UNIVERSITEIT LEIDEN
MATHEMATISCH INSTITUUT

UNIVERSITÄT REGENSBURG
FAKULTÄT DER MATHEMATIK

A LOCAL VERSION OF KASHIWARA'S CONJECTURE

HUGO J. ZOCK

SUPERVISOR:
PROF. MORITZ C. KERZ
UNIVERSITÄT REGENSBURG

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CONTENTS

1. INTRODUCTION

1.1. **Background: the complex Kashiwara conjecture.** In 1998, Masaki Kashiwara formulated several conjectures about *semisimple algebraic holonomic $\mathscr{D}$-modules* on smooth complex varieties in [Kas+98]. They were proved for *regular* holonomic $\mathscr{D}$-modules under the assumption of De Jong's conjecture (see [Jon01]) by Vladimir Drinfeld in [Dri01]. De Jong's conjecture was soon after established in sufficient generality by Gaitsgory and Böckle-Khare; see [Gai07] and [BK06]. The conjectures were also proved for regular holonomic $\mathscr{D}$-modules on quasi-projective varieties, using different methods, by Takurō Mochizuki in [Moc07].

Regular holonomic $\mathscr{D}$-modules correspond to *perverse sheaves* under the Riemann-Hilbert correspondence; see [HTT08, Theorem 7.2.5]. Particular examples of regular holonomic $\mathscr{D}$-modules are algebraic vector bundles with regular flat connections. Under the Riemann-Hilbert correspondence, these correspond to locally constant sheaves of finite dimensional $\mathbb{C}$-vector spaces in the analytic topology, also known as *complex local systems.* If the underlying variety X is connected, then these in turn correspond to finite-dimensional $\mathbb{C}$-linear representations of the fundamental group $\pi_1(X(\mathbb{C}))$. Under this correspondence, the notion of semisimplicity translates into the usual one for representations.

A particular corollary of Kashiwara's conjectures (specifically, (C2) in [Kas+98]) is the following theorem. For simplicity, we will refer to it as the "complex Kashiwara conjecture" in this thesis.

Theorem 1.1 (Complex Kashiwara conjecture)**.** *Let X and Y be smooth quasi-projective $\mathbb{C}$-varieties, and let $f\colon X \to Y$ be a morphism. If $\mathbb{L}$ on Y is a semisimple local system, then the local system $f^{-1}\mathbb{L}$ on X obtained by pulling back $\mathbb{L}$ along f is also semisimple.*

Hélène Esnault and Johan de Jong also provide an arithmetic proof of the Theorem in [JE23, Theorem 7.3]. They also explain that it suffices to assume that X and Y are *normal* and quasi-projective. The assumption of smoothness can, however, not be omitted altogether. Indeed, let Y be a proper rational curve with a single nodal point, and denote by Y° the curve Y punctured in two smooth points. Denote by $f\colon \mathbb{P}^1 - \{0,\infty\} = \mathbb{G}_m \to Y^\circ$ the normalization of Y°. Appendix A suggests an irreducible representation $\rho\colon \pi_1(Y^\circ) \to \mathrm{GL}_2(\mathbb{C})$ whose pullback to $\pi_1(\mathbb{G}_m)$ is not semisimple.

1.2. **The arithmetic local Kashiwara conjecture.** Inspired by Kashiwara's conjectures for semisimple holonomic $\mathscr{D}$-modules, Hélène Esnault and Moritz Kerz formulated arithmetic versions of the conjectures for *arithmetic semisimple perverse sheaves* in [EK23, Conjecture 9.7]. Their conjectures are highly relevant: they imply the monodromy weight conjecture in mixed characteristic. We will restrict our attention to particular examples of arithmetic perverse sheaves, namely arithmetic lisse $\overline{\mathbb{Q}}_\ell$-sheaves, which we refer to simply as *arithmetic ℓ-adic local systems.* Here ℓ always denotes a prime number invertible on the underlying scheme. Analogously to the situation for complex local systems, ℓ-adic local systems on a connected noetherian scheme X correspond to continuous finite-dimensional $\overline{\mathbb{Q}}_\ell$-linear representations of the étale fundamental group $\pi_1^{\text{ét}}(X)$. We will simply speak of *ℓ-adic representations* of $\pi_1^{\text{ét}}(X)$, leaving the other adjectives implicit. The theory of ℓ-adic local systems is recalled in detail in Section 2. Arithmetic ℓ-adic local systems are discussed in Section 6.1.

Let $\mathscr{O}$ be a strictly henselian discrete valuation ring, and let ℓ be a prime number such that $\ell \in \mathscr{O}^\times$. Denote by k the residue field of $\mathscr{O}$, by K the fraction field of $\mathscr{O}$, and let $\overline{K}$ be a separable closure of K. Write $S = \operatorname{Spec}\mathscr{O}$. Let $X \to S$ be a surjective smooth separated quasi-compact morphism of schemes. Denote by $X_{\overline{K}}$ the base change of X along $\operatorname{Spec}\overline{K} \to S$, and define X_k similarly. The conjectures of Esnault and Kerz imply the following conjecture, which will be referred to simply as the "arithmetic local Kashiwara conjecture".

Conjecture 1.2 (Arithmetic local Kashiwara conjecture)**.** *Let $\mathbb{L}$ be an ℓ-adic local system on X such that the pullback $\mathbb{L}_{\overline{K}} = \mathbb{L}|_{X_{\overline{K}}}$ of $\mathbb{L}$ to $X_{\overline{K}}$ is arithmetic*[1]*. Then $\mathbb{L}_k$ is semisimple.*

It does not suffice to assume that $\mathbb{L}_{\overline{K}}$ is only semisimple in the above conjecture. More precisely, the answer to the following question is "No".

Question 1.3 (Naive local Kashiwara conjecture, Question 4.1)**.** Let $\mathbb{L}$ be a semisimple ℓ-adic local system on X. Is it true that $\mathbb{L}_k$ is semisimple?

The assertion that the answer to Question 1.3 is "Yes", is referred to as the "naive local Kashiwara conjecture" in this thesis. Takurō Mochizuki answered a complex-geometric version of Question 1.3; see Question A.1. The counterexample he constructs is given in Appendix A.

As already indicated previously, Conjecture 1.2 is most interesting in the case that $\mathscr{O}$ has mixed characteristic. This case is presently out of reach. The following specific case of Conjecture 1.2 – where $\mathscr{O}$ is of equal positive characteristic – is proved in this thesis.

Theorem 1.4 (Theorem 6.13)**.** *Conjecture 1.2 is true if $\mathscr{O} = \mathscr{O}^{\text{hs}}_{C,\overline{c}}$ is the strict henselization of a normal connected $\mathbb{F}_q$-curve C at a geometric point $\overline{c} \in C(\overline{\mathbb{F}}_q)$, and if $X_{\overline{K}}$ is furthermore a connected curve.*

1.3. **Simpson's spreading argument.** Let G and H be profinite groups, and let G act continuously on H. Let $\mathscr{R}$ denote the set of semisimple ℓ-adic representations $H \to \mathrm{GL}(V)$ up to isomorphism. Then $\mathscr{R}$ is naturally equipped with a right G-action.

Theorem 1.5 (Theorem 5.6)**.** *Let $\rho\colon H \to \mathrm{GL}(V)$ be an irreducible ℓ-adic representation, and assume that the orbit $[\rho]\cdot G \subset \mathscr{R}$ of the isomorphism class of ρ is finite. Then there exists an open subgroup $U \subset G$ such that ρ extends to an ℓ-adic representation*

$$\widetilde{\rho}\colon H \rtimes U \to \mathrm{GL}(V).$$

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[1] In this thesis arithmetic local systems are always assumed to be semisimple, but this is an unusual convention in the literature.

If, furthermore, ρ has finite determinant, then also $\widetilde{\rho}$ can be chosen such that it has finite determinant.

The purpose of Theorem 1.5 in this thesis is twofold: it is the main ingredient in the proof of Theorem 1.4, and the simplified version – without the finiteness condition on determinants – is crucial for establishing the basic theory of arithmetic local systems.

Variants of Theorem 1.5 are well known and appear for instance in [Sim92] and [Lit21]. However, it does not seem to appear in such generality in the literature elsewhere.

1.4. **Goals and outline.** The first main goal of this thesis is to answer Question 1.3. Inspired by Mochizuki's example, we construct a similar counterexample in Section 4. It involves an elliptic fibration with singular fibre of type I_1 in Kodaira's classification of singular fibers. We explicitly compute the fundamental group of its special fibre, and then we use the specialization isomorphism from Section 3 to compute the fundamental group of the total space. This will allow us to construct explicit representations.

The second main goal of this thesis is to prove Theorem 1.4. In order to achieve this, we first devote a Section, Section 5, to the main ingredient used in the proof: Theorem 1.5. Our proof of Theorem 1.5 employs Galois cohomology with non-discrete and non-abelian coefficients, which we summarize in Appendix B, to strategically lift projective representations to proper representations.

The proof of Theorem 1.4 then proceeds by using the arithmeticity condition and Theorem 1.5 to reduce the problem to Theorem 6.15. The latter has a more global flavour and is subsequently proved using the work of Deligne [Del80] and Lafforgue [Laf02].

2. ℓ-ADIC LOCAL SYSTEMS AND THEIR MONODROMY REPRESENTATIONS

Throughout, ℓ denotes a prime and X denotes a separated noetherian scheme on which ℓ is invertible. Most of the material in this section is based on [FK13, Section 1.12], [Fu11, Section 10.1] and [Wei01, Appendix A]. ℓ-adic local systems are the étale analogue of locally constant sheaves of finite dimensional $\mathbb{C}$-vector spaces on complex manifolds, also known as *complex local systems.*

If $\mathbb{L}$ denotes a local system on a complex manifold M and $z \in M$ is a point, then there is an induced $\mathbb{C}$-linear action of the fundamental group $\pi_1(M,z)$ on the stalk $\mathbb{L}_z$, called the *monodromy action.* For a connected complex manifold M this induces an equivalence of categories:

$$\mathrm{Loc}_{\mathbb{C}}(M) \simeq \mathrm{Rep}_{\mathbb{C}}(\pi_1(M,z)), \tag{2.1}$$

where $\mathrm{Loc}_{\mathbb{C}}(M)$ denotes the category of complex local systems on M, and $\mathrm{Rep}_{\mathbb{C}}(\pi_1(M,x))$ denotes the category of finite-dimensional $\mathbb{C}$-linear representations of the topological fundamental group $\pi_1(M,z)$ of M based at z.

Example 2.1. Let $\Delta \subset \mathbb{C}$ denote the unit disc, and let Δ^* denote the punctured unit disc $\Delta \setminus \{0\}$. We let $f\colon M \to \Delta$ be an elliptic fibration with singular fiber of type I_1 in Kodaira's classification of singular fibers (see Appendix A). By Ehresmann's lemma, the map $M^* = M \setminus f^{-1}(0) \to \Delta^*$ is a fiber bundle. It follows that $R^1 f_* \mathbb{C}_{M^*}$ is a complex local system on Δ^*, where $\mathbb{C}_{M^*}$ denotes the constant sheaf with value $\mathbb{C}$ on M^*. Let $z \in \Delta^*$ be a point. By the proper base change theorem, the stalk $(R^1 f_* \mathbb{C}_{M^*})_z$ is the first singular cohomology group $H^1(f^{-1}(z),\mathbb{C})$ of the fiber $f^{-1}(z)$. The generator $\gamma \in \pi_1(\Delta^*, z)$ corresponding to a loop going around 0 counterclockwise once acts on $H^1(f^{-1}(z),\mathbb{C})$ via the matrix

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

after picking an appropriate basis of $H^1(f^{-1}(z),\mathbb{C}) \simeq \mathbb{C} \oplus \mathbb{C}$. See [Ach22].

Motivated by the topological situation, we desire an equivalence between ℓ-adic local systems and finite dimensional continuous $\overline{\mathbb{Q}}_\ell$-linear representations of the étale fundamental group $\pi_1^{\text{ét}}(X,\overline{x})$ based at a geometric point $\overline{x} \to X$ when X is connected. This equivalence is established in Section 2.4.

2.1. The category of π-adic sheaves.

Let $E/\mathbb{Q}_\ell$ be an ℓ-adic field with ring of integers $\mathcal{O}_E$, and let π be a uniformizer for $\mathcal{O}_E$.

Definition 2.2. A π-adic, or $\mathcal{O}_E$-, sheaf on X is a projective system $\mathscr{F} = (\mathscr{F}_n)_{n\geq 1}$ of sheaves on $X_{\text{ét}}$, where $\mathscr{F}_n$ is a constructible sheaf of $\mathcal{O}_E/\pi^n\mathcal{O}_E$-modules, and the transition map $\mathscr{F}_{n+1} \to \mathscr{F}_n$ induces an isomorphism

$$\mathscr{F}_{n+1} \otimes_{\mathcal{O}_E/\pi^{n+1}\mathcal{O}_E} \mathcal{O}_E/\pi^n\mathcal{O}_E \xrightarrow{\simeq} \mathscr{F}_n$$

for all $n \geq 1$.

For the notion of *constructibility* of sheaves on the étale site of a scheme, see [Sta24, Tag 05BE].

Example 2.3. (i) The projective system $\mathcal{O}_{E,X} = ((\mathcal{O}_E/\pi^n\mathcal{O}_E)_X)$ is a π-adic sheaf, where $(\mathcal{O}_E/\pi^n\mathcal{O}_E)_X$ denotes the constant sheaf with value $\mathcal{O}_E/\pi^n\mathcal{O}_E$ on $X_{\text{ét}}$. By abuse of notation, we will often just denote it by $\mathcal{O}_E$.

(ii) For $n \geq 1$, let

$$\mu_{\ell^n,X} = \ker \mathcal{O}_X^* \xrightarrow{\cdot \ell^n} \mathcal{O}_X^*.$$

Then $\mu_{\ell^n,X}$ is naturally a sheaf of finite $\mathbb{Z}/\ell^n\mathbb{Z}$-modules. It is represented by the finite étale cover $\underline{\operatorname{Spec}}\,\mathcal{O}_X[t]/(t^{\ell^n}-1) \to X$, and hence it is locally constant and constructible. We define the $\mathbb{Z}_\ell$-sheaf $\mathbb{Z}_\ell(1)$ to be the projective system $\mathbb{Z}_\ell(1) = (\mu_{\ell^n,X})$. For $m \geq 1$ we define $\mathbb{Z}_\ell(m) = (\mu_{\ell^n,X}^{\otimes m})$.

We say that a π-adic sheaf $\mathcal{F} = (\mathcal{F}_n)$ is *lisse* if each of the $\mathcal{F}_n$ is locally constant. In particular, the sheaves $\mathcal{O}_E$ and $\mathbb{Z}_\ell(m)$ from the examples above are lisse.

If $\mathcal{F} = (\mathcal{F}_n)$ and $\mathcal{G} = (\mathcal{G}_n)$ are π-adic sheaves, then a *morphism of π-adic sheaves $\mathcal{F} \to \mathcal{G}$* is a morphism $\varphi = (\varphi_n \colon \mathcal{F}_n \to \mathcal{G}_n)$ of projective systems, where each φ_n is a morphism of sheaves of $\mathcal{O}_E/\pi^n\mathcal{O}_E$-modules.

Example 2.4. Multiplication by π^n defines a morphism $\pi^n \colon \mathcal{F} \to \mathcal{F}$ of π-adic sheaves.

Proposition 2.5. *The category of π-adic sheaves is abelian.*

Proof. See [Sta24, Tag 03UO]. ■

Given a π-adic sheaf $\mathcal{F} = (\mathcal{F}_n)$ on X, and a geometric point $\overline{x} \to X$ of X, we define the *stalk of $\mathcal{F}$ at $\overline{x}$* to be

$$\mathcal{F}_{\overline{x}} = \varprojlim_{n \geq 1} \mathcal{F}_{n,\overline{x}}.$$

It is a module over $\mathcal{O}_E$. Additionally, it is finitely generated over $\mathcal{O}_E$; see [Sta24, Tag 03UQ]. Clearly a morphism of π-adic sheaves $\mathcal{F} \to \mathcal{G}$ induces a morphism between stalks $\mathcal{F}_{\overline{x}} \to \mathcal{G}_{\overline{x}}$. As for ordinary sheaves on the étale site, a sequence

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

of sheaves of $\mathcal{O}_E$-modules is exact if and only if the sequence of stalks

$$0 \to \mathcal{F}_{\overline{x}} \to \mathcal{G}_{\overline{x}} \to \mathcal{H}_{\overline{x}} \to 0$$

is exact for all geometric points $\overline{x} \to X$; see [Fu11, Proposition 10.1.17].

A π-adic sheaf is said to be *torsion* if all its stalks $\mathcal{F}_{\overline{x}}$ are torsion. Equivalently, $\pi^n \colon \mathcal{F} \to \mathcal{F}$ is zero for some $n \geq 1$, because X is noetherian.

Example 2.6. Fix an integer $m \geq 0$ and let $\mathcal{F}_m$ be a constructible $\mathcal{O}_E/\pi^m\mathcal{O}_E$-sheaf. Viewing $\mathcal{F}_m$ as a constructible sheaf of modules over $\mathcal{O}_E$, we can define $\mathcal{F} = (\mathcal{F}_m \otimes_{\mathcal{O}_E} \mathcal{O}_E/\pi^n\mathcal{O}_E)_{n\geq 1}$. Then $\mathcal{F}$ is a π-adic sheaf by [Sta24, Tag 0GKB]. It is in addition torsion, because it is killed by π^m.

2.1.1. *Functoriality of π-adic sheaves.* Let $f\colon Y \to X$ be a morphism of schemes with Y noetherian. By [Sta24, Tag 095G], the pullback of a constructible sheaf of modules over a ring $\mathscr{F}$ on $X_{\text{ét}}$ to $Y_{\text{ét}}$ along f is again constructible. This allows us to pull back a π-adic sheaf $\mathscr{F} = (\mathscr{F}_n)$ on X to a π-adic sheaf $f^{-1}\mathscr{F} = (f^{-1}\mathscr{F}_n)$ on Y. Here $f^{-1}\mathscr{F}_n \in \mathscr{O}_E/\pi^n\mathscr{O}_E(Y_{\text{ét}})$ denotes the usual inverse image sheaf (see [Sta24, Tag 00WX]). It is constructible by [Sta24, Tag 095G]. Notice also that the inverse image of a lisse sheaf is again lisse.

The assignment $\mathscr{F} \mapsto f^{-1}\mathscr{F}$ is of course functorial. We will often write $\mathscr{F}|_Y$ for the pullback $f^{-1}\mathscr{F}$ if no confusion can arise.

We now concern ourselves with pushforwards of lisse sheaves.

Proposition 2.7. *If $f\colon Y \to X$ is smooth and proper, then the higher derived image $R^i f_*\mathscr{F}$ of a locally constant constructible sheaf $\mathscr{F}$ on $Y_{\text{ét}}$ with torsion orders invertible on S is locally constant and constructible.*

In the proposition above, $R^i f_*\mathscr{F} \in \mathrm{Ab}(X_{\text{ét}})$ denotes the i-th higher direct image of $\mathscr{F}$ under f. See [Sta24, Tag 03PV].

Proof. This is [FK13, Theorem 8.9]. ■

In particular, since we assume ℓ to be invertible on X, the above proposition allows us to push forward a lisse π-adic sheaf $\mathscr{F} = (\mathscr{F}_n)$ from Y to a lisse π-adic sheaf $R^i f_*\mathscr{F} = (R^i f_*\mathscr{F}_n)$ on Y in a natural way, provided that f is smooth and proper.

2.2. **The category of E-sheaves.** By taking the quotient of the category of π-adic sheaves by the Serre subcategory (see [Wei13, Exercise 10.3.2]) of torsion π-adic sheaves, we obtain the category $M(X,E)$ of *E-sheaves* on X. It comes equipped with an exact functor from the category of π-adic sheaves. If $\mathscr{F}$ is a π-adic sheaf, then we denote its image in $M(X,E)$ by $\mathscr{F} \otimes_{\mathscr{O}_E} E$, or simply $\mathscr{F} \otimes E$.

We describe the category $M(X,E)$ more concretely. The objects of $M(X,E)$ are just π-adic sheaves. Morphisms in $M(X,E)$ are defined by

$$\mathrm{Hom}(\mathscr{F} \otimes E, \mathscr{G} \otimes E) = \mathrm{Hom}(\mathscr{F}, \mathscr{G}) \otimes_{\mathscr{O}_E} E.$$

If $\mathscr{F}, \mathscr{G}$ and $\mathscr{H}$ are π-adic sheaves, then the composition map

$$\mathrm{Hom}(\mathscr{F}, \mathscr{G}) \otimes_{\mathscr{O}_E} \mathrm{Hom}(\mathscr{G}, \mathscr{H}) \to \mathrm{Hom}(\mathscr{F}, \mathscr{H})$$

in the category of π-adic sheaves naturally induces the composition map

$$\mathrm{Hom}(\mathscr{F} \otimes E, \mathscr{G} \otimes E) \otimes_E \mathrm{Hom}(\mathscr{G} \otimes E, \mathscr{H} \otimes E) \to \mathrm{Hom}(\mathscr{F} \otimes E, \mathscr{H} \otimes E)$$

in $M(X,E)$.

If $\mathscr{F}$ is a π-adic sheaf, and $\overline{x} \to X$ is a geometric point of X, then the stalk of the E-sheaf $\mathscr{F} \otimes E$ at $\overline{x}$ is defined to be

$$(\mathscr{F} \otimes E)_{\overline{x}} = \mathscr{F}_{\overline{x}} \otimes E.$$

A morphism $\mathscr{F}\otimes E \to \mathscr{G}\otimes E$ of E-sheaves gives rise to an E-linear map of stalks $(\mathscr{F}\otimes E)_{\overline{x}} \to (\mathscr{G}\otimes E)_{\overline{x}}$ via

$$\begin{aligned}\mathrm{Hom}(\mathscr{F}\otimes E,\mathscr{G}\otimes E) &= \mathrm{Hom}(\mathscr{F},\mathscr{G})\otimes E\\ &\to \mathrm{Hom}(\mathscr{F}_{\overline{x}},\mathscr{G}_{\overline{x}})\otimes E\\ &\to \mathrm{Hom}(\mathscr{F}_{\overline{x}}\otimes E,\mathscr{G}_{\overline{x}}\otimes E).\end{aligned}$$

Example 2.8. We define the constant E-sheaf $E_X = \mathscr{O}_{E,X}\otimes E$, where $\mathscr{O}_{E,X}$ denotes the π-adic sheaf from Example 2.3. By abuse of notation we also denote E_X simply by E.

Example 2.9. Recall the definition of $\mathbb{Z}_\ell(m)$ from Example 2.3 (ii). We define

$$\mathbb{Q}_\ell(m) = \mathbb{Z}_\ell(m)\otimes\mathbb{Q}_\ell.$$

2.2.1. *Functoriality of E-sheaves.* If $\mathscr{F}$ is a torsion π-adic sheaf on X and $f\colon Y\to X$ is a morphism with Y noetherian, then the pullback $f^{-1}\mathscr{F}$ – which we constructed in 2.1.1 – is again a torsion sheaf. As a result, the functor f^{-1} from π-adic sheaves on X to π-adic sheaves on Y induces a functor

$$f^{-1}\colon M(X,E)\to M(Y,E). \tag{2.2}$$

An E-sheaf is said to be *lisse* if it is represented by a lisse π-adic sheaf. Lisse E-sheaves on X are also referred to as *local systems on X with coefficients in E*. We denote the full subcategory of $M(X,E)$ consisting of local systems on X with coefficients in E by $\mathrm{Loc}_E(X)$. If $f\colon Y\to X$ is smooth and proper, then we also obtain a functor

$$R^if_*\colon \mathrm{Loc}_E(Y)\to\mathrm{Loc}_E(X). \tag{2.3}$$

Notice that if $\mathscr{F}$ is a local system on X, and X is connected, then the dimension of $\mathscr{F}_{\overline{x}}$ as an E-vector space is independent of the chosen geometric point $\overline{x}$. It is referred to as the *rank* of the local system $\mathscr{F}$.

2.3. **The category of $\overline{\mathbb{Q}}_\ell$-sheaves.** Let $E\subset E'$ be an extension of ℓ-adic sheaves. If $\mathscr{F} = (\mathscr{F}_n)$ denotes an $\mathscr{O}_E$-sheaf, then $\mathscr{F}\otimes\mathscr{O}_{E'} = (\mathscr{F}_n\otimes_{\mathscr{O}_E/\mathfrak{m}_E^n}\mathscr{O}_{E'}/\mathfrak{m}_{E'}^n)$ is an $\mathscr{O}_{E'}$-sheaf. This gives an exact functor from $\mathscr{O}_E$-sheaves to $\mathscr{O}_{E'}$-sheaves that sends torsion sheaves to torsion sheaves. Hence, we obtain an induced functor

$$M(X,E)\to M(X,E')$$

on Serre quotients. Let $\overline{\mathbb{Q}}_\ell$ be an algebraic closure of $\mathbb{Q}_\ell$. We define the category of $\overline{\mathbb{Q}}_\ell$-sheaves as the direct limit[2]

$$M(X,\overline{\mathbb{Q}}_\ell) = \varinjlim M(X,E)$$

taken over all ℓ-adic fields $E\subset\overline{\mathbb{Q}}_\ell$. An object in $M(X,\overline{\mathbb{Q}}_\ell)$ is an E-sheaf $\mathscr{F}$ over some ℓ-adic field $E\subset\overline{\mathbb{Q}}_\ell$. We write $\mathscr{F}\otimes\overline{\mathbb{Q}}_\ell$ for its image in $M(X,\overline{\mathbb{Q}}_\ell)$. If $\mathscr{F}$, respectively $\mathscr{G}$, is an

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[2] Strictly speaking, this is a *2-colimit* of categories.

E-sheaf, respectively an E'-sheaf, then we can find $F \subset \mathbb{Q}_\ell$ an ℓ-adic field containing E and E', and we have

$$\mathrm{Hom}_{\overline{\mathbb{Q}}_\ell}(\mathscr{F} \otimes \overline{\mathbb{Q}}_\ell, \mathscr{G} \otimes \overline{\mathbb{Q}}_\ell) = \mathrm{Hom}_F(\mathscr{F} \otimes_E F, \mathscr{G} \otimes_{E'} F) \otimes_F \overline{\mathbb{Q}}_\ell.$$

Given a $\overline{\mathbb{Q}}_\ell$-sheaf on X, represented by an E-sheaf $\mathscr{F}$, its stalk at a geometric point $\overline{x} \to X$ is defined to be

$$(\mathscr{F} \otimes \overline{\mathbb{Q}}_\ell)_{\overline{x}} = \mathscr{F}_{\overline{x}} \otimes_E \overline{\mathbb{Q}}_\ell.$$

Assigning a $\overline{\mathbb{Q}}_\ell$ sheaf to its stalk at $\overline{x}$ is of course again functorial.

2.3.1. *Functoriality of* $\overline{\mathbb{Q}}_\ell$*-sheaves.* By taking a direct limit, the functor from (2.2) yields a functor

$$f^{-1}\colon M(X, \overline{\mathbb{Q}}_\ell) \to M(Y, \overline{\mathbb{Q}}_\ell). \tag{2.4}$$

A $\overline{\mathbb{Q}}_\ell$-sheaf is said to be lisse if it is represented by a lisse E-sheaf. Lisse $\overline{\mathbb{Q}}_\ell$-sheaves are also referred to as *local systems on X with coefficients in* $\overline{\mathbb{Q}}_\ell$, or simply *($\ell$-adic) local systems.* The full subcategory of $M(X, \overline{\mathbb{Q}}_\ell)$ consisting of local systems is denoted by $\mathrm{Loc}_{\overline{\mathbb{Q}}_\ell}(X)$, or simply $\mathrm{Loc}(X)$. If $f\colon Y \to X$ is smooth and proper with Y noetherian, then from (2.3) we obtain a functor

$$R^i f_*\colon \mathrm{Loc}(Y) \to \mathrm{Loc}(X). \tag{2.5}$$

On a connected scheme, as for local systems with coefficients in E, the *rank* of an ℓ-adic local system is well-defined.

Example 2.10 (Local systems coming from geometry)**.** Let $f\colon Y \to X$ be a smooth and proper morphism with Y noetherian. By abuse of notation, we denote by $\overline{\mathbb{Q}}_\ell = \overline{\mathbb{Q}}_{\ell,X}$ the ℓ-adic local system $\mathbb{Q}_{\ell,X} \otimes \overline{\mathbb{Q}}_\ell$ on X. Then $R^i f_* \overline{\mathbb{Q}}_\ell$ is a local system on X by the discussion above with, by proper base change, stalks

$$(R^i f_* \overline{\mathbb{Q}}_\ell)_{\overline{x}} = H^i_{\text{ét}}(Y \times_X \operatorname{Spec} \kappa(x)^{\text{sep}}; \overline{\mathbb{Q}}_\ell),$$

where $\kappa(x)^{\text{sep}}$ denotes the separable closure of $\kappa(x)$ in $\kappa(\overline{x})$. Here, we define the i-th étale cohomology group of a $\overline{\mathbb{Q}}_\ell$-sheaf $\mathscr{F}$, represented by an $\mathscr{O}_E$-sheaf $\mathscr{F}_{\mathscr{O}_E} = (\mathscr{F}_{\mathscr{O}_E,n})$, on a scheme S to be

$$H^i_{\text{ét}}(S; \mathscr{F}) = \varprojlim_n H^i_{\text{ét}}(S; \mathscr{F}_{\mathscr{O}_E,n}) \otimes_{\mathscr{O}_E} \overline{\mathbb{Q}}_\ell.$$

The local system $R^i f_* \overline{\mathbb{Q}}_\ell$ is an example of a local system *coming from geometry.*

More generally, we say that a local system $\mathscr{F}$ on X *comes from geometry* if there exists a dense open subscheme $U \subset X$ and a smooth proper morphism $f\colon Y \to U$ such that $\mathscr{F}|_U$ is a subquotient of $R^i f_* \overline{\mathbb{Q}}_\ell$, for some $i \geq 0$.

2.4. **Monodromy representations.** Recall that a locally constant constructible (i.e., with finite stalks) sheaf of sets $\mathscr{F}$ on $X_{\text{ét}}$ is represented by a finite étale cover $Y \to X$ by fpqc descent. In addition, this étale cover is unique by Yoneda's lemma. For a given geometric point $\overline{x} \to X$, the fiber $Y \times_X \overline{x} \to \overline{x}$ is precisely the stalk $\mathscr{F}_{\overline{x}}$, and so by the general theory of the étale fundamental group we obtain a continuous action of $\pi_1(X,\overline{x})$ on $\mathscr{F}_{\overline{x}}$. In case X is connected, this gives us an equivalence of categories between locally constant constructible sheaves of sets on $X_{\text{ét}}$ and finite sets with a continuous $\pi_1(X,\overline{x})$-action. Suppose now that $\mathscr{F}$ additionally caries the structure of a sheaf of R-modules over some ring R; then we even obtain a continuous action

$$\pi_1(X,\overline{x}) \to \operatorname{Aut}_R(\mathscr{F}_{\overline{x}})$$

of $\pi_1(X,\overline{x})$ on $\mathscr{F}_{\overline{x}}$ compatible with the structure of $\mathscr{F}_{\overline{x}}$ as an R-module.

Now let $\mathscr{F} = (\mathscr{F}_n)$ be a lisse $\mathscr{O}_E$-sheaf, and let $\overline{x} \to X$ denote a geometric point of X. For every $n \geq 1$, we obtain a continous homomorphism

$$\pi_1(X,\overline{x}) \to \operatorname{Aut}_{\mathscr{O}_E/\pi^n\mathscr{O}_E}(\mathscr{F}_{n,\overline{x}})$$

by the paragraph above. These are compatible since $\mathscr{F}_n = \mathscr{F}_{n+1}/\pi^n\mathscr{F}_{n+1}$. So we obtain a continuous homomorphism

$$\pi_1(X,\overline{x}) \to \varprojlim_n \operatorname{Aut}_{\mathscr{O}_E/\pi^n\mathscr{O}_E}(\mathscr{F}_{n,\overline{x}}) \simeq \operatorname{Aut}_{\mathscr{O}_E}(\mathscr{F}_{\overline{x}}),$$

where $\operatorname{Aut}_{\mathscr{O}_E}(\mathscr{F}_{\overline{x}})$ is equipped with the inverse limit topology. The proof of [Fu11, Theorem 10.23] shows that then also the induced map

$$\pi_1(X,\overline{x}) \to \operatorname{GL}(\mathscr{F}_{\overline{x}} \otimes E)$$

is continuous. We refer to it as the *monodromy representation* associated to $\mathscr{F} \otimes E$. We obtain a functor

$$\begin{aligned} \operatorname{Loc}_E(X) &\to \operatorname{Rep}_E(\pi_1(X,\overline{x})) \\ \mathscr{F} &\mapsto \mathscr{F}_{\overline{x}}, \end{aligned} \tag{2.6}$$

from $\operatorname{Loc}_E(X)$ to $\operatorname{Rep}_E(\pi_1(X,\overline{x}))$, the category of continuous finite-dimensional E-linear representations of $\pi_1(X,\overline{x})$.

Proposition 2.11. *If X is in addition connected, then the functor* (2.6) *is an equivalence of categories.*

Proof. This is [Fu11, Theorem 10.23]. ■

Suppose that G denotes a profinite group and $\rho\colon G \to \operatorname{GL}(V)$ is a representation of G, where V is a finite dimensional $\overline{\mathbb{Q}}_\ell$-vector space. Then ρ is said to be *continuous* if there exists an E-linear subspace $W \subset V$ for some ℓ-adic field $E \subset \overline{\mathbb{Q}}_\ell$, with a continuous E-linear action of G on W, such that $V \simeq W \otimes_E \overline{\mathbb{Q}}_\ell$ as G-representations.

Remark 2.12. One can show, using the Baire category theorem, that $\rho\colon G \to \mathrm{GL}(V)$ is a continuous $\overline{\mathbb{Q}}_\ell$-linear representation if and only if ρ is continous with respect to the ℓ-adic topology on $\mathrm{GL}(V)$. See [Chu18, Proposition 4.3].

Taking the limit over the functors of (2.6), we obtain a functor

$$\mathrm{Loc}(X) \to \mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(\pi_1(X,\overline{x})), \tag{2.7}$$

where $\mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(\pi_1(X,\overline{x}))$ denotes the category of *ℓ-adic representations* of $\pi_1(X,\overline{x})$.

Terminology 2.13. In this thesis, an *ℓ-adic representation* of a profinite group G is always supposed to be a $\overline{\mathbb{Q}}_\ell$-linear continuous finite-dimensional representation of G.

Corollary 2.14. *If X is in addition connected, then the functor of* (2.7) *is an equivalence of categories.* ■

Example 2.15. We consider again the setting of Example 2.10. We obtain a monodromy representation

$$\pi_1(X,\overline{x}) \to \mathrm{GL}(H^i_{\text{ét}}(Y \times_X \operatorname{Spec}\kappa(x)^{\text{sep}};\overline{\mathbb{Q}}_\ell)).$$

Let $\varphi\colon Y \to X$ be a morphism of schemes with Y noetherian. If $\mathscr{F}$ denotes a local system on X, then we obtain a local system $\varphi^{-1}\mathscr{F}$ on Y from (2.4). If $\rho\colon \pi_1(Y,\overline{y}) \to \mathrm{GL}(\mathscr{F}_{\overline{x}})$ denotes the monodromy representation of $\mathscr{F}$, where $\overline{y} \to Y$ is a geometric point lying over $\overline{x} \to X$, then the monodromy representation of $\varphi^{-1}\mathscr{F}$ is given by the composition

$$\pi_1(Y,\overline{y}) \xrightarrow{\varphi_*} \pi_1(X,\overline{x}) \xrightarrow{\rho} \mathrm{GL}(\mathscr{F}_{\overline{x}}),$$

where we identify the fibers $(\varphi^{-1}\mathscr{F})_{\overline{y}} = \mathscr{F}_{\overline{x}}$.

Terminology 2.16. In this thesis, unless otherwise specified, a local system is only considered up to isomorphism. We will then usually denote them by $\mathbb{L}$. When $\mathbb{L}$ is a local system on a connected scheme, we will often say that $\mathbb{L}$ "has property P" if its monodromy representation $\rho\colon \pi_1(X,\overline{x}) \to \mathrm{GL}(\mathbb{L}_{\overline{x}})$ "has property P". Of course, we only do this when this does not depend on the choice of geometric point $\overline{x}$. For instance, we will say that $\mathbb{L}$ *has finite determinant* if its monodromy representation has finite determinant, i.e., when the determinant character $\det\rho = \det\circ\rho\colon \pi_1(X,\overline{x}) \to \overline{\mathbb{Q}}_\ell^\times$ has finite image.

2.4.1. *(Semi-)simplicity.* If $\mathscr{F}$ is a local system on X, then $\mathscr{F}$ is said to be *simple*, or *irreducible*, if $\mathscr{F}$ has no non-trivial and non-zero lisse subobjects in the category of $\overline{\mathbb{Q}}_\ell$-sheaves. If $\mathscr{F}$ is a finite direct sum of simple lisse sheaves, then $\mathscr{F}$ is said to be *semisimple*. *Crucially*, these notions makes sense even when X is not connected. If X does happen to be connected, then by identifying lisse sheaves with their monodromy representations via the equivalence of Corollary 2.14, these notions coincide with the usual notions from representation theory. Hence, this terminology is also compatible with our conventions from Terminology 2.16.

Example 2.17. (i) The constant local system $\overline{\mathbb{Q}}_\ell$ on X is semisimple. It is irreducible if X is additionally connected.

(ii) Assume that X is in addition normal, connected and defined over an algebraically closed field k. Let $f\colon Y \to X$ be smooth and proper. We then know $\mathscr{F} = R^i f_* \overline{\mathbb{Q}}_\ell$ to be a local system by Example 2.10. By [Del80, Corollaire 3.4.13], $\mathscr{F}$ is semisimple. We also see that local systems coming from geometry are exactly those local systems $\mathscr{F}$ such that $\mathscr{F}|_U$ is a *direct summand* of $R^i f_* \overline{\mathbb{Q}}_\ell$, for some $i \geq 0$, some dense open subscheme U and some smooth and proper morphism $f\colon Y \to U$.

Lemma 2.18. *Let G be a profinite group and let $U \subset G$ be an open subgroup of G. For a given ℓ-adic representation $\rho\colon G \to \mathrm{GL}_r(\overline{\mathbb{Q}}_\ell)$, ρ is semisimple if and only if $\rho\,|_U$ is semisimple.*

Proof. The "only if" statement follows from an averaging argument akin to the one in the proof of Maschke's Theorem. For the "if" statement, let $V \subset U$ be an open normal subgroup of G. Then $\rho\,|_V$ is semisimple by Clifford's Theorem; see [Cli37]. The fact that $\rho\,|_U$ is semisimple then follows from the "only if" part of the lemma. ∎

Proposition 2.19. *Let $Y \to X$ be a surjective finite étale cover of X, and let $\mathbb{L}$ be a local system on X. Then $\mathbb{L}$ is semisimple if and only if the pullback $\mathbb{L}|_Y$ of $\mathbb{L}$ to Y is semisimple.*

Proof. If X and Y are connected, this follows from the previous lemma.

It is clear that a local system is semisimple if and only if its restriction to every connected component is semisimple, and so the general case reduces to the case where both X and Y are connected. ∎

3. SOME FACTS ABOUT FUNDAMENTAL GROUPS

The material in this section is based entirely on [GR06, Exposé XIII].

3.1. **The tame fundamental group.** Let $X \to S$ be a proper scheme of finite presentation over a scheme S with geometrically connected fibers. Let $D \rightarrowtail X$ be an effective Cartier divisor such that $D \rightarrowtail X \to S$ is smooth, and such that the suport of D lies in the smooth locus of $X \to S$. Denote by U the complement of $\operatorname{Supp} D$ in X, and let $\overline{x} \to U$ be a geometric point of U. Raynaud defines a notion of étale coverings of U, *tamely ramified along* D*, relative to* S in [GR06, Exposé XIII]. These tamely ramified coverings form a Galois category $\mathrm{Fib}^t_{\overline{x}} \colon \mathrm{Fét}^t_U \to \text{sets}$ with fiber functor the usual fiber functor associated with $\overline{x}$ restricted to the full subcategory of tame coverings. The automorphism group of the fiber functor $\mathrm{Fib}^t_{\overline{x}}$ is defined to be the *tame fundamental group* $\pi^t_1(U,\overline{x})$.

3.1.1. *Functoriality of the tame fundamental group.* The construction of the tame fundamental group is also functorial in the following sense. If $S' \to S$ denotes a morphism of schemes, write U', respectively D', for the pullback of U, respectively D, along $S' \to S$. Now, if $Y \to U$ denotes an étale cover, tamely ramified along D, then its pullback $Y' \to U'$ along $U' \to U$ is an étale cover, tamely ramified along D'. It follows that if $\overline{x}' \to U'$ denotes a geometric point of U', then we obtain a homomorphism of tame fundamental groups

$$\pi^t_1(U',\overline{x}') \to \pi^t_1(U,\overline{x}').$$

3.1.2. *The maximal pro-$\mathscr{L}$ quotient.* We will not get into the details of these constructions, but what is important to remark is that if $\mathscr{L}$ denotes the set of primes that do *not* occur as a residue characteristic of S, and if $Y \to U$ denotes an étale Galois cover whose degree is a product of primes in $\mathscr{L}$, then $Y \to U$ is tamely ramified. As a result, there is a surjective map of profinite groups

$$\pi^t_1(U,\overline{x}) \twoheadrightarrow \pi^{\mathscr{L}}_1(U,\overline{x}),$$

where $\pi^{\mathscr{L}}_1(U,\overline{x})$ denotes the *maximal pro-$\mathscr{L}$ quotient* of $\pi_1(U,\overline{x})$, which is also the maximal pro-$\mathscr{L}$ quotient of $\pi^t_1(U,\overline{x})$. If G is a profinite group, and $\mathscr{L}$ is any set of primes, then the maximal pro-$\mathscr{L}$ quotient of G is defined to be

$$G^{\mathscr{L}} = \varprojlim_U G/U, \tag{3.1}$$

where the projective limit runs over the open normal subgroups $U \subset G$ such that the index $[G:U]$ is a product of primes in $\mathscr{L}$. If only one prime p is not contained in $\mathscr{L}$, then $G^{\mathscr{L}}$ is usually denoted $G^{(p')}$ and called the *maximal prime-to-p quotient* of G.

3.2. **The specialization homomorphism.** Notation is as before. Let $\eta \in S$ and $s \in S$ be points of S such that there is a specialization of points $\eta \rightsquigarrow s$. Let $\overline{\eta} \to S$, respectively $\overline{s} \to S$, be a geometric point of S lying over η, respectively s. Let $\overline{x} \to U_{\overline{\eta}}$, respectively $\overline{y} \to U_{\overline{s}}$, be a geometric point of $U_{\overline{s}}$, respectively $U_{\overline{\eta}}$. Denote by $A = \mathscr{O}^{\mathrm{hs}}_{S,\overline{s}}$ the strict henselization of S at $\overline{s}$.

Proposition 3.1. *The homomorphism of tame fundamental groups*

$$\pi_1^t(U_{\overline{s}}, \overline{y}) \to \pi_1^t(U_A, \overline{y})$$

is an isomorphism. Here tameness is with respect to $D_{\overline{s}}$ and D_A.

Proof. This is [GR06, Exposé XIII, 2.10]. ■

By the fact that we have a specialization $\eta \rightsquigarrow s$, η defines a point of $\operatorname{Spec}\mathcal{O}_{S,s}$. Since $\operatorname{Spec} A \to \operatorname{Spec}\mathcal{O}_{S,s}$ is surjective, by the fact that $\mathcal{O}_{S,s} \to A$ is faithfully flat, we can lift η to a point of $\operatorname{Spec} A$ with residue field a separable extension of $\kappa(\eta)$. Because $\kappa(\overline{\eta})$ is separably closed, we can choose a morphism $\overline{\eta} \to \operatorname{Spec} A$ that gives rise to a commutative diagram

$$\begin{tikzcd}
\overline{\eta} \arrow[r] \arrow[dr] & \operatorname{Spec} A \arrow[d] & \overline{s} \arrow[l] \arrow[dl] \\
& S. &
\end{tikzcd}$$

We obtain the *specialization map*

$$\mathrm{sp}\colon \pi_1^t(U_{\overline{\eta}}, \overline{x}) \to \pi_1^t(U_A, \overline{x}) \simeq \pi_1^t(U_A, \overline{y}) \xrightarrow{\simeq} \pi_1^t(U_{\overline{s}}, \overline{y}). \tag{3.2}$$

Here the final isomorphism is the inverse of the one from Proposition 3.1. The specialization map is only defined up to an inner automorphism of $\pi_1^t(U_{\overline{s}}, \overline{y})$ because of the choice of an isomorphism $\pi_1^t(U_A, \overline{x}) \simeq \pi_1^t(U_A, \overline{y})$. It also depends on the choice of morphism $\overline{\eta} \to \operatorname{Spec} A$.

Theorem 3.2. *If $X \to S$ is furthermore smooth, then the specialization map of* (3.2) *induces an isomorphism on maximal prime-to-p quotients*

$$\pi_1^{(p')}(U_{\overline{\eta}}, \overline{x}) \xrightarrow{\simeq} \pi_1^{(p')}(U_{\overline{s}}, \overline{y}),$$

where p denotes the residue characteristic of S at s.

Proof. This is by [GR06, Exposé XIII, Corollaire 2.12]. ■

3.3. **A homotopy exact sequence.** Let S be a connected scheme. Let $X \to S$ be a smooth proper S-scheme with geometrically connected fibers. Suppose $f\colon U \hookrightarrow Z \to S$ is the complement of an effective cartier divisor $D \rightarrowtail X$ that is smooth over S and has support in the smooth locus of $X \to S$. Assume that $f\colon U \to S$ admits a section. Let $\mathscr{L}$ be the set of primes different from the residue characteristics of S.

Let $\overline{x} \to U$ be a geometric point of U. Write also $\overline{x}$ for the induced geometric point of S. Denote by K the kernel

$$K = \ker \pi_1(U, \overline{x}) \to \pi_1(S, \overline{x}),$$

and let N be the smallest normal subgroup of K such that K/N is a pro-$\mathscr{L}$ group. Then N is also normal in $\pi_1(U, \overline{x})$ and we define

$$\pi_1'(U, \overline{x}) = \pi_1(U, \overline{x})/N.$$

Let $\overline{s} \to S$ be a geometric point of S. Denote by $\overline{x}$ the geometric point of the geometric fiber $U_{\overline{s}}$ induced by $\overline{s} \to S \to U$. We obtain a sequence of homomorphisms

(3.3)
$$\begin{tikzcd}
1 \arrow[r] & \pi_1^{\mathscr{L}}(U_{\overline{s}}, \overline{x}) \arrow[r] & \pi_1'(U, \overline{x}) \arrow[r] & \pi_1(S, \overline{s}) \arrow[r] \arrow[l, dashed, bend right] & 1.
\end{tikzcd}$$

It is not hard to check that the composition $\pi_1^{\mathscr{L}}(U_{\overline{s}}, \overline{x}) \to \pi_1'(U, \overline{x}) \to \pi_1(S, \overline{s})$ is the trivial homomorphism.

Proposition 3.3. *Under the hypotheses above, the sequence in* (3.3) *constructed above is a split exact sequence.*

Proof. This is [GR06, Proposition 4.3] and [GR06, Exemple 4.4]. ■

Remark 3.4. Throughout, we have restricted ourselves to the case where D is smooth over the base S. We can generalize the results in this section by weakening this assumption to the assumption that D is a *normal crossing divisor, relative to* S, as defined in [GR06, Exposé XIII, 2.1].

4. A COUNTEREXAMPLE TO THE NAIVE LOCAL KASHIWARA CONJECTURE

Let R be a complete discrete valuation ring with algebraically closed residue field and write $S = \operatorname{Spec} R$. Denote the closed point of S by s and the generic point of S by η. Denote the residue field of R by k and let $p \geq 0$ be its characteristic. Throughout, ℓ denotes a prime different from p. The goal of this section is to give a counterexample to the naive local Kashiwara conjecture discussed in the introduction that works without any further restrictions on R. Specifically, we will answer the following question in the negative.

Question 4.1. Let $\mathscr{X} \to S$ be a surjective smooth separated quasi-compact morphism of schemes. Let $\mathbb{L}$ be a semi-simple ℓ-adic local system on $\mathscr{X}$. Is it true that the pullback of $\mathbb{L}$ to a local system on $\mathscr{X}_s$ is semi-simple?

Our construction in Section 4.2 to answer this question mimics the construction in Section A in many respects. In particular, it involves an elliptic fibration whose special fiber is a nodal curve. We proceed by first working out some facts regarding the fundamental group of this particular nodal curve.

4.1. **The fundamental group of a nodal curve.**

Throughout this section, k is any algebraically closed field of characteristic $p \geq 0$. We let C be a proper rational curve over k that is normal at every point with the exception of a *simple node* $n \in C(k)$. By this we mean that C looks étale locally around n like the origin of $\operatorname{Spec} k[x,y]/(xy)$. In other words, there is an isomorphism of local k-algebras $\mathscr{O}^{\mathrm{h}}_{C,n} \simeq (k[x,y]/(xy))^{\mathrm{h}}_{(x,y)}$, where $^{\mathrm{h}}$ denotes the henselization. Let $\pi \colon \mathbb{P}^1_k \to C$ be the normalization of C. There are *exactly* two points lying over n. This can be seen by first pulling back π along $\operatorname{Spec}\mathscr{O}^{\mathrm{h}}_{C,n} \to C$ and the fact that this pullback is the normalization of $\operatorname{Spec}\mathscr{O}^{\mathrm{h}}_{C,n}$ by [Sta24, Tag 0CBM]; then notice that the normalization of $\operatorname{Spec}\mathscr{O}^{\mathrm{h}}_{C,n} \simeq \operatorname{Spec}(k[x,y]/(xy))^{\mathrm{h}}_{(x,y)}$ is

$$\operatorname{Spec} k[x]^{\mathrm{h}}_{(x)} \sqcup \operatorname{Spec} k[y]^{\mathrm{h}}_{(y)} \to \operatorname{Spec}(k[x,y]/(xy))^{\mathrm{h}}_{(x,y)} \tag{4.1}$$

and that the fiber of this morphism over (x,y) contains exactly two points. Denote by a and b the points of $\mathbb{P}^1$ lying over the node n. There is the following "normalization sequence"

$$0 \to \mathscr{O}_C \to \pi_*\mathscr{O}_{\mathbb{P}^1} \to k_n \to 0, \tag{4.2}$$

where k_n is the skyscraper sheaf supported only at n with stalk k. Here the map $\pi_*\mathscr{O}_{\mathbb{P}^1} \to k_n$ is defined by $f \mapsto f(a) - f(b)$. It is a sequence of $\mathscr{O}_C$-modules.

Proposition 4.2. *The sequence* (4.2) *is exact.*

Proof. It is clear that for every point $c \in C$, the sequence of stalks at c is exact, except possibly at the node n. Consider the sequence of stalks at n

$$0 \to \mathscr{O}_{C,n} \to (\pi_*\mathscr{O}_{\mathbb{P}^1})_n \to k = \mathscr{O}_{C,n}/\mathfrak{m}_n \to 0.$$

Recall that the henselization $\mathscr{O}_{C,n} \to \mathscr{O}^{\mathrm{h}}_{C,n}$ is faithfully flat, and so we need only prove that the sequence we obtain after tensoring with $\mathscr{O}^{\mathrm{h}}_{C,n}$,

$$0 \to \mathscr{O}^{\mathrm{h}}_{C,n} \to (\pi_* \mathscr{O}_{\mathbb{P}^1})_n \otimes_{\mathscr{O}_{C,n}} \mathscr{O}^{\mathrm{h}}_{C,n} \to \mathscr{O}^{\mathrm{h}}_{C,n}/\mathfrak{m}_n \mathscr{O}^{\mathrm{h}}_{C,n} = k \to 0,$$

is exact. This sequence identifies with the sequence

$$0 \to (k[x,y]/(xy))^{\mathrm{h}}_{(x,y)} \to k[x]^{\mathrm{h}}_{(x)} \times k[y]^{\mathrm{h}}_{(y)} \to k \to 0 \tag{4.3}$$

by (4.1) and the argument preceeding it. The sequence (4.3) is clearly exact. ■

We can assume that neither a nor b is the point 0 or ∞. Denote by C° the curve C with the points $\pi(0)$ and $\pi(\infty)$ removed. We still denote the normalization $\mathbb{P}^1 \setminus \{0,\infty\} = \mathbb{G}_m \to C^\circ$ by π.

4.1.1. *Finite étale covers of the punctured nodal curve.* Denote the category of finite étale covers of C° by $\mathrm{Fét}_{C^\circ}$. Denote by $\mathscr{C}$ the category whose objects are finite étale covers $X \xrightarrow{f} \mathbb{G}_m$ equipped with an isomorphism $\varphi\colon f^{-1}(a) \xrightarrow{\simeq} f^{-1}(b)$ and whose morphisms $(f,\varphi) \to (f',\varphi')$ are morphisms of finite étale covers such that the square

$$\begin{tikzcd}
f^{-1}(a) \arrow[r] \arrow[d, "\varphi"] & f'^{-1}(b) \arrow[d, "\varphi'"] \\
f^{-1}(a) \arrow[r] & f'^{-1}(b)
\end{tikzcd}$$

commutes. The isomorphism φ is referred to as *a descent datum for f, relative to C°*. Given a finite étale cover $g\colon Y \to C^\circ$, we obtain a finite étale cover

$$\pi^* g\colon \pi^* Y := Y \times_{C^\circ} \mathbb{G}_m \to \mathbb{G}_m$$

by pulling the morphism g back along π. We have a canonical isomorphism

$$(\pi^* g)^{-1}(a) \simeq (\pi^* g)^{-1}(b)$$

and hence a canonical descent datum for $\pi^* g$. This gives us a functor

$$\pi^*\colon \mathrm{Fét}_{C^\circ} \to \mathscr{C}. \tag{4.4}$$

The rest of this section is devoted to constructing a pseudo-inverse to this functor.

4.1.2. *A glueing construction.* Given an object $(X \xrightarrow{f} \mathbb{G}_m, \varphi)$ of $\mathscr{C}$, we intend to construct a finite étale cover of C° by "glueing the fibers $f^{-1}(a)$ and $f^{-1}(b)$ together along the isomorphism φ". Define the ring $\mathscr{O}(\overline{X})$ by

$$\mathscr{O}(\overline{X}) = \{f \in \mathscr{O}(X) : f(x) = f(\varphi(x)) \text{ for all } x \in f^{-1}(a)\}.$$

It defines the coordinate ring of an affine scheme $\overline{X}$. Let $\rho\colon X \to \overline{X}$ denote the obvious map.

Proposition 4.3. *The square*

$$\begin{tikzcd}
f^{-1}(a) \sqcup f^{-1}(b) \arrow[r] \arrow[d, "\varphi \sqcup \mathrm{id}"] & X \arrow[d, "\rho"] \\
f^{-1}(b) \arrow[r] & \overline{X}
\end{tikzcd}$$

defines a pushout square in the category of affine k-schemes.

Proof. The corresponding map on rings defines a pullback square in the category of k-algebras. ■

By construction of the scheme $\overline{X}$, we also have an exact sequence of $\mathscr{O}_{\overline{X}}$-modules on $\overline{X}$

$$0 \to \mathscr{O}_{\overline{X}} \to \rho_* \mathscr{O}_X \to \bigoplus_{x \in f^{-1}(a)} k_{\rho(x)} \to 0, \tag{4.5}$$

where $\bigoplus_{x \in f^{-1}(a)} k_{\rho(x)}$ is the skyscraper sheaf on $\overline{X}$ supported at the points $\rho(x) = \rho(\varphi(x))$ with stalk k for $x \in f^{-1}(a)$. Letting $y = \varphi(x)$ and $z = \rho(x) = \rho(y)$, we obtain the exact sequence of stalks

$$0 \to \mathscr{O}_{\overline{X},z} \to (\rho_* \mathscr{O}_X)_z \to k \to 0.$$

Set $\mathscr{O}_{X,x \cup y} = (\rho_* \mathscr{O}_X)_z$. This is a semilocal ring with maximal ideals $\mathfrak{m}_x$ and $\mathfrak{m}_y$, corresponding to the points x and y. The ideal $\mathfrak{m}_z \mathscr{O}_{X,x \cup y}$ is precisely the Jacobson radical of $\mathscr{O}_{X,x \cup y}$. As a result, the $\mathscr{O}_{\overline{X},z}$-completion of $\mathscr{O}_{X,x \cup y}$ is isomorphic to $\widehat{\mathscr{O}}_{X,x} \times \widehat{\mathscr{O}}_{Y,y}$ by [MR86, Theorem 8.15]. Taking completions of $\mathscr{O}_{\overline{X},z}$-modules now yields the exact sequence

$$0 \to \widehat{\mathscr{O}}_{\overline{X},z} \to \widehat{\mathscr{O}}_{X,x} \times \widehat{\mathscr{O}}_{X,y} \to k \to 0. \tag{4.6}$$

4.1.3. *An equivalence of categories.* By Proposition 4.3, we obtain for each object $(X \xrightarrow{f} \mathbb{G}_m, \varphi)$ of $\mathscr{C}$ a commutative diagram

$$\begin{tikzcd}
X \arrow[r, "\rho"] \arrow[d, "f"] & \overline{X} \arrow[d, dashed, "\overline{f}"] \\
\mathbb{G}_m \arrow[r, "\pi"] & C^\circ.
\end{tikzcd} \tag{4.7}$$

Proposition 4.4. *The morpism $\overline{f} \colon \overline{X} \to C^\circ$ constructed in* (4.7) *is finite étale.*

Proof. It is affine by construction, and finite by finiteness of f and π and the fact that C° is noetherian (see [Sta24, Tag 00FP]). The fact that it is étale at all points outside the fiber over n is clear. To prove that it is étale at the point $z = \rho(x) = \rho(y)$ in the fiber over n, for some $x \in f^{-1}(a)$ and $y = \varphi(x)$, we show that the induced map on completed local rings $\widehat{\mathscr{O}}_{C^\circ,n} \to \widehat{\mathscr{O}}_{\overline{X},z}$ is an isomorphism. Then we can apply [Har10, Chapter 2, Exercise 10.4] and conclude that $\overline{f}$ is étale at z. Recall The exact sequence from (4.6). We can derive a similar such sequence for the completed local ring of C° at n from (4.2). We obtain a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \widehat{\mathscr{O}}_{\overline{X},z} \arrow[r] & \widehat{\mathscr{O}}_{X,x} \times \widehat{\mathscr{O}}_{X,y} \arrow[r] & k \arrow[r] & 0 \\
0 \arrow[r] & \widehat{\mathscr{O}}_{C^\circ,n} \arrow[r] \arrow[u] & \widehat{\mathscr{O}}_{\mathbb{G}_m,a} \times \widehat{\mathscr{O}}_{\mathbb{G}_m,b} \arrow[r] \arrow[u, "\simeq"] & k \arrow[r] \arrow[u, "="] & 0,
\end{tikzcd}$$

where the middle vertical map is an isomorphism by the fact that f is étale. It follows that $\widehat{\mathscr{O}}_{C^\circ,n} \to \widehat{\mathscr{O}}_{\overline{X},z}$ is an isomorphism. ■

This construction is clearly functorial, and so we obtain the functor

$$\begin{aligned} G\colon \mathscr{C} &\to \mathrm{Fét}_{C^\circ} \\ (X \xrightarrow{f} \mathbb{G}_m, \varphi) &\mapsto (\overline{X} \xrightarrow{\overline{f}} C^\circ). \end{aligned} \tag{4.8}$$

Theorem 4.5. *The functors from* (4.4) *and* (4.8) *are pseudo-inverse to each other. As a result, we obtain an equivalence of categories*

$$\mathrm{Fét}_{C^\circ} \simeq \mathscr{C}.$$

Proof. Let $Y \to C^\circ$ be a finite étale morphism. We naturally obtain a commutative triangle

$$\begin{tikzcd}
\overline{\pi^* Y} \arrow[rr, dashed] \arrow[dr] & & Y \arrow[dl] \\
& C^\circ. &
\end{tikzcd}$$

Since $\overline{\pi^* Y}$ and Y are finite étale of the same degree over C°, the dashed arrow above is finite étale of degree 1 because it is surjective. It follows that the dashed arrow is an isomorphism. So we have an isomorphism of functors $G \circ \pi^* \simeq \mathrm{id}$.

Conversely, if (X, φ) is a finite étale cover of $\mathbb{G}_m$ with a descent datum, then we naturally obtain a commutative triangle

$$\begin{tikzcd}
X \arrow[rr, dashed] \arrow[dr] & & \pi^* \overline{X} \arrow[dl] \\
& \mathbb{G}_m. &
\end{tikzcd}$$

By the same argument as before, the dashed arrow is an isomorphism. Hence, we obtain an isomorphism of functors $\pi^* \circ G \simeq \mathrm{id}$. ■

4.1.4. *Computing the fundamental group.* We first recall a few elementary facts about free products of profinite groups. Denote by Grp the category of groups, by PrGrp the category of profinite groups, and by $\mathrm{PrGrp}^{(p')}$ the category of profinite groups G such that (the supernatural number) $\#G = [G:1]$ is prime to p. We will call such groups *pro-prime-to-p*. Let $\widehat{(-)}\colon \mathrm{Grp} \to \mathrm{PrGrp}$ be the functor sending a group G to its profinite completion $\widehat{G}$. Let $(-)^{(p')}\colon \mathrm{PrGrp} \to \mathrm{PrGrp}^{(p')}$ denote the functor sending a profinite group G to its maximal prime-to-p quotient $G^{(p')}$. Recall that it is defined as

$$G^{(p')} = \varprojlim_{([G:U],p)=1} G/U,$$

where the projective limit ranges over the open normal subgroups U of G of index prime-to-p. It is often much easier to get a grip on the prime-to-p quotient of the fundamental group of a scheme over a field of characteristic p. This is because the prime-to-p quotient filters out *wildly ramified covers*, of which there are usually many. The following lemma illustrates this.

Lemma 4.6. *There is an isomorphism* $\pi_1^{(p')}(\mathbb{G}_m) \simeq \widehat{\mathbb{Z}}^{(p')}$.

Proof. Let $\varphi\colon Y \to \mathbb{G}_m$ be a degree d connected étale cover of $\mathbb{G}_m$, where d is prime-to-p. Denote by $\overline{Y}$ the unique smooth compactifcation of Y. Then φ extends to a degree-d morphism $\overline{\varphi}\colon \overline{Y} \to \mathbb{P}^1$ of curves. Denote by $e_1,\dots,e_r$, respectively $f_1,\dots,f_s$, the ramification indices of $\overline{\varphi}$ over 0, respectively ∞. Since $\overline{\varphi}$ is of degree prime-to-p, we can apply the Riemann Hurwitz formula ([Har10, Chapter IV, Corollary 2.4]). We obtain

$$2g_Y - 2 = -2d + \sum_{i=1}^{r}(e_i - 1) + \sum_{j=1}^{s}(f_j - 1),$$

where g_Y denotes the genus of Y. After some consideration, this shows that $g_Y = 0$ and $e_1 = f_1 = d$. This leaves exactly one option for Y and φ up to isomorphism, namely $Y \simeq \mathbb{G}_m$ and $\varphi\colon y \mapsto y^d$. It has automorphism group cyclic of order d. We find

$$\pi_1^{(p')}(\mathbb{G}_m) \simeq \varprojlim_{(d,p)=1} \mathbb{Z}/d\mathbb{Z} \simeq \widehat{\mathbb{Z}}^{(p')}.$$

■

We have the adjunctions of functors

$$\begin{tikzcd} \mathrm{Grp} \arrow[r, bend left, "\widehat{(-)}"] \arrow[r, phantom, "\perp"] & \mathrm{PrGrp} \arrow[l, bend left] \arrow[r, bend left, "(-)^{(p')}"] \arrow[r, phantom, "\perp"] & \mathrm{PrGrp}^{(p')}, \arrow[l, bend left] \end{tikzcd} \tag{4.9}$$

where the unnamed arrows are the evident forgetful functors.

Given groups G and G', we denote by $G * G'$ the free product of G and G'. If G and G' are profinite groups, then we also denote by $G * G'$ the *free profinite product* of G

and G'. If G and G' are pro-prime-to-p groups, then we denote by $G *^{(p')} G'$ the *free pro-prime-to-p product* of G and G'. In all three cases, this defines the coproduct of G and G' in their respective category. For a construction of the free profinite product and the free pro-prime-to-p product, see [NSW, Chapter IV, Section 1].

Lemma 4.7. *(i) For groups G and G' we have a canonical isomorphism*

$$\widehat{G * G'} \simeq \hat{G} * \hat{G}'.$$

(ii) For profinite groups G and G' we have a canonical isomorphism

$$(G * G')^{(p')} \simeq G^{(p')} *^{(p')} G'^{(p')}.$$

Proof. This is immediate by (4.9) and the fact that left adjoints preserve colimits. ■

We fix, once and for all, an étale path $\mathrm{Fib}_a \simeq \mathrm{Fib}_b$, where $\mathrm{Fib}_a, \mathrm{Fib}_b \colon \mathrm{Fét}_{\mathbb{G}_m} \rightrightarrows$ sets are the fiber functor over a and b. Via this étale path, we obtain the equivalences of categories

$$\begin{aligned} \mathrm{Fét}_{C^\circ} &\simeq \mathscr{C} \\ &\simeq \{\text{pairs } (f\colon X \to \mathbb{G}_m, \varphi \in \mathrm{Aut}(f^{-1}(a))) \text{ with } f \text{ finite étale}\} \\ &\simeq \{\text{finite sets } F \text{ with a continuous action of } \pi_1(\mathbb{G}_m, a) \text{ and of } \hat{\mathbb{Z}}\} \\ &\simeq \pi_1(\mathbb{G}_m, a) * \hat{\mathbb{Z}}\text{-sets}, \end{aligned} \tag{4.10}$$

where the last category consists of finite sets with a continuous action of $\pi_1(\mathbb{G}_m, a) * \hat{\mathbb{Z}}$. Here the third equivalence is induced by the functor Fib_a. We have a commutative triangle

$$\begin{tikzcd} \mathrm{Fét}_{C^\circ} \arrow[rr, "\simeq"] \arrow[dr, "\mathrm{Fib}_n"] & & \pi_1(\mathbb{G}_m, a) * \hat{\mathbb{Z}}\text{-sets} \arrow[dl] \\ & \text{sets,} & \end{tikzcd} \tag{4.11}$$

where the arrow $\pi_1(\mathbb{G}_m, a) * \hat{\mathbb{Z}}$-sets → sets is the evident forgetful functor.

Theorem 4.8. *(i) The diagram* (4.11) *induces a canonical isomorphism*

$$\pi_1(C^\circ, n) \simeq \pi_1(\mathbb{G}_m, a) * \hat{\mathbb{Z}}.$$

(ii) The maximal prime-to-p quotient of $\pi_1(C^\circ, n)$ is canonically isomorphic to

$$\pi_1(C^\circ, n)^{(p')} \simeq \widehat{\mathbb{Z} * \mathbb{Z}}^{(p')}.$$

(iii) The map $\pi_* \colon \pi_1(\mathbb{G}_m, a) \to \pi_1(C^\circ, n)$ is identified with the canonical inclusion

$$\pi_1(\mathbb{G}_m, a) \to \pi_1(\mathbb{G}_m, a) * \hat{\mathbb{Z}}$$

under the isomorphism of (i).

Proof. Part (i) is clear. Part (ii) follows from Lemma 4.7 and Lemma 4.6:

$$\begin{aligned}\pi_1(C^\circ, n)^{(p')} &\simeq (\pi_1(\mathbb{G}_m, a) * \hat{\mathbb{Z}})^{(p')} \\ &\simeq \pi_1(\mathbb{G}_m, a)^{(p')} *^{(p')} \hat{\mathbb{Z}}^{(p')} \\ &\simeq \hat{\mathbb{Z}}^{(p')} *^{(p')} \hat{\mathbb{Z}}^{(p')} \\ &\simeq (\hat{\mathbb{Z}} * \hat{\mathbb{Z}})^{(p')} \simeq \widehat{\mathbb{Z} * \mathbb{Z}}^{(p')}.\end{aligned}$$

Part (iii) follows from the commutative square

$$\begin{tikzcd}
\mathrm{Fét}_{C^\circ} \arrow[r, "\pi^*"] \arrow[d, "\simeq"] & \mathrm{Fét}_{\mathbb{G}_m} \arrow[d, "\simeq"] \\
\pi_1(\mathbb{G}_m, a) * \hat{\mathbb{Z}}\text{- sets} \arrow[r] & \pi_1(\mathbb{G}_m, a)\text{-sets},
\end{tikzcd}$$

where the horizontal arrow on the bottom is given by precomposing the action on a finite set by the canonical inclusion $\pi_1(\mathbb{G}_m, a) \to \pi_1(\mathbb{G}_m, a) * \hat{\mathbb{Z}}$. ■

4.2. **The construction.** We return to the question posed in the introduction. We use the same notation. Let $X \to S$ be an *elliptic fibration* whose special fiber is reduced and of type I_1 in the Kodaira classification of singular fibers. We take this to mean that X is regular and connected, $X \to S$ is proper, the generic fiber $X_\eta \to \operatorname{Spec} \kappa(\eta)$ is smooth of genus 1, and the special fiber $X_s \to \operatorname{Spec} k$ is a rational curve with a simple node as in the previous section. See also [Sil86, Appendix C.15].

Example 4.9. Let

$$X = \operatorname{Proj} \frac{\mathcal{O}[x, y, z]}{(y^2 z - x^3 - x^2 z - \pi z^3)} \to S.$$

One can check that $X \to S$ satisfies the properties mentioned above.

Denote by $n \in X_s(k)$ the nodal point of the special fiber.

Lemma 4.10. *The morphism $X \setminus \{n\} \to S$ is smooth.*

Proof. It is flat, because $X \to S$ is by the assumptions that it is surjective and that X is integral (see [Har10, Proposition 9.7]). The morphism $X \setminus \{n\} \to S$ is locally of finite presentation by the fact that $X \to S$ is. All the fibers of $X \setminus \{n\} \to S$ are smooth, because the generic fiber is X_η, which is smooth by assumption, and the special fiber is $X_s \setminus \{n\}$, which is regular over an algebraically closed field. The result follows. ■

Fix pairwise distinct non-singular points $x_0, x_1, x_2 \in X_s(k)$. By the above Lemma, we can now apply Hensel's Lemma as in [Mil16, Chapter I, Exercise 4.13] to obtain R-points

$$s_i \colon S \to X,$$

for $i = 1, 2$, such that x_i is equal to the composition

$$\operatorname{Spec} k \to S \to X.$$

Since $X \to S$ is separated, the s_i are closed immersions by [Har10, Chapter II, Exercise 4.8]. We obtain an effective Cartier divisor $D = s_1(S) + s_2(S)$ on X. Denote by U the scheme $X \setminus \operatorname{Supp} D \to S$ over S. The map $D \to S$ is finite étale, and D lies in the smooth locus of $X \to S$ by Lemma 4.10. As a result, we obtain the following lemma.

Lemma 4.11. *The canonical map of tame fundamental groups*

$$\pi_1^t(U_s, x_0) \xrightarrow{\simeq} \pi_1^t(U, x_0)$$

is an isomorphism. It induces an isomorphism

$$\pi_1^{(p')}(U_s, x_0) \xrightarrow{\simeq} \pi_1^{(p')}(U, x_0)$$

on maximal prime-to-p quotients.

Proof. This is by Proposition 3.1. ■

4.2.1. *Enter the representations.* Consider the normalization $\pi\colon \mathbb{P}^1_k \to X_s$. Without loss of generality, we assume that the points lying over x_1 and x_2 are 0 and ∞, respectively. Denote the point over x_0 by $\tilde{x}_0$. We obtain the normalization $\mathbb{G}_m \to U_s$ of U_s by restricting to $\mathbb{G}_m = \mathbb{P}^1 \setminus \{0, \infty\}$. In accordance with the previous section, we let a and b be the points over the node n. We fix, once and for all, an étale path $\mathrm{Fib}_{\tilde{x}_0} \simeq \mathrm{Fib}_a$. This induces an étale path $\mathrm{Fib}_{x_0} \simeq \mathrm{Fib}_n$. We obtain a commutative diagram

$$\begin{tikzcd}
\pi_1(\mathbb{G}_m, \tilde{x}_0) \arrow[r, "\simeq"] \arrow[d, "\pi_*"] & \pi_1(\mathbb{G}_m, a) \arrow[r, "="] \arrow[d, "\pi_*"] & \pi_1(\mathbb{G}_m, a) \arrow[d] \\
\pi_1(U_s, x_0) \arrow[r, "\simeq"] & \pi_1(U_s, n) \arrow[r, "\simeq"] & \pi_1(\mathbb{G}_m, a) * \hat{\mathbb{Z}}.
\end{tikzcd} \tag{4.12}$$

Here the isomorphisms of the left square come from the fixed étale paths, and the right hand square is by Theorem 4.8. It is by the isomorphisms in (4.12) that we identify $\pi_1(\mathbb{G}_m, \tilde{x}_0)$ with $\pi_1(\mathbb{G}_m, a)$ and $\pi_1(U_s, x_0)$ with $\pi_1(\mathbb{G}_m, a) * \hat{\mathbb{Z}}$. In particular, by Theorem 4.8, we obtain isomorphisms

$$\pi_1^{(p')}(\mathbb{G}_m, \tilde{x}_0) \simeq \hat{\mathbb{Z}}^{(p')} \quad \text{and} \quad \pi_1^{(p')}(U_s, x_0) \simeq \widehat{\mathbb{Z} * \mathbb{Z}}^{(p')},$$

and under these identifications the map $\pi_1^{(p')}(\mathbb{G}_m, \tilde{x}_0) \to \pi_1^{(p')}(U_s, x_0)$ corresponds to the map induced by the inclusion $\mathbb{Z} \hookrightarrow \mathbb{Z} * \mathbb{Z}$ of the left hand copy of $\mathbb{Z}$.

For any prime ℓ, $\mathrm{GL}_2(\mathbb{Z}_\ell)$ is a profinite group:

$$\mathrm{GL}_2(\mathbb{Z}_\ell) = \varprojlim_n \mathrm{GL}_2(\mathbb{Z}/\ell^n).$$

Consider now the profinite group

$$\begin{aligned} \Gamma_\ell &= \ker(\mathrm{GL}_2(\mathbb{Z}_\ell) \to \mathrm{GL}_2(\mathbb{Z}/\ell)) \\ &= \varprojlim_n \ker(\mathrm{GL}_2(\mathbb{Z}/\ell^n) \to \mathrm{GL}_2(\mathbb{Z}/\ell)). \end{aligned} \tag{4.13}$$

Lemma 4.12. *The profinite group* Γ_ℓ *from* (4.13) *is a pro-*ℓ *group.*

Proof. It is easily seen that each of the finite groups

$$\ker(\mathrm{GL}_2(\mathbb{Z}/\ell^n) \to \mathrm{GL}_2(\mathbb{Z}/\ell))$$

is an ℓ-group. ■

Remark 4.13. Replacing $\mathbb{Z}_\ell$ by the ring of integers $\mathcal{O}_E$ of an ℓ-adic field E, and 2 by an arbitary integer $r \geq 1$, we obtain by analogous reasoning a pro-ℓ group

$$\Gamma_\ell = \ker(\mathrm{GL}_r(\mathcal{O}_E) \to \mathrm{GL}_r(\mathcal{O}_E/\mathfrak{m}_E)).$$

In addition, it is *open* in $\mathrm{GL}_r(E)$.

As in the introduction, we assume ℓ is a prime different from p. It follows in particular that, by the above Lemma, Γ_ℓ is a pro-prime-to-p group. Consider the homomorphism

$$\mathbb{Z} * \mathbb{Z} \to \Gamma_\ell$$

defined by sending the first generator to $\left(\begin{smallmatrix} 1 & \ell \\ 0 & 1 \end{smallmatrix}\right)$ and the second generator to $\left(\begin{smallmatrix} 1 & 0 \\ \ell & 1 \end{smallmatrix}\right)$. We obtain a continuous representation

$$\rho^{(p')} \colon \pi_1^{(p')}(U_s, x_0) \simeq \widehat{(\mathbb{Z} * \mathbb{Z})}^{(p')} \to \Gamma_\ell \hookrightarrow \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell), \tag{4.14}$$

from the adjunctions in (4.9).

Lemma 4.14. *The representation $\rho^{(p')}$ from* (4.14) *is irreducible.*

Proof. The image of $\rho^{(p')}$ contains the subgroup of $\mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$ spanned by the matrices $\left(\begin{smallmatrix} 1 & \ell \\ 0 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 0 \\ \ell & 1 \end{smallmatrix}\right)$. It is easy to check that this subgroup does not fix any proper subspace of $\overline{\mathbb{Q}}_\ell^{\oplus 2}$. ■

Consider now the representation

$$\rho^{(p')} \circ \pi_* \colon \pi_1^{(p')}(\mathbb{G}_m, \tilde{x}_0) \to \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell). \tag{4.15}$$

Lemma 4.15. *The representation $\rho^{(p')} \circ \pi_*$ from* (4.15) *is not semi-simple.*

Proof. Identifying $\pi_1^{(p')}(\mathbb{G}_m, \tilde{x}_0)^{(p')}$ with $\hat{\mathbb{Z}}^{(p')}$ as before and restricting $\rho^{(p')} \circ \pi_*$ to $\mathbb{Z}$, we obtain the representation $\mathbb{Z} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$ sending 1 to $\left(\begin{smallmatrix} 1 & \ell \\ 0 & 1 \end{smallmatrix}\right)$. Clearly, this representation has a single non-trivial subrepresentation, and hence is not semi-simple. The same is then true for the represenation $\rho^{(p')} \circ \pi_*$ by continuity. ■

4.2.2. *Conclusion.* We now finally give a construction showing that the answer to Question 4.1 is *"No"*. Consider the S-scheme $\mathscr{X} := U \setminus \{n\} \to S$. It is smooth by Lemma 4.10. Clearly, it is surjective, separated and quasi-compact. Its special fiber is $\mathscr{X}_s = U_s \setminus \{n\} \to \operatorname{Spec} k$. Notice also that $\mathscr{X}$ is connected, so that we may define local systems on it in terms of monodromy representations. Define the representation $\rho \colon \pi_1(\mathscr{X}, x_0) \to \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$ to be the composition

$$\pi_1(\mathscr{X}, x_0) \xrightarrow{\simeq} \pi_1(U, x_0) \twoheadrightarrow \pi_1^{(p')}(U, x_0) \simeq \pi_1^{(p')}(U_s, x_0) \xrightarrow{\rho^{(p')}} \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell). \tag{4.16}$$

Here the first isomorphism is by the Zariski-Nagata purity Theorem, and the second isomorphism comes from Proposition 4.11.

Corollary 4.16. *The representation* ρ *from* (4.16) *is irreducible.*

Proof. This follows from Lemma 4.14 and the fact that ρ and $\rho^{(p')}$ share the same image. ■

To finish the construction, we only have to argue that

$$\rho \circ i_* \colon \pi_1(\mathscr{X}_s, x_0) \to \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell) \tag{4.17}$$

is *not* semi-simple. To this end, consider the diagram

$$\begin{tikzcd}
\pi_1(\mathbb{G}_m \setminus \{a,b\}, \tilde{x}_0) \arrow[r, "\simeq"] \arrow[d, two heads] & \pi_1(\mathscr{X}_s, x_0) \arrow[r, "i_*"] \arrow[d] & \pi_1(\mathscr{X}, x_0) \arrow[d, "\simeq"'] \arrow[ddr, bend left, "\rho"] & \\
\pi_1(\mathbb{G}_m, \tilde{x}_0) \arrow[r] \arrow[d, two heads] & \pi_1(U_s, x_0) \arrow[r] \arrow[d, two heads] & \pi_1(U, x_0) \arrow[d, two heads] & \\
\pi_1^{(p')}(\mathbb{G}_m, \tilde{x}_0) \arrow[r, "\pi_*"] & \pi_1^{(p')}(U_s, x_0) \arrow[r, "\simeq"] \arrow[rr, bend right, "\rho^{(p')}"'] & \pi_1^{(p')}(U, x_0) & \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)
\end{tikzcd} \tag{4.18}$$

Corollary 4.17. *The representation* $\rho \circ i_*$ *from* (4.17) *is not semi-simple.*

Proof. This follows from the fact that $\rho \circ i_*$ has the same image as $\rho^{(p')} \circ \pi_*$ by (4.18), and the fact that $\rho^{(p')} \circ \pi_*$ is not semi-simple by Lemma 4.15. ■

Remark 4.18. The representation ρ constructed here is not the exact analogue of the one constructed in the Complex-geometric version of this example in Section A. If we were to imitate that construction faithfully, we would get in trouble with the primes $p = 2$ and $p = 3$. We would also have to put additional conditions on ℓ.

5. SIMPSON'S SPREADING ARGUMENT

Let G and H be profinite groups and let G act continuously on H. Write $\widetilde{G} = H \rtimes G$ for the semi-direct product of G and H. It is again a profinite group, because the underlying topology of $\widetilde{G}$ is that of $G \times H$; hence, it is compact and totally disconnected. We think of G and H as living inside this semi-direct product. In particular, we denote the action of G on H by conjugation.

One scenario in which this situation arises naturally is described by the following standard result.

Proposition 5.1. *Suppose we have an exact sequence of profinite groups*

$$1 \to H \to K \to G \to 1$$

such that $K \to G$ admits a continuous section $G \to K$. Then G acts continuously on H by conjugation, and we have an isomorphism $K \simeq H \rtimes G$ fitting into the commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & H \arrow[r] \arrow[d, "="] & K \arrow[r] \arrow[d, "\simeq"] & G \arrow[r] \arrow[d, "="] & 1 \\
1 \arrow[r] & H \arrow[r] & H \rtimes G \arrow[r] & G \arrow[r] & 1.
\end{tikzcd}$$

■

The main result of this section is Theorem 5.6. The approach of the proof is partially inspired by [Sim92, Theorem 4].

5.1. **The space of ℓ-adic representations.** Recall our convention from Terminology 2.13. Throughout, we fix a prime ℓ. We define the set $\mathscr{R}$ to be the set of isomorphism classes of semisimple ℓ-adic representations of H:

$$\mathscr{R} = \{\text{semisimple } \ell\text{-adic representations } H \to \mathrm{GL}(V)\}/\simeq.$$

For $\rho\colon H \to \mathrm{GL}(V)$ a semisimple ℓ-adic representation, we denote by $[\rho] \in \mathscr{R}$ its isomorphism class. We have a natural right action of G on $\mathscr{R}$ defined by

$$[\rho]^g = [\rho^g] \quad (g \in G, [\rho] \in \mathscr{R}),$$

where

$$\rho^g(h) = \rho(ghg^{-1}) \quad (h \in H).$$

We intend to equip $\mathscr{R}$ with a topology such that the action described above is continuous. Denote by $\mathrm{Map}(H, \overline{\mathbb{Q}}_\ell)$ the set of continuous (set-)maps $H \to \overline{\mathbb{Q}}_\ell$. We obtain an injection

$$\begin{aligned} \mathscr{R} &\hookrightarrow \mathrm{Map}(H, \overline{\mathbb{Q}}_\ell) \\ [\rho] &\mapsto \mathrm{Tr}\,\rho := \mathrm{Tr} \circ \rho, \end{aligned}$$

where $\mathrm{Tr}\colon \mathrm{GL}(V) \to \overline{\mathbb{Q}}_\ell$ denotes the (continuous) trace map, by the following lemma.

Lemma 5.2 ([Wie12, Proposition 2.4.3])**.** *Let k be a field of characteristic* 0*, A a k-algebra, and V and V' two* semisimple *A-modules of finite k-dimension. If the characters* $\mathrm{Tr}_V\colon G \to k$ *and* $\mathrm{Tr}_{V'}\colon G \to k$ *obtained by sending $g \in G$ to* $\mathrm{Tr}(g|_V)$*, respectively* $\mathrm{Tr}(g|_{V'})$*, are equal, then V and V' are isomorphic as A-modules.* ■

We equip $\mathrm{Map}(H, \overline{\mathbb{Q}}_\ell)$ with the compact-open topology. Then $\mathscr{R}$ is equipped with the subspace topology inherited from $\mathrm{Map}(H, \overline{\mathbb{Q}}_\ell)$.

Proposition 5.3. *The action of G on $\mathscr{R}$ is continuous.*

Proof. By assumption, the map $G \times H \to H$ is continuous. By [Mun14, Theorem 46.11] the induced map

$$\varphi\colon G \to \mathrm{Map}(H, H)$$

is continuous if we equip $\mathrm{Map}(H, H)$ with the compact-open topology. The space H is locally compact and Hausdorff (H is profinite), and so by [Mun14, Exercise 7, §46] we find that the composition map

$$c\colon \mathrm{Map}(H, H) \times \mathrm{Map}(H, \overline{\mathbb{Q}}_\ell) \to \mathrm{Map}(H, \overline{\mathbb{Q}}_\ell)$$

is continuous. As a result, the map

$$G \times \mathrm{Map}(H, \overline{\mathbb{Q}}_\ell) \xrightarrow{\varphi\times\mathrm{id}} \mathrm{Map}(H, H) \times \mathrm{Map}(H, \overline{\mathbb{Q}}_\ell) \xrightarrow{c} \mathrm{Map}(H, \overline{\mathbb{Q}}_\ell)$$

is continuous, and hence so is

$$G \times \mathscr{R} \to \mathscr{R}.$$

■

Corollary 5.4. *Let* $[\rho] \in \mathscr{R}$*. Then the stabilizer* $\mathrm{Stab}_{[\rho]}$ *of* $[\rho]$ *is a closed subgroup of G. If the orbit of* $[\rho]$ *is finite, then* $\mathrm{Stab}_{[\rho]}$ *is also open.*

Proof. Write

$$\Phi\colon G \times \mathscr{R} \to \mathscr{R}$$

for the action map. We have

$$\mathrm{Stab}_{[\rho]} = \Phi^{-1}([\rho]) \cap G \times \{[\rho]\},$$

and so we only have to argue that $[\rho] \in \mathscr{R}$ is a closed point. This follows from the fact that $\mathrm{Map}(H, \overline{\mathbb{Q}}_\ell)$ is Hausdorff: $\overline{\mathbb{Q}}_\ell$ is Hausdorff so that we can apply [Mun14, Exercise 6, §46]. The second part of the corollary follows from the fact that $\mathrm{Stab}_{[\rho]}$ has finite index if the orbit of $[\rho]$ is finite. ■

Corollary 5.5. *Let* $[\rho] \in \mathscr{R}$ *such that orbit of* $[\rho]$ *in* $\mathscr{R}$ *is finite. Then the orbit of each of the irreducible constituents of ρ is finite.*

Proof. Let U be the stabilizer of $[\rho]$. By Corollary 5.4 it is open. The subgroup U permutes the irreducible constituents of ρ, and so there exists an open subgroup $V \subset U$ fixing all of them. ■

5.2. **Spreading ℓ-adic representations with finite determinant.** Notation is as before.

Theorem 5.6. *Let $\rho\colon H \to \mathrm{GL}_r(\overline{\mathbb{Q}}_\ell)$ be an irreducible ℓ-adic representation and assume that the orbit $[\rho]\cdot G \subset \mathscr{R}$ is finite. Then there exists an open subgroup $U \subset G$ such that ρ extends to a continuous representation*

$$\widetilde{\rho}\colon \widetilde{U} := H \rtimes U \to \mathrm{GL}_r(\overline{\mathbb{Q}}_\ell).$$

Furthermore, if $\det\rho$ *is finite (i.e., the determinant character of ρ has finite image), then $\widetilde{\rho}$ can be chosen such that* $\det\widetilde{\rho}$ *is finite.*

Proof. We can find E a finite extension of $\mathbb{Q}_\ell$ such that ρ factors as $H \to \mathrm{GL}_r(E) \to \mathrm{GL}_r(\overline{\mathbb{Q}}_\ell)$. By the assumption that $[\rho]\cdot G$ is finite, and Corollary 5.4, $\mathrm{Stab}_{[\rho]}$ is open. By potentially replacing G by the open subgroup $\mathrm{Stab}_{[\rho]}$, we can assume that G acts trivially on $[\rho]$. As a result, for every $g \in G$ there is an isomorphism $\rho^g \simeq \rho$ so that ρ and ρ^g are conjugate to each other by some $A_g \in \mathrm{GL}_r(E)$:

$$\rho^g = A_g \cdot \rho \cdot A_g^{-1}.$$

We can indeed take the A_g to be defined over E, because G acts trivially on the trace character of ρ, and hence trivially on the E-isomorphism class of ρ by Lemma 5.2. We define

$$\begin{aligned}\overline{A}\colon G &\to \mathrm{PGL}_r(E)\\ g &\mapsto \overline{A}_g,\end{aligned}$$

where $\overline{A}_g$ denotes the class of A_g in $\mathrm{PGL}_r(E)$. It is easily seen that $\overline{A}$ is a homomorphism by Schur's Lemma. We argue that $\overline{A}$ is additionally continuous by applying Lemma 5.9 below. We employ the notation introduced in that lemma. Notice first that, since ρ is irreducible over $\overline{\mathbb{Q}}_\ell$, we have

$$E[\rho(h)\colon h \in H] = \mathrm{Mat}(r \times r; E)$$

by [EG11, Theorem 3.2.2]. For $1 \le i, j \le r$ we can therefore write

$$e_{i,j} = \sum_{h\in H} \alpha_h^{i,j}\rho(h)$$

with $\alpha_h^{i,j} \in E$ zero for all but finitely many $h \in H$. Then for $g \in G$ we compute

$$\begin{aligned}(\mathrm{ev}_{i,j}\circ\overline{A})(g) &= A_g e_{i,j} A_g^{-1}\\ &= \sum_{h\in H} \alpha_h^{i,j}\rho^g(h)\\ &= \sum_{h\in H} \alpha_h^{i,j}\rho(ghg^{-1}).\end{aligned}$$

We see that $\mathrm{ev}_{i,j}\circ\overline{A}$ is a linear combination of continuous functions and hence is continuous. It follows that $\overline{A}$ is continuous by Lemma 5.9.

Our goal is to lift $\overline{A}\,|_U$ to a continuous homomorphism $A\colon U \to \mathrm{GL}_r(E)$ for some open subgroup $U \subset G$. To this end, we apply the theory of continuous non-abelian cohomology, which is recalled in Appendix B. Consider the strict exact sequence of topological G-groups (each with trivial G-action)

$$1 \to \mu_r \to \mathrm{SL}_r(E) \to \mathrm{PSL}_r(E) \to 1$$

from Corollary 5.12 below. By Theorem B.3, we obtain an exact sequence of pointed sets

$$H^1_{\text{cont}}(G;\mathrm{SL}_r(E)) \to H^1_{\text{cont}}(G;\mathrm{PSL}_r(E)) \xrightarrow{\delta} H^2_{\text{cont}}(G;\mu_r).$$

By potentially shrinking G to an open subgroup we can assume that the image of $\overline{A}$ lies in $\mathrm{PSL}_r(E)$ by Lemma 5.11. Let $U \subset G$ be an open subgroup such that $\mathrm{res}^G_U(\delta(\overline{A})) = 0$. This is possible by the fact that μ_r is discrete. Then, since restriction is compatible with connecting homomorphisms, we find

$$\delta(\mathrm{res}^G_U(\overline{A})) = 0 \in H^2_{\text{cont}}(U;\mu_r),$$

where now δ denotes the connecting homomorphism in the sequence

$$H^1_{\text{cont}}(U;\mathrm{SL}_r(E)) \to H^1_{\text{cont}}(U;\mathrm{PSL}_r(E)) \to H^2_{\text{cont}}(U;\mu_r).$$

It follows that there exists $A \in H^1_{\text{cont}}(U;\mathrm{SL}_r(E))$ lifting $\overline{A}\,|_U$[3]. We now set

$$\widetilde{\rho} = \rho \rtimes A\colon H \rtimes U \to \mathrm{GL}_r(E).$$

The last part of the proposition follows by construction since A takes values in $\mathrm{SL}_r(E)$. ■

Remark 5.7. Suppose $\rho\colon H \to \mathrm{GL}_r(\overline{\mathbb{Q}}_\ell)$ is irreducible with finite determinant. Then an extension $\widetilde{\rho}\colon H \rtimes U \to \mathrm{GL}_r(\overline{\mathbb{Q}}_\ell)$ of ρ with finite determinant is even *unique* up to a diminution of U. Indeed, let $\widetilde{\rho}, \widetilde{\rho}'\colon H \rtimes U \rightrightarrows \mathrm{GL}_r(\overline{\mathbb{Q}}_\ell)$ be two extensions of ρ with finite determinant. Then their "projectivizations" $H \rtimes U \rightrightarrows \mathrm{GL}_r(\overline{\mathbb{Q}}_\ell) \to \mathrm{PGL}_r(\overline{\mathbb{Q}}_\ell)$ must both equal $\rho \rtimes \overline{A}$, where $\overline{A}\colon U \to \mathrm{PGL}_r(\overline{\mathbb{Q}}_\ell)$ is the unique homomorphism such that $\rho^g = \overline{A}_g \rho \overline{A}_g^{-1}$ for all $g \in U$. Therefore, $\widetilde{\rho}$ and $\widetilde{\rho}'$ differ by a character $\chi\colon H \rtimes U \to \overline{\mathbb{Q}}_\ell^\times$. By finiteness of the determinants, this must be a finite character. Since χ is trivial on H, χ will vanish after shrinking U. Hence, $\widetilde{\rho}$ and $\widetilde{\rho}'$ will coincide after shrinking U.

Corollary 5.8. *Let $\rho\colon H \to \mathrm{GL}_r(\overline{\mathbb{Q}}_\ell)$ be a semisimple representation such that the orbit $[\rho] \cdot G \subset \mathscr{R}$ is finite. Then there exists an open subgroup $U \subset G$ such that ρ extends to a continuous representation*

$$\widetilde{\rho}\colon H \rtimes U \to \mathrm{GL}_r(\overline{\mathbb{Q}}_\ell).$$

Proof. Each of the irreducible constituents of ρ has finite G-orbit by Lemma 5.5. Then we apply Theorem 5.6 to spread each of the irreducible constituents. After taking an appropriate direct sum, we find a spreading of ρ. ■

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[3] Notice that by surjectivity of $\mathrm{SL}_r(E) \to \mathrm{PSL}_r(E)$ we can find an actual lift of $\overline{A}$ and not just of its conjugacy class in $H^1_{\text{cont}}(U;\mathrm{SL}_r(E))$.

5.3. **Some auxiliary results.** Let E be a finite extension of $\mathbb{Q}_\ell$. The projective general linear group $\mathrm{PGL}_r(E)$ is equipped with the quotient topology from $\mathrm{GL}_r(E) \twoheadrightarrow \mathrm{PGL}_r(E)$. Denote by $M_r(E)$ the algebra of $r \times r$-matrices over E.

Lemma 5.9 (Topology of PGL_r)**.** *The space* $\mathrm{PGL}_r(E)$ *has the coarsest topology making each of the evaluation maps*

$$\begin{aligned} \mathrm{ev}_{i,j}\colon \mathrm{PGL}_r(E) &\to M_r(E) \\ M &\mapsto Me_{i,j}M^{-1} \end{aligned}$$

continuous. Here $e_{i,j} \in M_r(E)$ denotes the matrix with a 1 in the (i,j)-th entry and zeroes everywhere else.

Proof. By the Skölem-Noether Theorem (see [GS17, Theorem 2.7.2]), we obtain a continuous bijection

$$\begin{aligned} \mathrm{PGL}_r(E) &\to \mathrm{Aut}_E(M_r(E)), \\ M &\mapsto (\varphi_M\colon N \mapsto MNM^{-1}), \end{aligned} \tag{5.1}$$

where $\mathrm{Aut}_E(M_r(E))$ denotes the set of E-algebra automorphisms of $M_r(E)$ with the subspace topology inherited from $\mathfrak{gl}(M_r(E)) \cong E^{\oplus r^4}$, the set of E-linear endomorphisms of $M_r(E)$. The space $\mathrm{Aut}_E(M_r(E))$ is a locally compact Hausdorff space, because it is a subspace of $\mathfrak{gl}(M_r(E))$. As a result, it is a Baire space. By [Ser92, Part II, Chapter IV, Section 4, Lemma 1], we conclude that the map from (5.1) is a homeomorphism. It is clear that the topology on $\mathfrak{gl}(M_r(E))$ is the coarsest one for which each of the maps

$$\begin{aligned} \mathrm{ev}_{i,j}\colon \mathfrak{gl}(M_r(E)) &\to M_r(E) \\ \varphi &\mapsto \varphi(e_{i,j}) \end{aligned}$$

is continuous. The result follows. ■

Denote by $\mathrm{PSL}_r(E)$ the image of $\mathrm{SL}_r(E)$ in $\mathrm{PGL}_r(E)$ equipped with the subspace topology.

Lemma 5.10. *The map* $\mathrm{SL}_r(E) \twoheadrightarrow \mathrm{PSL}_r(E)$ *admits a continuous (set-theoretic) section* $\mathrm{PSL}_r(E) \to \mathrm{SL}_r(E)$.

Proof. We show that the map $\mathrm{SL}_r(E) \to \mathrm{PGL}_r(E)$ of ℓ-adic Lie groups induces an isomorphism on Lie-algebras. The Lie algebra of $\mathrm{SL}_r(E)$ is $\mathfrak{sl}_r(E)$, the $r \times r$ matrices over E with trace 0. By construction of the quotient Lie group (see [Ser92, Part II, Chapter IV, Section 5]), the Lie algebra of $\mathrm{PGL}_r(E)$ is given by

$$\mathrm{Lie}(\mathrm{PGL}_r(E)) = M_r(E)/EI_r,$$

where I_r denotes the identity matrix. The induced map

$$\mathfrak{sl}_r(E) \to \mathrm{Lie}(\mathrm{PGL}_r(E))$$

is an isomorphism of Lie algebras. By the Inverse Function Theorem (see [Ser92, Part II, Chapter 2, Section 9]), $\mathrm{SL}_r(E) \to \mathrm{PGL}_r(E)$ is a local isomorphism. Certainly then, $\mathrm{SL}_r(E) \to \mathrm{PSL}_r(E)$ admits sections locally. Since $\mathrm{PSL}_r(E)$ has a basis consisting of open (hence closed) subgroups, we can extend such local sections to the whole of $\mathrm{PSL}_r(E)$. ■

Lemma 5.11. *The subspace* $\mathrm{PSL}_r(E) \subset \mathrm{PGL}_r(E)$ *is open.*

Proof. The proof of Lemma 5.10 shows that $\mathrm{SL}_r(E) \to \mathrm{PGL}_r(E)$ is a local isomorphism, from which it follows that the image of $\mathrm{SL}_r(E)$ in $\mathrm{PGL}_r(E)$ is open. ■

Corollary 5.12. *We have a strict exact sequence of topological groups*

$$1 \to \mu_r \to \mathrm{SL}_r(E) \to \mathrm{PSL}_r(E) \to 1, \tag{5.2}$$

where $\mu_r \subset E$ *denotes the set of* r*-th roots of unity in* E*. The sequence* (5.2) *satisfies properties (i) and (ii) of Section B.2.*

6. AN ARITHMETIC LOCAL KASHIWARA CONJECTURE IN EQUAL POSITIVE CHARACTERISTIC

In Section 4 we constructed a counterexample to the naive version of the local Kashiwara conjecture, Question 4.1. In this section we prove Theorem 1.4, a specific case of the arithmetic local Kashiwara conjecture, Conjecture 1.2, in equal positive characteristic. Recall our conventions from Terminology 2.13 and Terminology 2.16.

6.1. **Arithmetic local systems.**

Let k be a finitely generated field of characteristic $p \geq 0$, and let ℓ be a prime different from p. Pick an algebraically closed field Ω containing k and let $\overline{k}$ be the separable closure of k in Ω. Let X be an integral separated finite type $\overline{k}$-scheme.

Definition 6.1. A semisimple ℓ-adic local system $\mathbb{L}$ on X is said to be *arithmetic* if there exists an ℓ-adic local system $\mathbb{L}^0_{k'}$ on a spreading $X^0_{k'}$ of X to a scheme $X^0_{k'}$ over k', for some finite separable extension $k \subset_f k' \subset \overline{k}$, such that $\mathbb{L}^0_{k'}$ pulls back to $\mathbb{L}$.

Of course, when $\rho\colon \pi_1(X,\overline{x}) \to \mathrm{GL}(V)$ is a semisimple ℓ-adic representation of $\pi_1(X,\overline{x})$, we say that ρ is *arithmetic* if the associated lisse ℓ-adic sheaf is. Explicitly, ρ is arithmetic if there exists an ℓ-adic representation $\rho^0_{k'}\colon \pi_1(X^0_{k'},\overline{x}) \to \mathrm{GL}(V)$ that pulls back to ρ, where $X^0_{k'}$ is a spreading of X to a scheme over k', a finite separable extension $k \subset_f k' \subset \overline{k}$.

Remark 6.2. Our definition of an arithmetic local system notably differs from the definition in [Lit21], which drops the assumption of semisimplicity and requires instead that $\mathbb{L}$ arises only as a subquotient of the pullback of $\mathbb{L}^0_{k'}$. For semisimple local systems, the two notions coincide by [Lit21, Proposition 3.1.1].

Example 6.3. Assume that X is also normal and let $f\colon Y \to X$ be a smooth proper morphism. Then the local system $\mathbb{L} = R^i f_* \overline{\mathbb{Q}}_\ell$ is semisimple by Example 2.17 (ii)[4]. We argue that it is arithmetic. By a spreading argument[5], we find a finite separable extension $k \subset_f k' \subset \overline{k}$ and a smooth proper morphism $f^0_{k'}\colon Y^0_{k'} \to X^0_{k'}$ that pulls back to f. Then by the proper base change theorem, $\mathbb{L}$ is the pullback of the local system $\mathbb{L}^0_{k'} := R^i f^0_{k',*} \overline{\mathbb{Q}}_\ell$ on $X^0_{k'}$.

We aim to give a different characterization of arithmetic local systems in terms of their monodromy representations. By potentially replacing k by a finite separable extension, we can assume that there exists a spreading X^0 of X to a k-scheme. We fix X^0. Notice that a local system $\mathbb{L}$ on X is arithmetic if and only if there exists a finite separable extension $k \subset k' \subset \overline{k}$ and a local system $\mathbb{L}^0_{k'}$ on $X^0_{k'}$ such that $\mathbb{L}^0_{k'}$ pulls back to $\mathbb{L}$.

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[4] Deligne proves this when $\overline{k}$ is algebraically closed, but one obtains the same statement for separably closed base fields by [Sta24, Tag 0BTW].

[5] See [Gro67, §8, §17]. Such spreading arguments are employed multiple times in this section, often with details omitted.

By further enlarging k, we can also assume that X^0 admits a rational point $x\colon \operatorname{Spec} k \to X^0$. We fix x. Let $\overline{x}\colon \operatorname{Spec}\Omega \to \operatorname{Spec} \xrightarrow{x} X$ be the induced geometric point of X, and also write $\overline{x}$ for its composition with the map $X \to X^0$. There is a homotopy exact sequence

$$1 \to \pi_1(X,\overline{x}) \to \pi_1(X^0,\overline{x}) \to \operatorname{Gal}(\overline{k}/k) \to 1 \tag{6.1}$$

by [Sza09, Proposition 5.6.1]. It is split by the k-rational point $x\colon \operatorname{Spec} k \to X^0$. As a result, $\operatorname{Gal}(\overline{k}/k)$ acts continuously on $\pi_1(X,\overline{x})$ by conjugation. We consider the set of isomorphism classes of semisimple ℓ-adic representations of $\pi_1(X,\overline{x})$,

$$\mathscr{R} = \{\text{semisimple } \ell\text{-adic representations } \rho\colon \pi_1(X,\overline{x}) \to \mathrm{GL}(V)\}/\simeq.$$

As argued in Section 5, it injects into the set of continuous maps $\operatorname{Map}(\pi_1(X,\overline{x}),\overline{\mathbb{Q}}_\ell)$, from which it inherits the compact-open topology. The continuous conjugation action of $\operatorname{Gal}(\overline{k}/k)$ on $\pi_1(X,\overline{x})$ equips $\mathscr{R}$ with a continuous right $\operatorname{Gal}(\overline{k}/k)$-action by Proposition 5.3.

Proposition 6.4. *Let* $\rho\colon \pi_1(X,\overline{x}) \to \mathrm{GL}(V)$ *be a semisimple* ℓ*-adic representation. Then* ρ *is arithmetic if and only if the orbit of* $[\rho] \in \mathscr{R}$ *under* $\operatorname{Gal}(\overline{k}/k)$ *is finite.*

Proof. If the orbit of $[\rho]$ is finite, then by Corollary 5.8 there exists a finite separable extension $k \subset_f k' \subset \overline{k}$ and a representation $\rho^0_{k'}\colon \pi_1(X^0_{k'},\overline{x}) \to \mathrm{GL}(V)$ that pulls back to ρ. By definition, ρ is arithmetic.

Conversely, if ρ is arithmetic, then ρ descends to $\rho^0_{k'}\colon \pi_1(X_{k'},\overline{x}) \to \mathrm{GL}(V)$. We see that for $\sigma \in \operatorname{Gal}(\overline{k}/k')$ the representations ρ and ρ^σ are conjugate to each other:

$$\rho^\sigma = \rho^0_{k'}(\sigma)\cdot\rho\cdot\rho^0_{k'}(\sigma)^{-1}.$$

So, $[\rho]$ is fixed by the open subgroup $\operatorname{Gal}(\overline{k}/k') \subset \operatorname{Gal}(\overline{k}/k)$. It follows that the orbit of $[\rho]$ is finite. ■

Remark 6.5. The proof of the previous proposition is effectively the same as the proof of [Lit21, Proposition 3.1.1]; however, Litt omits some subtle details regarding the existence of spreadings of arithmetic representations, which are expanded on here via the results of Section 5.

Remark 6.6. Alternatively, we can let $\operatorname{Gal}(\overline{k}/k)$ act on the set of ℓ-adic local systems on X as follows: for $\sigma \in \operatorname{Gal}(\overline{k}/k)$ and $\mathbb{L}$ a semisimple local system on X, we define $\mathbb{L}^\sigma$ to be the pullback of $\mathbb{L}$ along the map $\mathrm{id}_X \times_k \operatorname{Spec}\sigma^{-1}\colon X \to X$. One can show that if $\rho\colon \pi_1(X,\overline{x}) \to \mathrm{GL}(\mathbb{L}_{\overline{x}})$ is the monodromy representation of $\mathbb{L}$, then the monodromy representation of $\mathbb{L}^\sigma$ is precisely ρ^σ. By the above proposition, it follows that $\mathbb{L}$ is arithmetic if and only if its orbit under $\operatorname{Gal}(\overline{k}/k)$ is finite.

By Corollary 5.5 and Proposition 6.4, we immediately obtain the following proposition.

Proposition 6.7. *Let* $\mathbb{L}$ *be an arithmetic local system on* X*. Then each of its irreducible constituents is arithmetic.* ■

Example 6.8 (local systems coming from geometry are arithmetic)**.** Assume again that X is normal. Recall the definition of a local system coming from geometry from Example 2.10. Let $\mathbb{L}$ be a local system on X coming from geometry. Then we can find a dense open subscheme $U \subset X$ such that $\mathbb{L}|_U$ is a direct summand of $R^i f_* \overline{\mathbb{Q}}_\ell$ for some $i \geq 0$ and some smooth proper morphism $f\colon Y \to U$ by Example 2.17 (ii). After potentially enlarging k, we can assume that our rational point x is contained in U. By Example 6.3 and Proposition 6.7, we see that $\mathbb{L}|_U$ is arithmetic as a local system on U. By surjectivity of $\pi_1(U,\overline{x}) \twoheadrightarrow \pi_1(X,\overline{x})$ and Proposition 6.4, we then deduce that also $\mathbb{L}$ itself is arithmetic. Alexander Petrov has conjectured that if X is a smooth variety and k is a number field, then *all* irreducible arithmetic local systems on X are of geometric origin; see [Pet23].

Proposition 6.9. *Assume that $X^0 \to \operatorname{Spec} k$ is a smooth* curve. *Let $\chi\colon \pi_1(X,\overline{x}) \to \overline{\mathbb{Q}}_\ell^\times$ be an arithmetic character. Then χ is* finite, *meaning that its image is finite. In particular, if $\mathbb{L}$ is any arithmetic local system on X, then its determinant is finite.*

Proof. We first reduce to the case that k is a finite field. It suffices to prove that $\chi(U)$ is finite for any open subgroup $U \subset \pi_1(X,\overline{x})$. By Remark 4.13, we can assume that χ factors over the maximal prime-to-p quotient of $\pi_1(X,\overline{x})$ after replacing X by a finite étale cover (see also the proof of Lemma 6.18). We write X_Ω for the base change of X along $\operatorname{Spec}\Omega \to \operatorname{Spec}\overline{k}$. We let $\overline{X}_\Omega \to \operatorname{Spec}\Omega$ be the unique smooth compactification of X_Ω. We can find a finite type $\mathbb{F}_p$-subalgebra $R \subset \Omega$ and a smooth proper spreading $\overline{X}_R \to \operatorname{Spec} R$ of $\overline{X}_\Omega \to \operatorname{Spec}\Omega$ with geometrically connected fibers. After enlarging R, we can also spread the open immersion $X_\Omega \hookrightarrow \overline{X}_\Omega$ to an open immersion $X_R \hookrightarrow \overline{X}_R$ over R such that the boundary of X_R in $\overline{X}_R$ is an effective Cartier divisor that is étale over $\operatorname{Spec} R$. Denote the fraction field of R by L and its separable closure in Ω by $\overline{L}$. We may assume that $k \subset L$ by potentially enlarging R, because k is assumed to be finitely generated.

By [Lan24], we have an isomorphism $\pi_1^{(p')}(X_{\overline{L}},\overline{x}) \simeq \pi_1^{(p')}(X,\overline{x})$ so that it suffices to prove that the pullback of χ to $\pi_1^{(p')}(X_{\overline{L}},\overline{x})$ is finite. Notice that this pullback is again arithmetic (see also the proof of Lemma 6.17). Pick a regular closed point $z \in \operatorname{Spec} R$ and a separable closure $\overline{\kappa(z)}$ of the *finite field* $\kappa(z)$. Write $\overline{z} = \operatorname{Spec}\overline{\kappa(z)} \to X_{\overline{\kappa(z)}}$. We obtain a specialization isomorphism

$$\operatorname{sp}\colon \pi_1^{(p')}(X_{\overline{L}},\overline{x}) \simeq \pi_1^{(p')}(X_{\overline{\kappa(z)}},\overline{z})$$

by Theorem 3.2. Hence, χ defines a character of $\pi_1^{(p')}(X_{\overline{\kappa(z)}},\overline{z})$.

It remains to be seen that this character is arithmetic. Let $\Phi \in \operatorname{Gal}(\overline{\kappa(z)}/\kappa(z))$ be the Frobenius element. We also denote its image under the isomorphism $\operatorname{Gal}(\overline{\kappa(z)}/\kappa(z)) \xrightarrow{\simeq} \pi_1(\operatorname{Spec} R_z^h,\overline{z})$ by Φ. Here R_z^h denotes the henselization of the localization of R at the maximal ideal corresponding to z. Via the composition

$$\operatorname{Gal}(\overline{L}/L^h) \twoheadrightarrow \pi_1(\operatorname{Spec} R_z^h,\operatorname{Spec}\Omega) \simeq \pi_1(\operatorname{Spec} R_z^h,\overline{z}),$$

Φ lifts to an element $\widetilde{\Phi} \in \mathrm{Gal}(\overline{L}/L^h)$, where $L^h = \mathrm{Frac}\, R_z^h$. Here, surjectivity of the first map above follows from normality. Now, the outer action of $\widetilde{\Phi}$ on $\pi_1^{(p')}(X_{\overline{L}}, \overline{x})$ is compatible with the outer action of Φ on $\pi_1^{(p')}(X_{\overline{\kappa(z)}}, \overline{z})$ under the specialization isomorphism. By arithmeticity of the character on $\pi_1(X_{\overline{L}}, \overline{x})$, we find that $\widetilde{\Phi}^m$ acts trivially on it for some $m \geq 1$. Hence, Φ^m acts trivially on the character on $\pi_1(X_{\overline{\kappa(z)}}, \overline{z})$. In other words, it is arithmetic.

So we may assume that k is a finite field. The proposition then follows from [Del80, Proposition 1.3.4]. ■

Remark 6.10. The proposition above can also be proved if X^0 is a smooth connected quasi-projective variety of some higher dimension by using a Lefschetz theorem to reduce to the case that X^0 is a smooth curve; see [EK15].

Remark 6.11. To prove that a given semsimple local system on X is arithmetic or not is in general difficult, but the above proposition does provide at least one necessary condition. It is not a sufficient condition. For an example of a semisimple local system with finite determinant that is *not* arithmetic, see Remark 6.14.

6.2. **A local Kashiwara conjecture.** Let q be a power of a prime p. Let $C/\mathbb{F}_q$ be a connected normal curve and let $c \in C$ be a closed point with residue field $\kappa = \kappa(c)$. Let k be an algebraic closure of κ. We denote by $\mathcal{O} = \mathcal{O}^{\mathrm{hs}}_{C,\overline{c}}$ the strict henselization of C at $\overline{c} = \mathrm{Spec}\, k \to \mathrm{Spec}\, \kappa \to C$, the geometric k-valued point lying over c. It is again a discrete valuation ring by [Sta24, Tag 07QL]. We denote by $K = \mathrm{Frac}\, \mathcal{O}$ the fraction field of $\mathcal{O}$. Notice that K is a separable extension of $\mathbb{F}_q(C)$, the field of rational functions on C. We fix an algebraically closed field Ω containing K and let $\overline{K}$ be the induced separable closure. Throughout, $X \to \mathrm{Spec}\, \mathcal{O}$ denotes a surjective smooth separated quasi-compact morphism of schemes. We assume that $X_{\overline{K}}$ is a *connected curve.*

Proposition 6.12. *The scheme X is regular and connected. In particular, it is integral.*

Proof. Connectedness of X follows by flatness over $\mathcal{O}$ and the fact that the generic fiber is connected by assumption. For regularity, notice that the points in the generic fiber are regular by smoothness and the fact that $X_K \hookrightarrow X$ is an open immersion. For a point $x \in X$ in the special fiber, the quotient $\mathcal{O}_{X,x}/(\pi)$ is regular by smoothness of the special fiber. Now notice that π is not a zero-divisor in $\mathcal{O}_{X,x}$ by flatness and apply [Sta24, Tag 00NU]. Integrality follows from the fact that X is connected, regular and noetherian. ■

Theorem 6.13 (Arithmetic local Kashiwara conjecture). *Let $\mathbb{L}$ be an ℓ-adic local system on X. Assume that the pullback $\mathbb{L}_{\overline{K}}$ of $\mathbb{L}$ to a local system on $X_{\overline{K}}$ is arithmetic. Then $\mathbb{L}_k = \mathbb{L}|_{X_k}$ is semisimple.*

Remark 6.14. By copying the construction of Section 4.2 verbatim, we can construct a surjective smooth separated quasi-compact $\mathcal{O}$-scheme, $\mathcal{X} \to \mathrm{Spec}\, \mathcal{O}$, whose geometric generic fiber is a connected curve, together with a semisimple ℓ-adic local system $\mathbb{L}$ on

$\mathscr{X}$ such that $\mathbb{L}_k$ is *not* semisimple. The local system $\mathbb{L}_{\overline{K}}$ is semisimple by Lemma 6.23. It is *not* arithmetic, because this would contradict Theorem 6.13. Its determinant is nonetheless finite by construction, which demonstrates that having a finite determinant is a necessary but insufficient condition for arithmeticity of a local system.

By potentially shrinking C to an étale neighborhood of $\overline{c}$, we can assume that $X \to \operatorname{Spec}\mathscr{O}$ spreads to a smooth separated quasi-compact C-scheme $X_C \to C$. It is connected by the same argument used to prove connectedness of X. The idea of the proof is to "spread" the problem to the following theorem of global nature.

Theorem 6.15. *If $\mathbb{L}$ denotes an irreducible local system on X_C with finite determinant, then $\mathbb{L}_k = \mathbb{L}|_{X_k}$ is semisimple.*

We will defer its proof to the next section.

Lemma 6.16. *Let $\mathbb{L}$ be an ℓ-adic local system on X such that $\mathbb{L}_{\overline{K}}$ is arithmetic. Let $Y \to X$ be a finite connected étale cover and write $\mathbb{L}_Y = \mathbb{L}|_Y$. Then $Y_{\overline{K}}$ is a connected curve, $\mathbb{L}_{Y,\overline{K}}$ is arithmetic, and $\mathbb{L}_{Y,k}$ is semisimple if and only if $\mathbb{L}_k$ is.*

Proof. By Lemma 6.23 below, we see that $Y_{\overline{K}}$ is connected. By Proposition 2.19, we see that semisimplicty of $\mathbb{L}_{Y,\overline{K}}$, respectively $\mathbb{L}_{Y,k}$, is equivalent to semisimplicity of $\mathbb{L}_{\overline{K}}$, respectively $\mathbb{L}_k$. Arithmeticity of $\mathbb{L}_{Y,\overline{K}}$ also follows from arithmeticity of $\mathbb{L}_{\overline{K}}$: we can spread $Y_{\overline{K}} \to X_{\overline{K}}$ to a morphism $Y_F \to X_F$, with $\mathbb{F}_q(C) \subset_f F \subset \overline{K}$ a finite separable extension, such that $\mathbb{L}_{\overline{K}}$ spreads to a local system $\mathbb{L}_F$ on X_F. Then $\mathbb{L}_{Y,F} = \mathbb{L}_F|_{Y_F}$ pulls back to $\mathbb{L}_{Y,\overline{K}}$. ■

Morally, the above lemma says that we are free to replace X by a finite étale cover in the proof of Theorem 6.13.

Lemma 6.17. *Let $\mathbb{L}$ be an ℓ-adic local system on X such that $\mathbb{L}_{\overline{K}}$ is arithmetic. Let $C' \to C$ be a non-constant morphism of connected normal $\mathbb{F}_q$-curves and let $\overline{c}' \to C'$ be a geometric point over $\overline{c}$. Write $\mathscr{O}' = \mathscr{O}^{\mathrm{hs}}_{C',\overline{c}'}$, $\operatorname{Spec} K' = \operatorname{Frac}\mathscr{O}'$, $X' = X \times_{\mathscr{O}} \operatorname{Spec}\mathscr{O}'$ and $\mathbb{L}' = \mathbb{L}|_{X'}$. Let $K' \subset \Omega$ be an embedding extending the embedding $K \subset \Omega$ and let $\overline{K'}$ be the separable closure of K' in Ω. Then $\mathbb{L}'_{\overline{K'}}$ is arithmetic.*

Proof. We have $\pi_1(X_{\overline{K'}}) \simeq \pi_1(X_{\overline{K}})$, because both are isomorphic to $\pi_1(X_{K^{\mathrm{alg}}})$ by [Sta24, Tag 0BTW]. As a result, $\mathbb{L}'_{\overline{K'}}$ is semisimple. To see that $\mathbb{L}'_{\overline{K'}}$ is arithmetic, let $\mathbb{F}_q(C) \subset_f F \subset \overline{K}$ be a finite separable extension such that $\mathbb{L}_{\overline{K}}$ spreads to a local system on X_F; then $\mathbb{L}'_{\overline{K'}}$ spreads to a local system on $X_{\mathbb{F}_q(C')F}$. ■

In the lemma above, notice that $X'_k = X_k$ and $\mathbb{L}'_k = \mathbb{L}_k$. Furthermore, $X'_{\overline{K'}} = X_{\overline{K'}}$ is again a connected curve. Morally, the above lemma says that we are free to replace C by a curve over it in the proof of Theorem 6.13.

The proof of Theorem 6.13 now proceeds by repeatedly applying Lemma 6.16 and 6.17 to show that we can, without loss of generality, impose additional conditions on the objects of Theorem 6.13 until we have reduced to Theorem 6.15.

Lemma 6.18. *In the context of Theorem 6.13, we can assume without loss of generality to the theorem as stated that $\mathbb{L}$, respectively $\mathbb{L}_{\overline{K}}$, factors over the maximal prime-to-p quotient of $\pi_1(X)$, respectively $\pi_1(X_{\overline{K}})$.*

Proof. Denote by $\overline{x} \to X_{\overline{K}}$ a geometric point of $X_{\overline{K}}$. We find $E/\mathbb{Q}_\ell$ an ℓ-adic field so that the monodromy representation of $\mathbb{L}$, $\rho\colon \pi_1(X,\overline{x}) \to \mathrm{GL}_r(\overline{\mathbb{Q}}_\ell)$, factors as $\pi_1(X,\overline{x}) \to \mathrm{GL}_r(E) \hookrightarrow \mathrm{GL}_r(\overline{\mathbb{Q}}_\ell)$. By Remark 4.13, there is an open pro-ℓ subgroup $\Gamma_\ell \subset \mathrm{GL}_r(E)$. So we can find $U \subset \pi_1(X,\overline{x})$ an open subgroup such that U lands in Γ_ℓ under ρ. Let $Y \to X$ be the the finite étale cover corresponding to U. Denote by $\mathbb{L}_Y$ the local system on Y obtained by pulling back $\mathbb{L}$ to a local system on Y, and denote by $\rho_Y\colon \pi_1(Y,\overline{y}) \to \mathrm{GL}_r(\overline{\mathbb{Q}}_\ell)$ its monodromy representation. Here $\overline{y}$ is a geometric point of $Y_{\overline{K}}$ lifting $\overline{x}$. Then by construction $\rho_Y(\pi_1(Y,\overline{y})) \subset \Gamma_\ell$. Since ℓ is assumed to be different from p, the representation ρ_Y factors over the maximal prime-to-p quotient of $\pi_1(Y,\overline{y})$. Then, being the pullback of ρ_Y, the monodromy representation of $\mathbb{L}_{Y,\overline{K}}$ factors over the maximal prime-to-p quotient of $\pi_1(Y_{\overline{K}},\overline{y})$. By Lemma 6.16, we may assume that $Y = X$ without loss of generality to Theorem 6.13. ■

Lemma 6.19. *In the context of Theorem 6.13, we can assume without loss of generality to the theorem as stated that $X_{C\setminus\{c\}} = X_C \times_C C\setminus\{c\} \to C\setminus\{c\}$ admits a* nice compactification $X_{C\setminus\{c\}} \hookrightarrow \overline{X}_{C\setminus\{c\}} \to C\setminus\{c\}$ *with geometrically connected fibers; i.e., there is an open immersion $X_{C\setminus\{c\}} \hookrightarrow \overline{X}_{C\setminus\{c\}}$ of $C\setminus\{c\}$-schemes such that*

(i) $\overline{X}_{C\setminus\{c\}} \to C\setminus\{c\}$ is smooth and proper with geometrically connected fibers;
(ii) the boundary of $X_{C\setminus\{c\}}$ in $\overline{X}_{C\setminus\{c\}}$ is an effective Cartier divisor that is étale over $C\setminus\{c\}$;
(iii) the geometric fibers of $\overline{X}_{C\setminus\{c\}} \to C\setminus\{c\}$ are conneceted.

Additionally, we can assume the existence of a section $C\setminus\{c\} \to X_{C\setminus\{c\}}$.

The relevance of the above lemma comes from the fact that we obtain the split homotopy exact sequence

$$\begin{tikzcd} 1 \arrow[r] & \pi_1^{(p')}(X_{\overline{K}},\overline{x}) \arrow[r] & \pi_1'(X_{C\setminus\{c\}},\overline{x}) \arrow[r] & \pi_1(C\setminus\{c\},\overline{x}) \arrow[r] \arrow[l, dashed, bend right] & 1 \end{tikzcd} \tag{6.2}$$

from Proposition 3.3. Here $\overline{x} \to X_{\overline{K}}$ denotes the geometric point induced by $\operatorname{Spec}\Omega \to C\setminus\{c\} \to X_{C\setminus\{c\}}$.

Proof of Lemma 6.19. Denote by K^{perf} the perfect closure of K in Ω. Then there exists a unique regular compactification $\overline{X}_{K^{\mathrm{perf}}} \to \operatorname{Spec} K^{\mathrm{perf}}$, which is smooth by the fact that K^{perf} is perfect. We can find a finite extension $K \subset_f K' \subset K^{\mathrm{perf}}$ such that the compactification $\overline{X}_{K^{\mathrm{perf}}} \to \operatorname{Spec} K^{\mathrm{perf}}$ spreads to a smooth proper morphism $\overline{X}_{K'} \to \operatorname{Spec} K'$ that defines a smooth compactification of $X_{K'}$. Next, we find a finite (inseparable) extension $\mathbb{F}_q(C) \subset \mathbb{F}_q(C')$ such that K' is the compositum of K and $\mathbb{F}_q(C')$ inside Ω. By potentially enlarging K', we can assume that $\mathbb{F}_q(C) \subset \mathbb{F}_q(C')$ is a normal extension. The extension $\mathbb{F}_q(C) \subset \mathbb{F}_q(C')$ corresponds to a map of $\mathbb{F}_q$-curves $C' \to C$. We let $\overline{c}' \to C'$ be a geometric point over $\overline{c}$ and we set $\mathscr{O}' = \mathscr{O}^{\mathrm{hs}}_{C',\overline{c}'}$. After choosing an embedding $\operatorname{Frac}\mathscr{O}' \subset \Omega$ extending the embedding $K \subset \Omega$, we get $K' = K\mathbb{F}_q(C') \subset \operatorname{Frac}\mathscr{O}'$; hence, by enlarging K', we may assume that $\operatorname{Frac}\mathscr{O}' = K'$. By Lemma 6.17, we can assume that $C' = C$ without loss of generality to Theorem 6.13.

So, we have a smooth compactification $\overline{X}_K \to \operatorname{Spec} K$ of $X_K \to \operatorname{Spec} K$. We spread this to a smooth compactification of $X_{C\setminus\{c\}}$. Indeed, we have

$$\operatorname{Spec} K = \varprojlim C' \setminus \{c', c'_2, \dots, c'_n\},$$

where the limit ranges over $(C', \overline{c}') \to (C, \overline{c})$, the connected étale neighborhoods of $\overline{c}$, and $c', c'_2, \dots, c'_n$ are the points of C' over c. So, by potentially shrinking C, we can assume the existence of a smooth proper morphism $\overline{X}_{C\setminus\{c\}} \to C \setminus \{c\}$ that is a smooth compactification of $X_{C\setminus\{c\}} \to C\setminus\{c\}$ in such a way that the boundary of $X_{C\setminus\{c\}}$ in $\overline{X}_{C\setminus\{c\}}$ is an effective Cartier divisor that is étale over $C \setminus \{c\}$. By shrinking C a little further, we may also assume that the fibers of $\overline{X}_{C\setminus\{c\}} \to C\setminus\{c\}$ are geometrically connected by [Sta24, Tag 055G]. Finally, by spreading a K-point of X_K, we can assume the existence of a section $C\setminus\{c\} \to X_{C\setminus\{c\}}$. ■

Remark 6.20. In the proof of the lemma above, we crucially use that X_K is a curve to guarantee the existence of a nice compactification. If X_K is instead some higher dimensional variety, we would need to be able to assume that X_K is an open subscheme of a smooth proper K-scheme $\overline{X}_K$ such that the boundary $\overline{X}_K \setminus X_K$ is a normal crossing divisor. If K were a field of characteristic 0, this would be possible by Hironaka's Theorem, but it is unknown whether such a compactification always exists in positive characteristic.

Lemma 6.21. *In the context of Theorem 6.13, we can assume without loss of generality to the theorem as stated that each of the irreducible constituents $\mathbb{L}_{1,\overline{K}}, \dots, \mathbb{L}_{m,\overline{K}}$ of $\mathbb{L}_{\overline{K}}$ can be spread to a local system on $X_{C\setminus\{c\}}$ with finite determinant.*

Proof. Let $\overline{x} \to X_{\overline{K}}$ be the geometric point defined below (6.2). Let $\rho_{i,\overline{K}} \colon \pi_1(X_{\overline{K}}, \overline{x}) \to \mathrm{GL}(\mathbb{L}_{i,\overline{K},\overline{x}})$ be the monodromy representation of $\mathbb{L}_{i,\overline{K}}$. The representations $\rho_{i,\overline{K}}$ are precisely the irreducible constituents of the monodromy representation of $\mathbb{L}_{\overline{K}}$. By Lemma 6.18, we can assume that each of them factors over the maximal prime-to-p quotient $\pi_1^{(p')}(X_{\overline{K}}, \overline{x})$ of $\pi_1(X_{\overline{K}}, \overline{x})$. The exact sequence (6.2) and the theory in Section 5 gives us an action of $\pi_1(C\setminus\{c\}, \overline{x})$ on the space of isomorphism classes of semisimple ℓ-adic representations of $\pi_1^{(p')}(X_{\overline{K}}, \overline{x})$.

We argue that the orbit of $[\rho_{i,\overline{K}}]$ under $\pi_1(C\setminus\{c\}, \overline{x})$ is finite. The section $C\setminus\{c\} \to X_{C\setminus\{c\}}$ from Lemma 6.19 gives rise to an $\mathbb{F}_q(C)$-rational point $\operatorname{Spec}\mathbb{F}_q(C) \to X_{\mathbb{F}_q(C)}$ on $X_{\mathbb{F}_q(C)} \to \operatorname{Spec}\mathbb{F}_q(C)$, and $\overline{x}$ is a geometric point lying over this rational point. As a result, we obtain a homomorphism of exact sequences with compatible splittings

$$\begin{tikzcd}
1 \arrow[r] & \pi_1^{(p')}(X_{\overline{K}},\overline{x}) \arrow[r] & \pi_1'(X_{C\setminus\{c\}},\overline{x}) \arrow[r] & \pi_1(C\setminus\{c\},\overline{x}) \arrow[r] \arrow[l, dashed, bend right] & 1 \\
1 \arrow[r] & \pi_1(X_{\overline{K}},\overline{x}) \arrow[r] \arrow[u] & \pi_1(X_{\mathbb{F}_q(C)},\overline{x}) \arrow[r] \arrow[u] & \mathrm{Gal}(\overline{K}/\mathbb{F}_q(C)) \arrow[r] \arrow[u, two heads] \arrow[l, dashed, bend left] & 1.
\end{tikzcd}$$

It follows that the conjugation action of $\mathrm{Gal}(\overline{K}/\mathbb{F}_q(C))$ on $\pi_1(X_{\overline{K}},\overline{x})$ is compatible with the conjugation action of $\pi_1(C\setminus\{c\},\overline{x})$ on $\pi_1^{(p')}(X_{\overline{K}},\overline{x})$. Because $C\setminus\{c\}$ is normal, the homomorphism $\mathrm{Gal}(\overline{K}/\mathbb{F}_q(C)) \twoheadrightarrow \pi_1(C\setminus\{c\},\overline{x})$ is surjective. By Proposition 6.7, the representations $\rho_{i,\overline{K}}$ are arithmetic; hence, their orbit under $\mathrm{Gal}(\overline{K}/\mathbb{F}_q(C))$ is finite. Therefore, the orbit of $\rho_{i,\overline{K}}$ under $\pi_1(C\setminus\{c\},\overline{x})$ is finite. By Theorem 5.6, we find an open subgroup $U\subset\pi_1(C\setminus\{c\},\overline{x})$ such that every representation $\rho_{i,\overline{K}}$ spreads to a representation of $\pi_1^{(p')}(X_{\overline{K}},\overline{x})\rtimes U \hookrightarrow \pi_1'(X_{C\setminus\{c\}},\overline{x})$.

The subgroup $U\subset\pi_1(C\setminus\{c\},\overline{x})$ corresponds to a finite cover $C'\to C$, étale over $C\setminus\{c\}$, but perhaps ramified over c. By Lemma 6.17 we can assume $C'=C$, and hence that $\rho_{i,\overline{K}}$ spreads to a representation of $\pi_1'(X_{C\setminus\{c\}},\overline{x})$. Thus, $\mathbb{L}_{i,\overline{K}}$ spreads to a local system on $X_{C\setminus\{c\}}$. Notice also that the local systems $\mathbb{L}_{i,\overline{K}}$ have finite determinant by Proposition 6.9, and so by Theorem 5.6 we can assume that the resulting local systems on $X_{C\setminus\{c\}}$ have finite determinant. ■

Lemma 6.22. *In the context of Theorem 6.13, we can assume without loss of generality to the theorem as stated that each of the irreducible constituents* $\mathbb{L}_{1,\overline{K}},\ldots,\mathbb{L}_{m,\overline{K}}$ *of* $\mathbb{L}_{\overline{K}}$ *spreads to a local system on* X_C *with finite determinant.*

Proof. By Lemma 6.21, we can spread the irreducible constituents $\mathbb{L}_{1,\overline{K}},\ldots,\mathbb{L}_{m,\overline{K}}$ of $\mathbb{L}_{\overline{K}}$ to local systems $\mathbb{L}_1,\ldots,\mathbb{L}_m$ on $X_{C\setminus\{c\}}$ with finite determinant. We next argue that the local systems $\mathbb{L}_i$ can be extended to local systems on X_C, or, in other words, that they are *unramified over* X_C. By Lemma 6.25 below, it suffices to prove that each of the local systems $\mathbb{L}_{i,K}$ is unramified over X. Let $\overline{x}=\operatorname{Spec}\Omega\to X_{\overline{K}}$ be a geometric point. Denote by $\rho_i\colon \pi_1(X_{C\setminus\{c\}},\overline{x})\to \mathrm{GL}(\mathbb{L}_{i,\overline{x}})$ the monodromy representation of $\mathbb{L}_i$, and by $\rho\colon\pi_1(X,\overline{x})\to \mathrm{GL}(\mathbb{L}_{\overline{x}})$ the monodromy representation of $\mathbb{L}$. Define $\mathrm{Gal}(\overline{K}/K)$-representations $V_1,\ldots,V_m$ by

$$V_i = \mathrm{Hom}(\rho_i|_{\pi_1(X_{\overline{K}},\overline{x})}, \rho|_{\pi_1(X_{\overline{K}},\overline{x})}).$$

A priori, these are only $\pi_1(X_K,\overline{x})$-representations, with the action of $\pi_1(X_K,\overline{x})$ on V_i defined as in (6.5); but this action factors over the quotient by $\pi_1(X_{\overline{K}},\overline{x})$. By Lemma 6.24, we have an isomorphism of $\pi_1(X_K,\overline{x})$-representations

$$\rho|_{\pi_1(X_K,\overline{x})} \simeq \rho_1|_{\pi_1(X_K,\overline{x})}\otimes V_1\oplus\ldots\oplus\rho_m|_{\pi_1(X_K,\overline{x})}\otimes V_m. \tag{6.3}$$

Consider the diagram with exact rows and columns

$$\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & & \\
1 \arrow[r] & H_{\overline{K}} \arrow[r] \arrow[d] & \pi_1(X_{\overline{K}},\overline{x}) \arrow[r] \arrow[d] & \pi_1(X,\overline{x}) \arrow[r] \arrow[d, "="] & 1 \\
1 \arrow[r] & H_K \arrow[r] \arrow[d] & \pi_1(X_K,\overline{x}) \arrow[r] \arrow[d] & \pi_1(X,\overline{x}) \arrow[r] & 1 \\
 & Q \arrow[r, dashed, "\simeq"] \arrow[d] & \mathrm{Gal}(\overline{K}/K) \arrow[d] & & \\
 & 1 & 1. & &
\end{tikzcd} \tag{6.4}$$

A diagram chase shows that the dashed arrow $Q \dashrightarrow \mathrm{Gal}(\overline{K}/K)$ in (6.4) is an isomorphism.

We know that $\rho|_{\pi_1(X_K,\overline{x})}$ is trivial on H_K. So, for $\sigma \in H_K$, $\rho_i(\sigma) \otimes \sigma|_{V_i}$ is the identity by (6.3). This implies that $\rho_i(\sigma) = \alpha \cdot \mathrm{id}$ and $\sigma|_{V_i} = \alpha^{-1} \cdot \mathrm{id}$ for some $\alpha \in \overline{\mathbb{Q}}_\ell^\times$. Furthermore, α must be a root of unity, because ρ_i has finite determinant. Hence; the representation $\rho_i|_{H_K}$ is just a finite character, and so the subgroup $U_i \subset H_K$ consisting of $\sigma \in H_K$ such that $\rho_i(\sigma)$ is trivial, is open in H_K. Furthermore, we have $H_{\overline{K}} \subset U_i$, because $\sigma|_{V_i}$ is trivial for $\sigma \in H_{\overline{K}}$.

The group U_i defines an open subgroup of Q, denoted $\overline{U_i}$. By Galois theory, $\overline{U_i} \subset Q \simeq \mathrm{Gal}(\overline{K}/K)$ corresponds to a finite extension $K \subset_f K'_i \subset \overline{K}$. It follows that $U_i = \ker \pi_1(X_{K'_i},\overline{x}) \to \pi_1(X,\overline{x})$.

Setting K' equal to the compositum of all the K'_i, we can conclude that all the representations $\rho_i|_{\pi_1(X_{K'},\overline{x})}$ vanish on $\ker \pi_1(X_{K'},\overline{x}) \to \pi_1(X,\overline{x})$, and hence that the local systems $\mathbb{L}_{i,K'}$ extend to local systems on X. By an analogous argument to the one in Lemma 6.19, we can now assume without loss of generality that $K' = K$ and that each of the local systems $\mathbb{L}_{i,K}$ are unramified over X. By Lemma 6.25, we then find that the local systems $\mathbb{L}_i$ are unramified over X_C. ■

Proof of Theorem 6.13. By Lemma 6.22, we obtain local systems $\mathbb{L}_1,\dots,\mathbb{L}_m$ on X_C with finite determinant by spreading the irreducible constituents $\mathbb{L}_{1,\overline{K}},\dots,\mathbb{L}_{m,\overline{K}}$ of $\mathbb{L}_{\overline{K}}$. We find

$$\mathbb{L} \simeq \mathbb{L}_1^{\oplus e_1}|_X \oplus \dots \oplus \mathbb{L}_m^{\oplus e_m}|_X,$$

where e_i is the multiplicity with which $\mathbb{L}_{i,\overline{K}}$ occurs in $\mathbb{L}|_{X_{\overline{K}}}$, by the surjectivity of $\pi_1(X_{\overline{K}}) \to \pi_1(X)$ (Lemma 6.23). We set

$$\mathbb{L}_C = \mathbb{L}_1^{\oplus e_1} \oplus \dots \oplus \mathbb{L}_m^{\oplus e_m}.$$

Theorem 6.15 shows that each of the pullbacks $\mathbb{L}_{i,k}$ is semisimple, and hence the local system $\mathbb{L}_{C,k} = \mathbb{L}_k$ is as well. ■

6.2.1. *Some auxiliary results.*

Lemma 6.23. *The homomorphism $\pi_1(X_{\overline{K}}) \to \pi_1(X)$ is surjective.*

Proof. We're to prove that if $Y \to X$ is a connected étale cover, its pullback $Y_{\overline{K}} \to X_{\overline{K}}$ is connected. Notice that since X is normal and irreducible by Proposition 6.12, the map $\pi_1(X_K) \to \pi_1(X)$ is surjective. Indeed, $X_K \hookrightarrow X$ is an open immersion into an irreducible normal scheme; hence, this follows from [Sza09, Proposition 5.4.9]. It follows that $Y_K \to X_K$ is connected.

The composition $Y \to X \to \operatorname{Spec}\mathcal{O}$ is smooth. By assumption, the special fiber of X is non-empty, and hence the special fiber of Y is non-empty. By [Mil16, Chapter I, Exercise 4.13], there exists a section $\operatorname{Spec}\mathcal{O} \to Y$. This section induces a K-point $\operatorname{Spec} K \to Y_K$. By [Sta24, Tag 04KV], the étale cover $Y_{\overline{K}} \to X_{\overline{K}}$ is connected. ■

Let G be a profinite group, and let $H \subset G$ be a closed subgroup. Suppose we have an ℓ-adic representation $\rho\colon G \to \mathrm{GL}_r(\overline{\mathbb{Q}}_\ell)$ such that its restriction to an H-representation, ρ_H, is semisimple with (pairwise non-isomorphic) irreducible constituents $\rho_{1,H},\dots,\rho_{m,H}$. Suppose now that $\rho_{1,H},\dots,\rho_{m,H}$ extend to continuous representations $\rho_1,\dots,\rho_m\colon G \to \mathrm{GL}_r(\overline{\mathbb{Q}}_\ell)$. Set $V_i = \mathrm{Hom}_H(\rho_i|_H, \rho|_H)$. It has a continuous $\overline{\mathbb{Q}}_\ell$-linear action of G given by

$$^{\sigma}\varphi(x) = \sigma\varphi(\sigma^{-1}x) \tag{6.5}$$

for $\sigma \in G$ and $\varphi \in V_i$. Notice that H acts trivially on V_i. We obtain a homomorphism of G-representations

$$\rho_1 \otimes V_1 \oplus \ldots \oplus \rho_m \otimes V_m \to \rho \tag{6.6}$$

given by $x \otimes \varphi \mapsto \varphi(x)$.

Lemma 6.24. *The homomorphism of 6.6 is an isomorphism of G-representations.*

Proof. It is easily seen to be an isomorphism of representations over H. So it is a G-equivariant homomorphism and an isomorphism on the level of vector spaces; hence, it is an isomorphism of G-representations. ■

Lemma 6.25. *Let $\mathbb{L}$ be an ℓ-adic local system on $X_{C\setminus\{c\}} = X_C \times_C C \setminus \{c\}$, and denote by $\mathbb{L}_K = \mathbb{L}|_{X_K}$ the pullback of $\mathbb{L}$ to X_K. Then $\mathbb{L}$ is unramified over X_C if and only if $\mathbb{L}_K$ is unramified over X.*

Proof. We need only prove that if $\mathbb{L}_K$ is unramified over X, then also $\mathbb{L}$ is uramified over X_C. Let $\eta \in X_c$ be a generic point, and let $\eta' \in X_k$ be a generic point lying over η. Let $\overline{\eta}' \to X$ be a geometric point lying over η', and let $\overline{\eta} \to X_C$ be the induced geometric point of X_C. By purity of the branch locus, it suffices to prove that the pullback of $\mathbb{L}$ along $\operatorname{Spec} K^{\mathrm{hs}}_{\overline{\eta}} \to X_{C\setminus\{c\}}$ is trivial. Here $K^{\mathrm{hs}}_{\overline{\eta}} = \operatorname{Frac}\mathcal{O}^{\mathrm{hs}}_{X_C,\overline{\eta}}$ is the fraction field of the strict henselization of X_C at $\overline{\eta}$. Set $K^{\mathrm{hs}}_{\overline{\eta}'} = \operatorname{Frac}\mathcal{O}^{\mathrm{hs}}_{X,\overline{\eta}'}$. By the fact that $X \to X_C$ is weakly étale and [Sta24, Tag 094Z], we have an isomorphism $\operatorname{Spec} K^{\mathrm{hs}}_{\overline{\eta}'} \xrightarrow{\simeq} \operatorname{Spec} K^{\mathrm{hs}}_{\overline{\eta}}$ that fits into the commutative square

$$\begin{tikzcd}
\operatorname{Spec} K^{\mathrm{hs}}_{\overline{\eta}'} \arrow[r] \arrow[d] & X_K \arrow[d] \\
\operatorname{Spec} K^{\mathrm{hs}}_{\overline{\eta}} \arrow[r] & X_{C\setminus\{c\}}.
\end{tikzcd}$$

By the assumption that $\mathbb{L}_K$ is unramified, the pullback $\mathbb{L}_K|_{\operatorname{Spec} K^{\mathrm{hs}}_{\overline{\eta}'}}$ is trivial. Therefore also $\mathbb{L}|_{\operatorname{Spec} K^{\mathrm{hs}}_{\overline{\eta}}}$ is trivial. ■

6.3. **Proof of Theorem 6.15.** Notation is as before.

6.3.1. *Weights.* Let $\mathscr{X}$ be a separated scheme of finite type over $\mathbb{F}_q$ and let $\mathscr{F}$ be a $\overline{\mathbb{Q}}_\ell$-sheaf on $\mathscr{X}$. Given a closed point $x \in \mathscr{X}$, the *geometric Frobenius* $F_x \in \operatorname{Gal}(k/\kappa(x))$ at x is defined to be the inverse of the usual Frobenius $\alpha \mapsto \alpha^{N(x)}$. Here $N(x)$ denotes the cardinality of the residue field $\kappa(x)$. The geometric Frobenius F_x gives rise to a $\overline{\mathbb{Q}}_\ell$-linear operator

$$F_x\colon \mathscr{F}_{\overline{x}} \to \mathscr{F}_{\overline{x}},$$

where $\overline{x} = \operatorname{Spec} k \to \mathscr{X}$ denotes the k-valued geometric point induced by x.

Definition 6.26 ([Del80, Définition 1.2.2])**.** Let $n \in \mathbb{Z}$ be a integer. The sheaf $\mathscr{F}$ is said to be *pure of weight n* if for every closed point $x \in \mathscr{X}$, all the eigenvalues α of

$$F_x\colon \mathscr{F}_{\overline{x}} \to \mathscr{F}_{\overline{x}}$$

are algebraic numbers, and such that for every embedding $\iota\colon \mathbb{Q}(\alpha) \hookrightarrow \mathbb{C}$ we have

$$|\iota(\alpha)|^2 = N(x)^n.$$

If $\mathscr{F}$ is pure of some weight n, then we say that $\mathscr{F}$ is pure.

Remark 6.27. More recent definitions of weights allow weights to be arbitrary real numbers and do not require the eigenvalues of the Frobenii to be algebraic. See, for instance, [Wei01, Chapter I, Definition 2.1].

6.3.2. *Proof of Theorem 6.15.*

Lemma 6.28. *Let $\mathscr{X}$ be a normal variety of finite type over $\mathbb{F}_q$. Let $\mathbb{L}$ be an irreducible local system with finite determinant. Then $\mathbb{L}$ is pure of weight* 0.

Proof. This is [Laf02, Proposition VII.7]. Lafforgue proves this by first reducing to the case that $\mathscr{X}$ is a curve. This reduction step is, however, flawed. See [Del12, Section 0.7 and Sections 1.5-1.9] for the corrected proof. ■

Lemma 6.29. *Let $\mathscr{X}_0$ be a normal finite type scheme over $\mathbb{F}_q$. Let $\mathbb{L}_0$ be a pure local system on $\mathscr{X}_0$. Then $\mathbb{L}$, the pullback of $\mathbb{L}_0$ to $\mathscr{X} := \mathscr{X}_0 \times_{\mathbb{F}_q} \operatorname{Spec}\overline{\mathbb{F}}_q$, is semisimple.*

Proof. This is [Del80, Theorem 3.4.1(iii)]. ■

Proof of Theorem 6.15. By Lemma 6.28, $\mathbb{L}$ is pure. Then also $\mathbb{L}|_{X_c}$, the pullback of $\mathbb{L}$ to a local system on the fiber of $X_C \to C$ over c, is pure. By Lemma 6.29, $\mathbb{L}_k$ is semisimple. ■

APPENDIX A. MOCHIZUKI'S COUNTEREXAMPLE TO A COMPLEX-GEOMETRIC VERSION OF THE NAIVE LOCAL KASHIWARA CONJECTURE

Let Δ denote the complex unit disc, and let $\Delta^* = \Delta \setminus \{0\}$ be the punctured unit disc. In this appendix, we give a counterexample to the complex-geometric version of the naive local Kashiwara conjecture suggested in the introduction. Specifically, we show that the answer to Question A.1 below is "No".

Question A.1. Given a smooth map $f \colon \mathscr{X} \to \Delta$, with $\mathscr{X}$ a complex manifold, and a semisimple complex local system $\mathbb{L}$ on $\mathscr{X}$. Is the pullback $\mathbb{L}_0$ of $\mathbb{L}$ to a local system on $\mathscr{X}_0$ again semisimple? Here $\mathscr{X}_0$ denotes the fiber of f over 0.

The counterexample constructed in this section is originally due to Takurō Mochizuki. Recall the correspondence between complex local systems and complex monodromy representations from (2.1).

A.1. **Topological aspects of the construction.**

Let $f \colon X \to \Delta$ be an elliptic fibration such that the fibre X_0 over 0 is reduced and of type I_1 in the Kodaira classification of singular fibers of elliptic fibrations described in [Kod63]. We take this to mean that f is a proper holomorphic map, f is smooth over Δ^*, the fiber over any point of Δ^* is a smooth connected genus one curve, and the special fiber X_0 is a rational curve with a single node (or ordinary double point). We denote the singular point of X_0 by p_0. Specifically, p_0 has a neighborhood that is complex-analytically isomorphic to a neighborhood of the origin in the zero locus cut out by $xy = 0$ in $\mathbb{C}^2$.

Example A.2. Consider

$$X = \{((x:y:z),\lambda) \in \mathbb{P}^2(\mathbb{C}) \times \Delta : y^2z = 4x^3 + (\lambda-3)xz^2 + (\lambda-1)z^3\}$$

with the obvious projection map $X \to \Delta$. It is an elliptic fibration. Its fiber over 0 is the nodal cubic

$$X_0 \colon y^2z = 4x^3 - 3xz^2 - z^3 = (2x+z)^2(x-z).$$

Denote by $\varphi \colon \mathbb{P}^1 \to X_0$ the normalization. We can assume that $\varphi^{-1}(p_0) = \{0,\infty\}$. Define points in $\mathbb{P}^1$ by

$$\widetilde{z}_0 = 1, \widetilde{z}_1 = 1+\sqrt{-1} \text{ and } \widetilde{z}_2 = 1-\sqrt{-1},$$

and set

$$z_i = \varphi(\widetilde{z}_i) \quad (i = 0,1,2).$$

Notice that these are all smooth points. By potentially shrinking Δ, we can assume that there exist holomorphic sections $s_1 \colon \Delta \to X$ and $s_2 \colon \Delta \to X$ of f such that $z_1 \in s_1(\Delta), z_2 \in s_2(\Delta)$ and $s_1(\Delta) \cap s_2(\Delta) = \emptyset$. Indeed, $X \setminus \{p_0\} \to \Delta$ is smooth, and smooth maps between complex manifolds always admit local sections at all points.

Consider the loops $\gamma_i : ([0,1],\{0,1\}) \to (\mathbb{P}^1 \setminus \{\widetilde{z}_1, \widetilde{z}_2\}, \widetilde{z}_0)$ for $i = 1,2$ given by

$$\gamma_1(t) = 1 + \sqrt{-1} - \sqrt{-1}\exp(2\pi\sqrt{-1}t) \quad (0 \le t \le 1),$$
$$\gamma_2(t) = 1 - \sqrt{-1} + \sqrt{-1}\exp(2\pi\sqrt{-1}t) \quad (0 \le t \le 1).$$

We denote the induced elements in $\pi_1(\mathbb{P}^1 \setminus \{\widetilde{z}_1, \widetilde{z}_2\}, \widetilde{z}_0)$ by $[\gamma_1]$ and $[\gamma_2]$ repectively. They both generate $\pi_1(\mathbb{P}^1 \setminus \{\widetilde{z}_1, \widetilde{z}_2\}, \widetilde{z}_0)$. Furthermore, $[\gamma_1]$ and $[\gamma_2]$ and are inverse to each other. Define paths ρ_1 and ρ_2 by

$$\begin{aligned} \rho_1 \colon [0,1/2] &\to \mathbb{P}^1 \\ t &\mapsto 1 - 2u, \end{aligned}$$

and

$$\begin{aligned} \rho_2 \colon [1/2,1] &\to \mathbb{P}^1 \\ t &\mapsto \frac{1-u}{u-\frac{1}{2}} + 1. \end{aligned}$$

Here $\rho_2(\frac{1}{2})$ is sent to $\infty = (1:0) \in \mathbb{P}^1$. Composing both paths with φ and glueing them together we obtain a loop $\rho \colon ([0,1],\{0,1\}) \to (X_0 \setminus D, z_0)$.

Lemma A.3. *The group $\pi_1(X_0 \setminus D, z_0)$ is freely generated by $\varphi_*[\gamma_1]$ and $[\rho]$.*

Proof. Topologically, $\varphi \colon \mathbb{P}^1 \setminus \{\widetilde{z}_1, \widetilde{z}_2\} \to X \setminus D$ is the quotient map for the equivalence relation on $\mathbb{P}^1 \setminus \{\widetilde{z}_1, \widetilde{z}_2\}$ glueing 0 and ∞ together. The result then follows by the Van Kampen theorem. ■

Lemma A.4. *The inclusion of the fiber $i_0 \colon X_0 \setminus D \hookrightarrow X \setminus D$ is a homotopy equivalence.*

Proof. This uses a slightly more general version of [Cle77, Theorem 5.7]. ■

A.2. **Aspects of the construction from representation theory.** We define the representation $\kappa_0 \colon \pi_1(X_0 \setminus D, z_0) \to \mathrm{GL}_2(\mathbb{C})$ by

$$\kappa_0(\varphi_*[\gamma_1]) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \kappa_0([\rho]) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Lemma A.5. *(i) The representation κ_0 is irreducible.*
(ii) The representation $\kappa_0 \circ \varphi_* \colon \pi_1(\mathbb{P}^1 \setminus \{\widetilde{z}_1, \widetilde{z}_2\}, \widetilde{z}_0) \to \mathrm{GL}_2(\mathbb{C})$ is not semi-simple.

Proof. (i) This is clear by the fact that the image of κ_0 does not fix any non-trivial subspaces of $\mathbb{C}^2$.
(ii) The image $\kappa_0 \circ \varphi_*$ fixes only the subspace $\mathbb{C} \cdot (1,0)$ of $\mathbb{C}^2$.

■

Applying Lemma A.4, we obtain an irreducible representation

$$\pi_1(X \setminus D, z_0) \to \mathrm{GL}_2(\mathbb{C}).$$

To find an answer to Question A.1, we now proceed as follows. Let $\mathscr{X} := (X \setminus D)^{\mathrm{sm}} \to \Delta$ be the smooth locus of the relative curve $X \setminus D \to \Delta$. It is precisely $X \setminus D$ with the nodal point p_0 of the singular fiber removed. Its fiber over 0 is $\mathscr{X}_0 = (X_0 \setminus D)^{\mathrm{sm}} := X_0 \setminus (D \cup \{p_0\})$. Let $i_0 \colon \mathscr{X}_0 \hookrightarrow \mathscr{X}$ denote the inclusion map. The fundamental group $\pi_1(\mathscr{X}, z_0)$ is isomorphic to $\pi_1(X \setminus D, z_0)$, and so we obtain from κ_0 an irreducible representation

$$\kappa : \pi_1(\mathscr{X}, z_0) \to \mathrm{GL}_2(\mathbb{C}).$$

We argue that the restriction of this representation along the fiber over 0 is *not* semisimple. We have the commutative diagram of groups

$$\begin{tikzcd}
\pi_1(\mathbb{P}^1 \setminus \{\widetilde{z}_1, \widetilde{z}_2, 0, \infty\}, \widetilde{z}_0)\}) \arrow[r, "\simeq"] \arrow[d, two heads] & \pi_1(\mathscr{X}_0, z_0) \arrow[r, "i_{0,*}"] \arrow[d] & \pi_1(\mathscr{X}, z_0) \arrow[d, "\simeq"] \\
\pi_1(\mathbb{P}^1 \setminus \{\widetilde{z}_1, \widetilde{z}_2\}, \widetilde{z}_0) \arrow[r, "\varphi_*"] & \pi_1(X_0 \setminus D, z_0) \arrow[r, "\simeq"] & \pi_1(X \setminus D, z_0).
\end{tikzcd}$$

This diagram shows that the representation $\kappa \circ i_{0,*} \colon \pi_1((X_0 \setminus D)^{\mathrm{sm}}, z_0) \to \mathrm{GL}_2(\mathbb{C})$ has the same image as $\kappa_0 \circ \varphi_*$. By Lemma A.5 we see that $\kappa \circ i_{0,*}$ is not semi-simple, since $\kappa_0 \circ \varphi_*$ is not. This gives a negative answer to Question A.1.

APPENDIX B. CONTINUOUS NON-ABELIAN GALOIS COHOMOLOGY

Throughout, G is a profinite group. We collect a few facts regarding continuous non-abelian cohomology used in Section 5. Although there are many references on both continuous cohomology and non-abelian cohomology, the author was unable to find any references regarding the cohomology of profinite groups with non-discrete and non-abelian coefficients. Many of the statements in this appendix are straightforward generalizations of well known results. In particular, we mimic [Ser79, Appendix to Chapter VII].

B.1. **The cohomology groups.**

Definition B.1. A G-group T is a topological group T (perhaps non-abelian) with a continuous action of G. A morphism of topological G-groups is a continuous homomorphism compatible with the actions of G.

If T is a G-group, $t \in T$ is an element of T and $\sigma \in G$, is an element of G, then we will denote the image of t under the action of σ by ${}^{\sigma}t$.

Definition B.2. Let T be a G-group. We define a continuous one-cocycle of G with coefficients in T to be a continuous map of spaces

$$c\colon G \to T \quad \sigma \mapsto c_\sigma$$

such that for all $\sigma, \tau \in G$ we have

$$c_{\sigma\tau} = c_\sigma {}^{\sigma}c_\tau.$$

Two one-cocyles c and b are said to be cohomologous if there is $t \in T$ such that for all $\sigma \in G$ we have

$$c_\sigma = t^{-1} b_\sigma {}^{\sigma}t.$$

For a G-group T, "being cohomologous" defines an equivalence relation $\sim$ on the set of continuous one-cocyles of G with coefficients in T. We define

$$H^1_{\mathrm{cont}}(G;T) = \{\text{continuous one-cocycles } c\colon G \to T\}/\sim.$$

Notice that $H^1_{\mathrm{cont}}(G,T)$ is equipped with a canonical basepoint: the class of the trivial one-cocycle $\sigma \mapsto 1$.

If T happens to be an abelian G-group, then $H^2_{\mathrm{cont}}(G;T)$ is defined in the usual way as continuous 2-cocyles modulo continuous 2-coboundaries; see [Tat76].

If $H \subset G$ is a closed subgroup, then we can define restriction

$$\mathrm{res}^G_U\colon H^i_{\mathrm{cont}}(G,T) \to H^i_{\mathrm{cont}}(U,T)$$

as usual.

B.2. **An analogue of the long exact sequence.** Suppose now that we have a *strict exact sequence* of topological G-groups

$$1 \to T' \to T \to T'' \to 1.$$

This means in particular that T' carries the subspace topology inherited from T and T'' carries the quotient topology inherited from T. Assume also that

(i) T' lands in the center of T;
(ii) we have a continuous set-theoretic section $s\colon T'' \to T$.

Notice that T' is abelian so that $H^2_{\mathrm{cont}}(G;T')$ is defined. We construct a boundary map

$$\delta\colon H^1_{\mathrm{cont}}(G;T'') \to H^2_{\mathrm{cont}}(G;T') \tag{B.1}$$

as follows: for the class of a continous one-cocyle $c\colon G \to T''$, we define

$$\delta(c)_{\sigma,\tau} = b_\sigma {}^\sigma b_\tau b^{-1}_{\sigma\tau} \in T' \quad (\sigma,\tau \in G), \tag{B.2}$$

where $b\colon G \to T$ is a continuous lift of c. Such a lift always exists, since we can compose c with the section s from (ii). As shown in [Ser79], it is a 2-cocycle. Furthermore, $\delta(c)$ is continuous by continuity of b. If we pick a different lift $\sigma \mapsto a'_\sigma b_\sigma$, with $a'_\sigma \in T'$, then $\sigma \mapsto a'_\sigma$ is continuous. Now, the two-cocycle $\delta(c)$ is replaced by $(\sigma,\tau) \mapsto a_{\sigma,\tau}\delta(c)_{\sigma,\tau}$, where

$$a_{\sigma,\tau} = (\partial a')_{\sigma,\tau} = a'_\sigma {}^\sigma a'_\tau a'^{-1}_{\sigma\tau}.$$

It follows that the class of $\delta(c)$ in $H^2_{\mathrm{cont}}(G;T')$ is independent of the choice of the lift.

We show that δ does not depend on the choice of representative for the cocycle class of c. Indeed, if c' is a continuous one-cocycle cohomologous to c, then there is $t'' \in T''$ such that

$$c'_\sigma = t''^{-1} c_\sigma {}^\sigma t'' \quad (\sigma \in G).$$

Let $t \in T$ such that $t \mapsto t''$. We can lift c' to $\sigma \mapsto t^{-1} b_\sigma {}^\sigma t$. Clearly, this is again a continuous lift. As shown in [Ser79], the resulting continuous two-cocycles of G with coefficients in T' are the same. We conclude that the map δ is well-defined.

Theorem B.3. *The sequence*

$$H^1_{\mathrm{cont}}(G;T) \to H^1_{\mathrm{cont}}(G;T'') \xrightarrow{\delta} H^2_{\mathrm{cont}}(G;T'),$$

with δ the map from (B.1), *is an exact sequence of pointed sets.*

Proof. The fact that the composition of the two maps is the trivial map is exactly as in the classical discrete case. Suppose we have a one-cocycle $c \in H^1_{\mathrm{cont}}(G;T'')$ such that $\delta(c) = 0 \in H^2_{\mathrm{cont}}(G;T')$. Then there is $a \in C^1_{\mathrm{cont}}(G;T')$ a continuous map such that

$$\delta(c)_{\sigma,\tau} = a_\sigma {}^\sigma a_\tau a^{-1}_{\sigma\tau} \quad (\sigma,\tau \in G).$$

By property (i) above we get

$$(b_\sigma a^{-1}_\sigma)^\sigma (b_\tau a^{-1}_\tau)(b_{\sigma\tau} a^{-1}_{\sigma\tau})^{-1} = 1 \in T' \quad (\sigma \in G). \tag{B.3}$$

Define now $b'\colon G \to T$ by $\sigma \mapsto b_\sigma a^{-1}_\sigma$. Then b' is continuous by the fact that a and b are. By (B.3), b' is a cocycle. We clearly have $b' \mapsto c$ and so we win. ■

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