1. Grothendieck's Galois Theory

Fix a field k and a separable closure k^s, i.e., k^s is the compositum of all separable subextensions of $k \subset \Omega$, where Ω is some algebraically closed field containing k. Write G_k for the absolute Galois group $\mathrm{Gal}(k^s/k)$.

Definition 1.1. An algebra A over k is called *finite étale* if A is a finite product of separable extensions of k (which need not be contained in k^s). A morphism of finite étale algebras is just a morphism of k-algebras. We write $\mathrm{Fét}_k$ for the category of finite étale k-algebras. ▲

If A is a finite étale algebra over k, then the set of morphisms of k-algebras $\mathrm{Hom}(A, k^s)$ caries a natural left-action of G_k. This action is continuous. If we write sets for the category of finite sets, and G_k-sets for the category of finite sets with a continuous action of G_k, then we have a contravariant functor

$$\mathrm{Hom}(-, k^s)\colon \mathrm{Fét}_k^{\mathrm{opp}} \to \mathrm{sets}.$$

This functor factors over G_k-sets:

$$\begin{tikzcd} \mathrm{Fét}_k \arrow[rr, "\mathrm{Hom}(-,k^s)"] \arrow[dr, dashed] & & \mathrm{sets} \\ & G_k\text{-}\mathrm{sets}. \arrow[ur] & \end{tikzcd} \tag{1.1}$$

Here the right diagonal arrow is the natural forgetful functor.

Theorem 1.2 (Grothendieck's version of Galois theory, [Sza09, Theorem 1.5.4])**.** *The dashed arrow from* (1.1)

$$\mathrm{Hom}(-, k^s)\colon \mathrm{Fét}_k \xrightarrow{\simeq} G_k\text{-}\mathrm{sets}$$

is an equivalence of categories. Under this equivalence the finite separable extensions of k correspond to sets with a transitive action of G_k. The Galois extensions of k correspond to the G_k-sets of the form G_k/U, where $U \subset G_k$ is an open subgroup of G_k.

We finish with a few remarks to motivate the theorem above. It turns out that a very similar theorem holds true for the category of coverings of topological spaces. If X is a connected topological space, $x \in X$ is a fixed basepoint, and $\pi_1(X,x)$ is the fundamental group, then we can consider the *profinite completion* of $\pi_1(X,x)$

$$\widehat{\pi}_1(X,x) = \varprojlim \pi_1(X)/U.$$

Here the projective limit is taken over the normal subgroups of finite index in $\pi_1(X)$. If $Y \xrightarrow{f} X$ denotes a *finite* covering of X, then the fiber $f^{-1}\{x\}$ over x naturally obtains a continuous action of $\widehat{\pi_1}(X,x)$; the so called *monoodromy action.* In this way we obtain a functor

$$\mathrm{Cov}_X \to \widehat{\pi}_1(X,x)\text{-sets},$$

for which an analogous theorem to the one above holds true. Specifically, this is proved in [Sza09, Corollary 2.3.9].

When one encounters many similar such statement, the natural question to ask is "what makes the clock tick?" In other words, is there some general framework which encapsulates all such statements? It turns out that there is. It is aptly named the *theory of Galois categories.* The canonical reference for Galois categories is [GR06, Exposé V]. A nice reference in English is Lenstra's text [Len, Chapter 3].

2. Goals for a report

A report should include a proof of Theorem 1.2. It would of course be nice if you could also relate this to the usual Theorem of infinite Galois category. If you would like to take the report into a more categorical direction, you could try to treat the theory of Galois categories with some examples. If you know a little bit about algebraic geometry, you can apply this to finite étale covers of schemes and obtain the theory of the étale fundamental group.

2.1. **prerequisites.** It would be very useful if you are already familiar with (infinite) Galois theory. For the theory of Galois categories, you really only need to know a little bit about category theory, but in order to appreciate its usefulness, you should know about covering spaces in topology, or even about étale covers of schemes.

References

[GR06] Alexander Grothendieck and Michèle Raynaud. *Revêtements Etales et Groupe Fondamental.* fr. Vol. 224. Lecture Notes in Mathematics. Springer Berlin Heidelberg, 2006.

[Len] Hendrik Lenstra. *Galois theory for schemes.* URL: `https://websites.math.leidenuniv.nl/algebra/GSchemes.pdf`.

[Sza09] Tamás Szamuely. *Galois Groups and Fundamental Groups.* 1st ed. Vol. 117. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2009.