This prime corresponds to an associated point x' of X, and by the assertion $\mathfrak{p} \ni \varphi_x^\# \pi$ it lies in the fiber of φ over y. Since y is not the generic point of Y and x' is associated, this leads to a contradiction. ■

Remark 16. In case X is also reduced in the above theorem, we see that $\varphi : X \to Y$ is flat if and only if all irreducible components of X dominate Y. One might say that all irreducible components of X "lie flat over Y".

Example 17. We consider a family of conics. Let $A = \mathbb{C}[t]$ and let X be the scheme $\operatorname{Proj} A[x, y, z]/(xy - tz^2)$. Then the morphism $X \to \operatorname{Spec} A = \mathbb{A}^1_{\mathbb{C}}$ is faitfhfully flat by the above remark. The fibers over the closed points of $\mathbb{A}^1_{\mathbb{C}}$ are as follows.

- Over $t \neq 0$, the fiber is a non-degenerate conic in $\mathbb{P}^2_{\mathbb{C}}$.
- Over $t = 0$, the fiber is a degenerate conic in $\mathbb{P}^2_{\mathbb{C}}$.

This example illustrates that even if the general member of a flat family of varieties is irreducible, the special member can be reducible.

The following example shows that neither the assumption about regularity, nor the assumption about the dimension of Y can be dropped.

Example 18. (i) Let $X = \operatorname{Spec} k[x, y]/(y^2 - x^3)$ be the cuspidal curve and let $\mathbb{A}^1_k \to X$ be the map from Example 8 (iii). The domain of this morphism is integral, and the map is surjective, but it is not flat. Indeed, the scheme X has a singular point and so Theorem 15 does not apply.

(ii) Suppose $2 \neq 0$ in k[1].Let $A = k[x, y]$ and $B = A[u, v]/IJ$, where $I = (x - u, y - v)$ and $J = (x + u, y + v)$ are ideals of $k[x, y, u, v]$. It can be shown that the map $X = \operatorname{Spec} B \to \operatorname{Spec} A = \mathbb{A}^2_k$ is not flat. Geometrically, X consists of two planes meeting in a point, both mapping isomorphically to a plane. The irreducible components of X do dominate $\mathbb{A}^2_k$, but $\mathbb{A}^2_k$ is not of dimension 1, and so Theorem 15 does not apply.

The notion of flatness for morphisms of schemes is partly motivated by the fact that they have "nicely varying" fibers. The rest of these notes will give this statement some meaning.

Proposition 19 (flat maps are generizing)**.** *Let $\varphi : X \to Y$ be a flat map of schemes, let $x \in X$ and write $y = \varphi(x)$. If y' is a generization of y (i.e. $y \in \overline{\{y'\}}$), then there is a generization x' of x such that $\varphi(x') = y'$. One also says that generizations lift along φ, or that φ is generizing.*

[1] I'm unsure if this assumption is necessary.

The proof of this proposition comes down to the following algebraic lemma, which is reminiscent of the situation for integral extensions.

Lemma 20 (Going-down for flat maps)**.** *Let $A \to B$ be a flat map of rings and write $\varphi : \operatorname{Spec} B \to \operatorname{Spec} A$ for the induced map on spectra. If $\mathfrak{q}' \subset \mathfrak{q}$ are primes of A, and $\mathfrak{p}$ is a prime of B such that $\varphi(\mathfrak{p}) = \mathfrak{q}$, then there exists a prime $\mathfrak{p}' \subset \mathfrak{p}$ of B such that $\varphi(\mathfrak{p}') = \mathfrak{q}'$.*

Proof. The induced map $A_{\mathfrak{q}} \to B_{\mathfrak{p}}$ is a flat map of local rings and hence faithfully flat; thus, there exists a prime of $B_{\mathfrak{p}}$ pulling back to $\mathfrak{q}'A_{\mathfrak{q}}$. This prime is of the form $\mathfrak{p}'B_{\mathfrak{p}}$, where $\mathfrak{p}'$ is a prime of B pulling back to $\mathfrak{q}'$. ∎

Proof of Proposition 19. Apply the Going-down lemma to the flat map of rings $\varphi_x^\# : \mathscr{O}_{Y,y} \to \mathscr{O}_{X,x}$. ∎

Remark 21. Another example of generizing maps are open maps of schemes. It turns out that a flat map of schemes $\varphi : X \to Y$ is also (universally) open, as long as it is locally of finite presentation. See [Sta22, Tag 01UA].

Corollary 22 (fibre dimension)**.** *Let $\varphi : X \to Y$ be a map of locally Noetherian schemes. Let $x \in X$ and write $y = \varphi(x)$. Then*

$$\dim \mathscr{O}_{X,x} \leq \dim \mathscr{O}_{Y,y} + \dim \mathscr{O}_{X,x} \otimes_{\mathscr{O}_{Y,y}} \kappa(y).$$

If, in addition, φ is flat, then the above inequality becomes an equality.

Proof. For the inequality "≤" we refer to [Mat86, Theorem 15.1]. We prove the inequality

$$\dim \mathscr{O}_{X,x} \geq \dim \mathscr{O}_{Y,y} + \dim \mathscr{O}_{X,x} \otimes_{\mathscr{O}_{Y,y}} \kappa(y)$$

using the Going-down lemma for flat maps. Let

$$\mathfrak{r}_0 \subsetneq \mathfrak{r}_1 \subsetneq \ldots \subsetneq \mathfrak{r}_r = \mathfrak{m}_x \mod \mathfrak{m}_y \mathscr{O}_{X,x}$$

be a chain of prime ideals in $\mathscr{O}_{X,x} \otimes_{\mathscr{O}_{Y,y}} \kappa(y) = \mathscr{O}_{X,x}/\mathfrak{m}_y\mathscr{O}_{X,x}$ of length r, and let

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \ldots \subsetneq \mathfrak{p}_m = \mathfrak{m}_y$$

be a chain of prime ideals in $\mathscr{O}_{Y,y}$ of length m. The chain in $\mathscr{O}_{X,x}/\mathfrak{m}_y\mathscr{O}_{X,x}$ lifts to a chain

$$\mathfrak{m}_y \mathscr{O}_{X,x} \subset \mathfrak{q}_m \subsetneq \ldots \subsetneq \mathfrak{q}_{r+m} = \mathfrak{m}_x$$

in $\mathscr{O}_{X,x}$. Using the Going-down lemma for flat maps we now construct a chain of prime ideals

$$\mathfrak{q}_0 \subsetneq \ldots \subsetneq \mathfrak{q}_m$$

in $\mathscr{O}_{X,x}$ pulling back to the chain $\mathfrak{p}_0 \subsetneq \ldots \subsetneq \mathfrak{p}_m$. We've now found a chain of prime ideals $\mathfrak{q}_0 \subsetneq \ldots \subsetneq \mathfrak{q}_{r+m}$ in $\mathscr{O}_{X,x}$ of length $r+m$. We conclude that

$$\dim \mathscr{O}_{X,x} \geq \dim \mathscr{O}_{Y,y} + \dim \mathscr{O}_{X,x} \otimes_{\mathscr{O}_{Y,y}} \kappa(y).$$

∎

Remark 23. The proof of the above corollary only relies on the fact that φ is generizing.

There are more theorems regarding the continuous variation of numbers which can be associated to the fibers of a flat morphism. See, for example, [Vak, Section 24.7].

References

[Gör20] Ulrich Görtz. *Algebraic Geometry I: Schemes With Examples and Exercises*. 2nd ed. 2020. Springer Studium Mathematik - Master. Wiesbaden: Springer Fachmedien Wiesbaden, 2020.

[Har10] Robin Hartshorne. *Algebraic geometry*. Graduate texts in mathematics 52. New York: Springer Science+Business Media, Inc, 2010.

[Mat86] Hideyuki Matsumura. *Commutative ring theory*. Cambridge studies in advanced mathematics 8. Cambridge University Press, 1986.

[Sta22] The Stacks project authors. *The Stacks project*. `https://stacks.math.columbia.edu`. 2022.

[Vak] Ravi Vakil. *MATH 216: Foundations of Algebraic Geometry*. URL: `https://math.stanford.edu/%20vakil/216blog/`.