1. Fundamental groups of Riemann surfaces

This project is very similar in spirit to the project "Grothendieck's Galois theory", in the sense that it provides a connection between the Galois theory of algebra and that of topological covers. We let X be a *connected compact Riemann surface*. In other words, X is a connected compact complex manifold of dimension 1. Really, this means that X is covered by opens which are homeomorphic to open balls of $\mathbb{C}$, and such that the associated transition functions are holomorphic. Topologically, X is an oriented surface of some genus $g \geq 0$.

Example 1.1. The projective line $\mathbb{P}^1_{\mathbb{C}}$ is an example of a Riemann surface. Topologically it is just a sphere: an oriented surface of genus 0. ■

Let $S \subset X$ be a finite set of points on X, and let $X' = X \setminus S$ be the complement of S in X. Fix a basepoint $x \in X'$. We are interested in the fundamental group $\pi_1(X', x)$. The point of this project is to show that, remarkably, this is essentially the Galois group of a certain field extension assoicated to X'.

Let $\mathcal{M}(X)$ be the *field of meromorphic functions on* X. The elements of $\mathcal{M}(X)$ are the smooth functions $\varphi\colon X \to \mathbb{P}^1_{\mathbb{C}}$ which are locally given by holomorphic functions. We fix an algebraic closure $\overline{\mathcal{M}(X)}$ of $\mathcal{M}(X)$. For any Riemann surface Y and any holomorphic map $Y \to X$, we obtain a finite extension of fields $\mathcal{M}(X) \subset \mathcal{M}(Y)$ by pulling back meromorphic functions on X to Y along the holomorphic map $Y \to X$. A holomorphic map $Y \to X$ is said to be *unramified* over X' if the restriction of Y to the inverse image of X' is a topological covers of X'. We define $K_{X'}$ to be the compositum of all extensions of $\mathcal{M}(X)$ in $\overline{\mathcal{M}(X)}$ which arise from holomorphic maps $Y \to X$ unramified over X'.

Theorem 1.2 ([Sza09, Theorem 3.4.1])**.** *The field extension $K_{X'}/\mathcal{M}(X)$ is a Galois extension of fields. We have an isomorphism of profinite groups*

$$\mathrm{Gal}(K_{X'}/\mathcal{M}(X)) \simeq \widehat{\pi}_1(X', x).$$

In the theorem above, $\widehat{\pi}_1(X', x)$ denotes the *profinite completion* of the usual fundamental group $\pi_1(X', x)$. It is defined as

$$\widehat{\pi}_1(X', x) \simeq \varprojlim \pi_1(X', x)/U,$$

where the projective limit ranges over $U \subset \pi_1(X', x)$ open normal of finite index.

The key idea in the proof of Theorem 1.2 is to establish a correspondence between finite extensions of $\mathcal{M}(X)$ and connected branched covers of X. This is done in [Sza09, Theorem 3.3.7]

Remark 1.3. The finite extensions $\mathcal{M}(X) \subset L \subset \overline{\mathcal{M}(X)}$ which arise from covers unramified over X' can also be characterized by the following property. For every $p \in X'$ there is an associated discrete valuation v_p on $\mathcal{M}(X)$. This valuation does not ramifiy in the extension $\mathcal{M}(X) \subset L$.

2. Goals for a report

The notion of a Riemann surface and morphisms between Riemann surfaces should be briefly discussed. Most of the important properties can be stated without proof or with only brief motivation. [Sza09, Chapter 3, Section 1] should treat everything you need, but there are also other nice references for Riemann surfaces such as [Mir95]. Like mentioned in the proof above, it is key to establish the correspondence between connected branched covers and finite extensions of the field of meromorphic functions. It might be more natural to remove the word "connected", and replace "finite extensions of the field of meromorphic functions" by "finite étale algebras over the field of meromorphic functions", which are also discussed in the description of the project "Grothendieck's Galois theory". You are of course somewhat constrained by the permitted length of your report, and so it is important to think carefully about which proofs of partial results to omit and which to write out in detail.

2.1. Prerequisites.

Fundamental groups of topological spaces and Galois groups. Ideally you already know at least the definition of a Riemann surface.

References

[Mir95] Rick Miranda. *Algebraic curves and Riemann surfaces*. Vol. 5. Graduate studies in mathematics. Providence, R.I: American Mathematical Society, 1995.

[Sza09] Tamás Szamuely. *Galois Groups and Fundamental Groups*. 1st ed. Vol. 117. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2009.