A VARIETY WITH TRIVIAL CANONICAL BUNDLE, DEGENERATE HODGE-TO-DE RHAM SPECTRAL SEQUENCE, AND OBSTRUCTED DEFORMATIONS

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ABSTRACT. We exhibit a variety as in the title, resolving a question of Brantner and Taelman [BT24, Remark 1.1].

1. INTRODUCTION

A smooth and proper variety X over an algebraically closed field k is said to be *Calabi-Yau* if its canonical bundle ω_X is trivial. In [BT24] it is demonstrated that such an X has unobstructed mixed characteristic formal deformations, provided that the Hodge-to-de Rham spectral sequence $E_1^{pq} = H^q(X, \Omega_X^p) \Rightarrow H_{\mathrm{dR}}^{p+q}(X/k)$ of X degenerates and that the crystalline cohomology $H^*_{\mathrm{cris}}(X/W(k))$ is torsion-free.

In [BT24, Remark 1.1] Brantner and Taelman raise the following question.

Question 1.1. Is there a Calabi-Yau variety X such that the Hodge-to-de Rham spectral sequence degenerates, *and* such that X has obstructed deformations?

By the results of Brantner and Taelman, if such an X exists, it must necessarily have torsion in the crystalline cohomology $H^*_{\mathrm{cris}}(X/W(k))$.

In this short note we answer Question 1.1 *affirmatively*. The variety X will be a certain bielliptic surface in characteristic 2.

Presumably, one can also construct a bielliptic surface in characteristic 3 to answer Question 1.1; but beyond characteristic 3 all bielliptic surfaces will have torsion-free crystalline cohomology groups. This raises the following question.

Question 1.2. Is there a Calabi-Yau variety X as in Question 1.1 in any characteristic $p > 0$?

An affirmative answer to this question would necessarily involve varieties of dimension > 2.

It should also be mentioned that the definition of a Calabi-Yau variety used here is not the generally accepted definition; usually the additional condition

$$H^i(\mathcal{O}_X) = 0 \quad \text{for} \quad 0 < i < \dim X \tag{1.1}$$

is also imposed. The variety X we introduce will *not* satisfy this additional condition. It is then natural to ponder the question below.

Question 1.3. Is there an X as in Question 1.1 that satisifies the condition of (1.1)?

Such an X would necessarily be of dimension > 2. Indeed, Calabi-Yau surfaces satisfying (1.1) are $K3$ surfaces, for which the deformations are known to be unobstructed.

2. THE VARIETY X IN QUESTION

Throughout we assume that k is of characterstic 2.

Let E be an ordinary elliptic curve over k; i.e., the for the k-points of the 2-torsion subgroup $E[2] \subset E$ we have

$$E[2](k) \simeq \mathbb{Z}/2\mathbb{Z}.$$

Let $P \in E[2](k)$ be the unique non-trivial 2-torsion point of E. Consider the product $E \times E$ and let $\mathbb{Z}/2\mathbb{Z}$ act on $E \times E$ by

$$(x, y) \mapsto (x + P, -y).$$

Here the "$+$" and "$-$" are those of the group law of E. The action of $\mathbb{Z}/2\mathbb{Z}$ on $E \times E$ is free, and so the quotient $X = (E \times E)/(\mathbb{Z}/2\mathbb{Z})$ will be a smooth proper surface over k. It is an example of a bielliptic surface, which is a family of algebraic surfaces of Kodaira dimension 0.

We recall some standard properties of bielliptic surfaces (see e.g. [BM77]). The euler characteristic of the structure sheaf of X is

$$\chi(\mathscr{O}_X) = 0. \tag{2.1}$$

The Betti numbers of X are given by

$$b_0 = b_4 = 1, \quad b_1 = b_2 = b_3 = 2. \tag{2.2}$$

Proposition 2.1. *The cotangent bundle Ω^1_X is trivial: $\Omega^1_X \simeq \mathscr{O}_X^{\oplus 2}$.*

Proof. It suffices to prove that $\mathbb{Z}/2\mathbb{Z}$ acts trivially on $\Omega^1_{E\times E} \simeq p_1^*\Omega^1_E \oplus p_2^*\Omega^1_E$. Given $\alpha \in \Omega^1_E$, pulling back along the map $x \mapsto x + P$ sends α to α regardless of the characteristic of k, and pulling back along $x \mapsto -x$ sends α to $-\alpha = \alpha$ (recall that 2 = 0). ■

The above proposition in particular implies $\omega_X = \Omega^2_X \simeq \mathscr{O}_X$, and so by Serre-duality

$$h^{02} = h^2(\mathscr{O}_X) = h^0(\omega_X) = 1.$$

From (2.1), we deduce

$$h^{01} = h^1(\mathscr{O}_X) = 2.$$

We find the Hodge diamond of X to be

$$\begin{array}{ccccc} & & 1 & & \\ & 2 & & 2 & \\ 1 & & 4 & & 1 \\ & 2 & & 2 & \\ & & 1. & & \end{array} \tag{2.3}$$

2.1. **The Hodge-to-de Rham spectral sequence.** Let $\Omega_X^\bullet$ denote the algebraic de Rham complex of X and recall that the de Rham cohomology of X is defined by

$$H^i_{\mathrm{dR}}(X/k) = H^i R\Gamma(X, \Omega_X^\bullet).$$

We also write $h^i_{\mathrm{dR}} = \dim_k H^i_{\mathrm{dR}}(X/k)$. The naive filtration on $\Omega_X^\bullet$ gives rise to the Hodge-to-de Rham spectral sequence

$$E_1^{ij} = H^j(X, \Omega_X^i) \Rightarrow H^{i+j}_{\mathrm{dR}}(X/k). \tag{2.4}$$

In order to show that the spectral sequence of (2.4) degenerates we need only show that for all n we have

$$h^n_{\mathrm{dR}} = \sum_{i+j=n} h^{ij}. \tag{2.5}$$

The inequality $\leqslant$ in 2.5 is already clear from the spectral sequence 2.4. The following proposition therefore reduces to proving the inequality $\geqslant$.

Proposition 2.2. *The Hodge-to-de Rham spectral sequence* (2.4) *degenerates at the* E_1*-page.*

Proof. Write $Y = X \times X$ so that we have an étale $\mathbb{Z}/2\mathbb{Z}$-torsor $Y \to X$. We obtain the Hochschild-Serre spectral sequence

$$E_2^{ij} = H^i(\mathbb{Z}/2\mathbb{Z}, H^j(Y, \Omega_Y^\bullet)) \Rightarrow H^{i+j}(X, \Omega_X^\bullet).$$

The action of $\mathbb{Z}/2\mathbb{Z}$ on $H^j(Y, \Omega_Y^\bullet)$ is trivial, and so, since we know the de Rham cohomology of the abelian variety Y, we can compute all the terms on the E_2-page. In particular we deduce that

$$E_2^{01} \simeq k^{\oplus 4}, \quad E_2^{10} \simeq E_2^{20} \simeq k.$$

It follows that $h^1_{\mathrm{dR}}(X/k) = \dim_k E_\infty^{01} + \dim_k E_\infty^{10} \geqslant 3+1 = 4$. Since $h^1_{\mathrm{dR}}(X/k) \leqslant h^{10} + h^{01} = 2+2 = 4$ we conclude that $h^1_{\mathrm{dR}}(X/k) = 4$. Poincaré duality also gives $h^3_{\mathrm{dR}} = 4$.

To see that $h^2_{\mathrm{dR}} = h^{20} + h^{11} + h^{02} = 1+4+1 = 6$, one inspects the Hodge-to-de Rham spectral sequence of X directly, using the already computed Hodge and de Rham numbers. ■

Remark 2.3. We provide an alternative proof for the above proposition using [Suw83]. One can show that there is an exact sequence

$$0 \to \mathrm{Pic}^{0,\mathrm{red}}_{X/k} \to \mathrm{Pic}^\tau_{X/k} \to E[2] \to 0.$$

Since E is an ordinary elliptic curve we have $E[2] \simeq \mathbb{Z}/2\mathbb{Z} \times \mu_2$. We have

$$\mathrm{rk}_2\, E[2] = \mathrm{rk}_2\, {}_F E[2] + \mathrm{rk}_2\, {}_V E[2]. \tag{2.6}$$

Here ${}_F E[2]$, respectively ${}_V E[2]$, denotes the kernel of the Frobenius, respectively the Verschiebung, morphism on $E[2]$, and for a given finite group scheme G over k we write $\mathrm{rk}_2 G = \log_2 h^0(\mathscr{O}_G)$. We conclude from (2.6) and [Suw83, Corollary 3] that the Hodge-to-de Rham spectral sequence of X degenerates at E_1.

2.2. **Torsion in the crystalline cohomology.** Write $W = W(k)$ for the ring of Witt vectors. For the crystalline cohomology $H^*(X/W)$ of X there is the following "universal coefficient exact sequence":

$$0 \to H^1(X/W) \otimes_W k \to H^1_{\mathrm{dR}}(X/k) \to \mathrm{Tor}_1^W(H^2(X/W), W) \to 0. \tag{2.7}$$

Since $\dim_k H^1(X/W) \otimes_W k = b_1$, as $H^1(X/W)$ is torsion-free, the term $\mathrm{Tor}_1^W(H^2(X/W), W)$ is non-trivial if and only if

$$h^1_{\mathrm{dR}} > b_1.$$

The betti numbers of X are recorded in (2.2), and the de Rham numbers are computed from the Hodge numbers (2.3) and the fact that the Hodge-to-de Rham spectral sequence degenerates by Proposition 2.2. We observe that

$$h^1_{\mathrm{dR}} = 4 > 2 = b_1,$$

and hence conclude that $H^2(X/W)$ has non-trivial $p = 2$-torsion. This in combination with the remarks of the introduction makes the variety X a potential candidate to answer Question1.1.

Remark 2.4. One can in fact show much more: $H^*(X/W)_{\mathrm{tors}}$ is always killed by 2 and from this we can deduce the precise structure of the integral crystaline cohomology groups.

3. The deformation theory of X

As before, $W = W(k)$. Denote by Art_W the category of local Artin W-algebras. Let

$$\begin{aligned} \mathrm{Def}_X \colon \mathrm{Art}_W &\to \mathrm{Sets} \\ R &\mapsto \{f \colon \mathscr{X} \to \operatorname{Spec} R \mid f \text{ is flat and } \mathscr{X} \otimes_R k \simeq X\}/\simeq \end{aligned} \tag{3.1}$$

denote the deformation functor of X. We argue that X has obstructed deformations; i.e., that the functor Def_X is not formally smooth. In light of Proposition 2.2 this answers Question 1.1. Practically all of the leg work is done by Holger Partsch in [Par13].

Consider the projection onto the first coordinate

$$p \colon X \to E/\langle x \mapsto x + P\rangle.$$

It is an elliptic fibration over the elliptic curve $A := E/\langle x \mapsto x + P\rangle$, and it can easily be shown that this is the Albanese fibration. Consider the problem of deforming $\mathscr{X}$ together with the elliptic fibration $p\colon X \to A$: for a given $R \in \mathrm{Art}_W$, a flat morphism of R-schemes $\mathscr{X} \to \mathscr{A}$ is said to be a *deformation of* p if there is a commutative diagram of k-schemes

$$\begin{array}{ccc} \mathscr{X} \otimes_R k & \xrightarrow{\simeq} & X \\ \Big\downarrow{\scriptstyle \pi\otimes_R k} & & \Big\downarrow{\scriptstyle p} \\ \mathscr{A} \otimes_R k & \xrightarrow{\simeq} & A. \end{array}$$

We introduce a second deformation functor:

$$\begin{aligned} \mathrm{Fib}_{X/A} = \mathrm{Fib}_X \colon \mathrm{Art}_W &\to \mathrm{Sets} \\ R &\mapsto \{\pi\colon \mathscr{X} \to \mathscr{A} \text{ deformation of } p\}/\simeq . \end{aligned}$$

There is obviously a natural map

$$\mathrm{Fib}_X \to \mathrm{Def}_X . \tag{3.2}$$

Proposition 3.1. *The map of* (3.2) *is an isomorphism.*

Proof. See [Par13, Proposition 6.6]. ∎

3.1. **Deformations of a Jacobian fibration.** The elliptic fibration $p\colon X \to A$ is *Jacobian*: it admits a section $A \dashrightarrow X$. We fix one such section $e\colon A \to X$. This makes $p\colon X \to A$ into an elliptic curve over A, and we now want to consider deformations of $X \to A$ as an elliptic curve. These are deformations $\pi\colon \mathscr{X} \to \mathscr{A}$ of p that are Jacobian (they admit a section $\varepsilon\colon \mathscr{A} \to \mathscr{X}$)[1]. Consider the deformation functor

$$\begin{aligned} \mathrm{Jac}_X \colon \mathrm{Art}_W &\to \mathrm{Sets} \\ R &\mapsto \{\pi\colon \mathscr{X} \to \mathscr{A} \text{ a Jacobian deformation of } p\colon X \to A\}/\simeq . \end{aligned}$$

A given deformation $\mathscr{X} \to \mathscr{A}$ of p might not admit a section, but $\mathrm{Pic}^0_{\mathscr{X}/\mathscr{A}} \to \mathscr{A}$ is a deformation of p that always does: the unit section $\varepsilon\colon \mathscr{A} \to \mathrm{Pic}^0_{\mathscr{X}/\mathscr{A}}$ is one. This gives us a natural map

$$\begin{aligned} \mathrm{Fib}_X &\to \mathrm{Jac}_X . \\ (\mathscr{X} \to \mathscr{A}) &\mapsto \mathrm{Pic}^0_{\mathscr{X}/\mathscr{A}} . \end{aligned} \tag{3.3}$$

Since for any jacobian fibration $\mathscr{X} \to \mathscr{A}$ we have $\mathscr{X} \simeq \mathrm{Pic}^0_{\mathscr{X}/\mathscr{A}}$ over $\mathscr{A}$, the map of (3.3) canonically admits a section $\mathrm{Jac}_X \dashrightarrow \mathrm{Fib}_X$.

The goal now is to analyze the deformation functor Jac_X and to show that it is *not* smooth.

[1]We do not need to consider the section as part of the datum of a Jacobian fibration, because any section can be sent by an automorphism of $\mathscr{X} \to \mathscr{A}$ to any other section.

We introduce two more auxiliary deformation functors: Def_E is the deformation functor of E as an elliptic curve, and $\mathrm{Def}_{E,p}$ is defined by

$$R \mapsto \{(\mathscr{E}, \mathscr{P} \in \mathscr{E}[2](R)) \mid \mathscr{E} \otimes_R k \simeq E \text{ and } \mathscr{P} \mapsto P \text{ under this isomorphism}\}.$$

Now consider the map

$$\mathrm{Def}_{E,P} \times \mathrm{Def}_E \to \mathrm{Jac}_X \tag{3.4}$$

that "caries out the Igusa construction"; i.e., it sends a pair $((\mathscr{E}, \mathscr{P}), \mathscr{F})$ to $\mathscr{E} \times \mathscr{F} / \langle (x, y) \mapsto (x + \mathscr{P}, -y) \rangle$.

Proposition 3.2. *The map of* (3.4) *is an isomorphism.*

Proof. This is the content of [Par13, Proposition 3.2]. ∎

The deformation functor Def_E is smooth, but $\mathrm{Def}_{E,P}$ is not:

Proposition 3.3. *The deformation functor* $\mathrm{Def}_{E,P}$ *is not smooth.*

Proof. In [Par13, Section 6.1.2] the base of a versal deformation of $\mathrm{Def}_{E,P}$ is computed, using Serre-Tate theory, to be $\mathrm{Spf}\, W[[q-1]][\sqrt{q}]$, which is not a smooth formal W-scheme. ∎

Corollary 3.4. *The deformation functor* Jac_X *is not formally smooth.*

Proof. Combine the previous two propositions. ∎

We finally settle Question 1.1:

Corollary 3.5. *The deformation functor* Def_X *is not formally smooth.*

Proof. By the isomorphism $\mathrm{Fib}_X \simeq \mathrm{Def}_X$ from Proposition 3.1, we are reduced to showing that Fib_X is not formally smooth. If this were the case, then we could deduce that Jac_X is formally smooth using the section $\mathrm{Jac}_X \to \mathrm{Fib}_X$ mentioned right below (3.3). This contradicts the previous corollary. ∎

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