# A LOCAL VERSION OF KASHIWARA'S CONJECTURE

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## CONTENTS

1. Introduction 2
1.1. Background: the complex Kashiwara conjecture 2
1.2. The arithmetic local Kashiwara conjecture 3
1.3. A spreading argument for $\ell$-adic representations 5
1.4. Contents 6
1.5. Acknowledgements 6
2. Some facts about tame fundamental groups 6
2.1. The tame fundamental group 6
2.2. The specialization homomorphism 7
2.3. A homotopy exact sequence 8
3. Preliminaries on $\ell$-adic local systems 9
3.1. The category of (A-R) $\pi$-adic sheaves 10
3.2. The category of $E$-sheaves 12
3.3. The category of $\overline{\mathbb{Q}}_\ell$-sheaves 13
3.4. Monodromy representations 14
3.5. Weights 17
3.6. Purity of the branch locus 17
4. A spreading argument for $\ell$-adic representations 18
4.1. The space of $\ell$-adic representations 19
4.2. Spreading $\ell$-adic representations with finite determinant 21
4.3. Some auxiliary results 23
5. Arithmetic local systems 24
5.1. Basic properties 25
5.2. Finiteness of arithmetic characters 26
6. A counterexample to the naive local Kashiwara conjecture 28
6.1. The (tame) fundamental group of a nodal curve 28
6.2. A counterexample (after Mochizuki) 34
7. The arithmetic local Kashiwara conjecture in equal characteristic 36
7.1. Proof of Theorem 7.1 37
Appendix A. Mochizuki's counterexample 39
A.1. Topological aspects of the construction 39

A.2. Aspects of the construction from representation theory 41
Appendix B. Continuous non-abelian Galois Cohomology 41
B.1. The cohomology groups 42
B.2. An analogue of the long exact sequence 42
References 43

## 1. INTRODUCTION

1.1. **Background: the complex Kashiwara conjecture.** In 1998, Masaki Kashiwara formulated several conjectures about *semisimple holonomic $\mathscr{D}$-modules* on smooth complex algebraic varieties [Kas98]. They were proved for *regular* holonomic $\mathscr{D}$-modules under the assumption of De Jong's conjecture by Drinfeld [Jon01; Dri01]. De Jong's conjecture was soon after established in sufficient generality by Gaitsgory and Böckle-Khare [Gai07; BK06]. The conjectures were also proved for regular holonomic $\mathscr{D}$-modules on quasi-projective varieties, using different methods, by Takurō Mochizuki [Moc07].

Regular holonomic $\mathscr{D}$-modules correspond to *perverse sheaves* under the Riemann-Hilbert correspondence. Particular examples of regular holonomic $\mathscr{D}$-modules are vector bundles with regular flat connections. Under the Riemann-Hilbert correspondence these correspond to locally constant sheaves of finite dimensional $\mathbb{C}$-vector spaces in the analytic topology, also known as *complex local systems.* If the underlying variety $X$ is connected, then these in turn correspond to finite-dimensional $\mathbb{C}$-linear representations of the fundamental group $\pi_1(X(\mathbb{C}))$. Under this correspondence, the notion of semisimplicity translates into the usual one for representations.

We discuss briefly the conjecture of Kashiwara relevant for this thesis. Let $X$ be a smooth complex algebraic variety, and let $f\colon X \to \mathbb{A}^1$ be a regular function. Given a perverse sheaf $\mathsf{F}$ on $X$, the *perverse sheaf of nearby cycles* $\Psi_f \mathsf{F} = R\psi_f \mathsf{F}[-1]$ on the fiber $X_0 = f^{-1}(0)$ comes equipped with a monodromy operator $T \in \operatorname{Aut} \Psi_f \mathsf{F}$. The perverse sheaf $\Psi_f \mathsf{F}$ admits a unique spliting [Rei10, Lemma 1.1]

$$\Psi_f \mathsf{F} = \Psi_f^{\mathrm{nil}} \mathsf{F} \oplus \Psi_f^{\mathrm{inv}} \mathsf{F}$$

such that $T - 1$ restricts to both factors, and acts nilpotently on $\Psi_f^{\mathrm{nil}} \mathsf{F}$ and as an automorphism on $\Psi_f^{\mathrm{inv}} \mathsf{F}$. The nilpotent operator $T - 1$ on $\Psi_f^{\mathrm{nil}} \mathsf{F}$ gives rise to a *monodromy filtration* $M_\bullet$.

**Theorem 1.1** ([Dri01, Section 1.2, Conjecture 3])**.** *If $\mathsf{F}$ is a semisimple perverse sheaf on a smooth complex variety $X$ and $f\colon X \to \mathbb{A}^1$ is a regular function, then*

$$\mathrm{gr}^M \Psi_f^{\mathrm{nil}} \mathsf{F} = \bigoplus_{i \in \mathbb{Z}} \mathrm{gr}_i^M \Psi_f^{\mathrm{nil}} \mathsf{F}$$

*is semisimple.*

In the special case where $f$ is smooth and $\mathsf{F} = \mathbb{L}[\dim X]$ comes from a local system on $X$ the theorem above takes the following particularly simple form.

**Corollary 1.2.** *If* $\mathbb{L}$ *on* $X$ *is a semisimple local system on a smooth complex variety* $X$ *and* $f\colon X \to \mathbb{A}^1$ *is smooth, then* $\mathbb{L}|X_0$ *is semisimple.*

*Proof.* Write $d = \dim X$. Set $\mathsf{F} = \mathbb{L}[d]$. Since $f$ is smooth, the complex of nearby cycles simplifies to $\Psi_f\mathsf{F} = \mathsf{F}[-1]|X_0$. Furthermore, the monodromy operator $T\colon \Psi_f\mathsf{F} \to \Psi_f\mathsf{F}$ is trivial, so that $\Psi_f^{\mathrm{nil}}\mathsf{F} = \Psi_f\mathsf{F}$. We find

$$\mathrm{gr}^M\Psi_f^{\mathrm{nil}}\mathsf{F} = \mathbb{L}[d-1]|X_0.$$

Now apply Theorem 1.1. ■

We mention that the Corollary above also implies the following corollary.

**Corollary 1.3** ([Moc07, Theorem 25.30][1])**.** *Let* $X$ *and* $Y$ *be smooth complex varieties, and let* $f\colon X \to Y$ *be a morphism. If* $\mathbb{L}$ *on* $Y$ *is a semisimple local system, then the local system* $f^{-1}\mathbb{L}$ *on* $X$ *obtained by pulling back* $\mathbb{L}$ *along* $f$ *is also semisimple.*

Esnault and De Jong also provide an arithmetic proof of the Corollary [JE24, Theorem 7.3]. They also explain that it suffices to assume that $X$ and $Y$ are *normal.* The assumption of smoothness can, however, not be omitted altogether. Indeed, let $Y$ be a proper rational curve smooth outside of a single simple node, and denote by $Y^\circ$ the curve $Y$ punctured in two smooth points. Denote by $f\colon \mathbb{P}^1 - \{0,\infty\} = \mathbb{G}_m \to Y^\circ$ the normalization of $Y^\circ$. Appendix A suggests an irreducible representation $\rho\colon \pi_1(Y^\circ) \to \mathrm{GL}_2(\mathbb{C})$ whose pullback to $\pi_1(\mathbb{G}_m)$ is not semisimple.

1.2. **The arithmetic local Kashiwara conjecture.** Inspired by Kashiwara's conjectures for semisimple holonomic $\mathscr{D}$-modules, Esnault and Kerz formulated arithmetic versions of the conjectures for *arithmetic semisimple perverse* $\ell$*-adic sheaves* [EK23, Conjecture 9.7]. A perverse sheaf is *arithmetic* if it descends to a variety over a finitely generated subfield. Their conjectures are highly relevant: they imply the monodromy weight conjecture in mixed characteristic [EK23, Corollary 9.8].

The setup is as follows. Let $\mathscr{O}$ be a strictly henselian discrete valuation ring, and let $\ell$ be a prime number such that $\ell \in \mathscr{O}^\times$. Write $S = \operatorname{Spec}\mathscr{O}$. Let $s \in S$ be the closed point, $\eta \in S$ the generic point, and $\overline{\eta} \to S$ a geometric point over the generic point. Let $f\colon \mathscr{X} \to S$ be a separated morphism of finite type. We define the dimension function

$$\begin{aligned} \delta_f\colon \mathscr{X} &\to \mathbb{Z} \\ x &\mapsto \mathrm{trdeg}(\kappa(x)/\kappa(f(x)) + \dim\overline{\{f(x)\}}. \end{aligned} \tag{1.1}$$

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[1] In loc. cit. smoothness is erroneously omitted. The quasi-projectivity assumption that is in place in loc. cit. is irrelevant, because semisimplicity of a local system on a normal variety can be checked on a dense affine open.

It induces a $t$-structure on $D^b_c(\mathscr{X},\overline{\mathbb{Q}}_\ell)$ [Gab04] and its heart we call the category of *perverse ($\ell$-adic) sheaves on $\mathscr{X}$*. For $\mathsf{F} \in D^b_c(\mathscr{X},\overline{\mathbb{Q}}_\ell)$ a perverse sheaf, we let $\Psi^{\mathrm{nil}}_f\mathsf{F}$ be the *perverse sheaf of unipotent nearby cycles on $\mathscr{X}_s$* as defined in [EK23, Appendix A.3]. It comes equipped with a unipotent action of the maximal pro-$\ell$ quotient $\mathrm{Gal}(\overline{\eta}/\eta) \to \mathbb{Z}_\ell(1)$. This action is encoded in a nilpotent operator $N\colon \Psi^{\mathrm{nil}}_f\mathsf{F} \to \Psi^{\mathrm{nil}}_f\mathsf{F}(-1)$, which in turn induces a monodromy filtration $M_\bullet$ on $\Psi^{\mathrm{nil}}_f\mathsf{F}$ [EK23, Section 7]. The local analogue of Theorem 1.1 is now the following.

**Conjecture 1.4** ([EK23, Conjecture 9.7])**.** *Let $f\colon \mathscr{X} \to S$ be a proper scheme, and let $\mathsf{F}$ be an $\ell$-adic perverse sheaf on $\mathscr{X}$ such that $\mathsf{F}|\mathscr{X}_{\overline{\eta}}$ is semisimple and arithmetic. Then*

$$\mathrm{gr}^M\Psi^{\mathrm{nil}}_f\mathsf{F} = \bigoplus_{i\in\mathbb{Z}}\mathrm{gr}^M_i\Psi^{\mathrm{nil}}_f\mathsf{F}$$

*is semisimple.*

We will again restrict our attention from perverse sheaves to local systems. Analogously to the situation for complex local systems, $\ell$-adic local systems on a connected scheme $X$ correspond to continuous finite-dimensional $\overline{\mathbb{Q}}_\ell$-linear representations of the étale fundamental group $\pi_1^{\text{ét}}(X)$. We will simply speak of *$\ell$-adic representations* of $\pi_1^{\text{ét}}(X)$, leaving the other adjectives implicit. Let $f\colon \mathscr{X} \to S$ be a smooth quasi-compact separated morphism. Using the dimension function of (1.1), we define the locally constant function $\delta_{\mathscr{X}}\colon \mathscr{X} \to \mathbb{Z}$ sending $x \in \mathscr{X}$ to $\delta_f(y)$, where $y \in \mathscr{X}$ is the maximal point specializing to $x$. For a local system $\mathbb{L}$ on $\mathscr{X}$, the complex $\mathsf{F} = \mathbb{L}[\delta_{\mathscr{X}}]$ defines a perverse sheaf. We have $\Psi^{\mathrm{nil}}_f\mathsf{F} = \mathbb{L}_s[\delta_{\mathscr{X}}-1]$, where $\mathbb{L}_s = \mathbb{L}|\mathscr{X}_s$. Indeed, $\Psi_f\mathsf{F} = \mathsf{F}[-1]|\mathscr{X}_s$ by local acyclicity of smooth morphisms. Furthermore, the nilpotent operator $N\colon \Psi_f\mathsf{F} \to \Psi_f\mathsf{F}(-1)$ is zero, so that $\Psi_f\mathsf{F} = \Psi^{\mathrm{nil}}_f\mathsf{F}$. Triviality of $N$ also implies

$$\mathrm{gr}^M\Psi^{\mathrm{nil}}_f\mathsf{F} = \Psi^{\mathrm{nil}}_f\mathsf{F} = \mathbb{L}_s[\delta_{\mathscr{X}} - 1].$$

**Conjecture 1.5.** *Let $\mathbb{L}$ be an $\ell$-adic local system on $f\colon \mathscr{X} \to S$ a smooth quasi-compact separated $S$-scheme such that the pullback $\mathbb{L}_{\overline{\eta}} = \mathbb{L}|\mathscr{X}_{\overline{\eta}}$ of $\mathbb{L}$ to $\mathscr{X}_{\overline{\eta}}$ is semisimple and arithmetic. Then $\mathbb{L}_s$ is semisimple.*

*Conjecture 1.4 ⇒ Conjecture 1.5.* Let $\mathscr{X} \xrightarrow{j} \overline{\mathscr{X}} \xrightarrow{\overline{f}} S$ be a Nagata compactification over $S$. Set $\mathsf{F} = j_{!*}\mathbb{L}[\delta_{\mathscr{X}}]$. Then $\mathsf{F}_{\overline{\eta}} = j_{\overline{\eta}!*}\mathsf{F}$ is semisimple [KW01, Corollary III.5.5], and arithmetic by the fact that $\mathbb{L}_{\overline{\eta}}$ is arithmetic. We find

$$\mathrm{gr}^M\Psi^{\mathrm{nil}}_{\overline{f}}\mathsf{F}|\mathscr{X}_s = \mathrm{gr}^M\Psi^{\mathrm{nil}}_f\mathbb{L}[\delta_{\mathscr{X}}] = \mathbb{L}_s[\delta_{\mathscr{X}} - 1],$$

where the second equality is by the discussion above. Conjecture 1.4 applied to $\mathsf{F}$, in combination with the fact that the restrition of a semisimple perverse sheaf to a Zariski open is semisimple [KW01, Corollary III.5.4], yields Conjecture 1.5. ■

It does not suffice to assume that $\mathbb{L}_{\overline{\eta}}$ is only semisimple in Conjecture 1.5. Takurō Mochizuki has constructed an example to show that arithmeticity cannot be omitted for a complex geometric version of the local Kashiwara conjecture; see Appendix A. We work out the algebro-geometric version of Mochizuki's example in Section 6. It involves an elliptic fibration with singular fiber a rational nodal curve.

As already indicated previously, Conjecture 1.5 is most interesting in the case that $\mathscr{O}$ has mixed characteristic. This case is presently out of reach. The following specific case of Conjecture 1.5 with $\mathscr{O}$ of equal characteristic is proved in this thesis.

**Theorem 1.6** (Theorem 7.1)**.** *Let $F$ be a finite field or the complex numbers. Conjecture 1.5 is true if $\mathscr{O} = \mathscr{O}^{\mathrm{hs}}_{C,\overline{c}}$ is the strict henselization of a normal curve $C$ over $F$ at a closed geometric point $\overline{c} \to C$, and if furthermore $\mathscr{X}_{\overline{\eta}}$ is a curve.*

The proof proceeds by using the arithmeticity condition and the spreading argument outlined in Section 1.3 to strategically spread the problem to one of global nature. If $F = \mathbb{C}$, Theorem 1.3 can then be invoked to conclude. If $F$ is a finite field, then the Theorem will follow from results of Deligne [Del80] and results of the Langlands program proved by Lafforgue [Laf02].

We note that Theorem 1.6 combined with the afformentioned counterexample of Section 6 produces a local systems that is not arithmetic for which there seems to be no other obvious way to rule out arithmeticity; see also Remark 7.2. That is, Conjecture 1.4 could serve as an interesting obstruction to arithmeticity of local systems on varieties over local fields.

## 1.3. **A spreading argument for $\ell$-adic representations.**

We state here the crucial spreading argument for the proof of Theorem 1.6. Let $G$ and $H$ be profinite groups, and let $G$ act continuously on $H$. Let $\mathscr{R}$ denote the set of semisimple $\ell$-adic representations $H \to \mathrm{GL}(V)$ up to isomorphism. Then $\mathscr{R}$ is naturally equipped with a right $G$-action.

**Theorem 1.7** (Theorem 4.8)**.** *Let $\rho\colon H \to \mathrm{GL}(V)$ be an irreducible $\ell$-adic representation, and assume that the orbit $[\rho] \cdot G \subset \mathscr{R}$ of the isomorphism class of $\rho$ is finite. Then there exists an open subgroup $U \subset G$ such that $\rho$ extends to an $\ell$-adic representation*

$$\widetilde{\rho}\colon H \rtimes U \to \mathrm{GL}(V).$$

*If, furthermore, $\rho$ has finite determinant, then also $\widetilde{\rho}$ can be chosen such that it has finite determinant.*

The purpose of Theorem 1.7 in this thesis is twofold: it is the main ingredient in the proof of Theorem 1.6, and the simplified version without the finiteness condition on determinants, is crucial for establishing the basic theory of arithmetic local systems.

Our proof of Theorem 1.7 employs Galois cohomology with non-discrete and non-abelian coefficients, which we recall in Appendix B, to strategically lift projective representations to actual representations.

Variants of Theorem 1.7 are well known and appear for instance in [Sim92; Lit21]. However, it does not seem to appear in such generality in the literature elsewhere.

### 1.4. Contents.

The basic facts we will need about tame fundamental groups are summarized in Section 2. The basic theory of $\ell$-adic local systems together with the most important constructions and facts that will be used throughout the thesis are the contents of Section 3. In particular, the results from [Del80; Laf02] that we rely on are recalled there.

In Section 4 we prove the crucial spreading argument stated in Theorem 1.7. It relies on the theory of Galois cohomology with non-discrete non-abelian coefficients, whose necessary facts, for lack of a good reference, are stated in Appendix B.

The basic theory of arithmetic local systems is worked out in detail in Section 5.

The first main objective of this thesis is completed in Section 6, where we construct an example showing that the arithmeticity condition of Conjecture 1.5 cannot be omitted. It is inspired by the counterexample of Mochizuki, which is contained in Appendix A.

The second main objective of this thesis is to prove Theorem 1.6. This is the content of Section 7.

### 1.5. Acknowledgements.

This work is a slightly altered version of the author's master's thesis, written under the superversion of Moritz Kerz at the university of Regensburg. Some mistakes have been corrected, some material has been added, and some has been moved around. Firstly, the introduction now contains the original local Kashiwara Conjecture 1.4 of Esnault and Kerz and the proof that it implies the simplified Conjecture 1.5, the main focus of this thesis. Further notable changes are: Section 5 used to be part of Section 7 and has been expanded upon, and the proof of Theorem 7.1 has been streamlined.

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## 2. SOME FACTS ABOUT TAME FUNDAMENTAL GROUPS

The material in this section is based entirely on [GR71, Exposé XIII].

### 2.1. The tame fundamental group.

Let $X \to S$ be a proper scheme of finite presentation over a scheme $S$ with geometrically connected fibers. Let $D \rightarrowtail X$ be an effective Cartier divisor such that $D \rightarrowtail X \to S$ is smooth, and such that the suport of $D$ lies in the smooth locus of $X \to S$. Denote by $U$ the complement of $\operatorname{Supp} D$ in $X$, and let $\overline{x} \to U$ be a geometric point of $U$. Raynaud defines a notion of étale coverings of $U$, *tamely ramified along $D$, relative to $S$* [GR71, Exposé XIII]. These tamely ramified coverings form a Galois category $\mathrm{Fib}^t_{\overline{x}}\colon \mathrm{Fét}^t_U \to$ sets with fiber functor the usual fiber functor associated with $\overline{x}$ restricted to the full subcategory of tame coverings. The automorphism group of the fiber functor $\mathrm{Fib}^t_{\overline{x}}$ is defined to be the *tame fundamental group* $\pi_1^t(U,\overline{x})$.

2.1.1. *Functoriality of the tame fundamental group.* The construction of the tame fundamental group is also functorial in the following sense. If $S' \to S$ denotes a morphism of schemes, write $U'$, respectively $D'$, for the pullback of $U$, respectively $D$, along $S' \to S$. Now, if $Y \to U$ denotes an étale cover, tamely ramified along $D$, then its pullback $Y' \to U'$ along $U' \to U$ is an étale cover, tamely ramified along $D'$. It follows that if $\overline{x}' \to U'$ denotes a geometric point of $U'$, then we obtain a homomorphism of tame fundamental groups

$$\pi_1^t(U',\overline{x}') \to \pi_1^t(U,\overline{x}').$$

2.1.2. *The maximal pro-$\mathscr{L}$ quotient.* We will not get into the details of these constructions, but what is important to remark is that if $\mathscr{L}$ denotes a set of primes that do *not* occur as a residue characteristic of $S$, and if $Y \to U$ denotes an étale Galois cover whose degree is a product of primes in $\mathscr{L}$, then $Y \to U$ is tamely ramified. As a result, there is a surjective map of profinite groups

$$\pi_1^t(U,\overline{x}) \twoheadrightarrow \pi_1^{\mathscr{L}}(U,\overline{x}),$$

where $\pi_1^{\mathscr{L}}(U,\overline{x})$ denotes the *maximal pro-$\mathscr{L}$ quotient* of $\pi_1(U,\overline{x})$, which is also the maximal pro-$\mathscr{L}$ quotient of $\pi_1^t(U,\overline{x})$. If $G$ is a profinite group, and $\mathscr{L}$ is any set of primes, then the maximal pro-$\mathscr{L}$ quotient of $G$ is defined to be

$$G^{\mathscr{L}} = \varprojlim_{U} G/U, \tag{2.1}$$

where the projective limit runs over the open normal subgroups $U \subset G$ such that the index $[G:U]$ is a product of primes in $\mathscr{L}$. If $\mathscr{L}$ consists of only one prime $\ell$, then we will usually write $G^{(\ell)}$ for $G^{\mathscr{L}}$. If only one prime $p$ is not contained in $\mathscr{L}$, then $G^{\mathscr{L}}$ is usually denoted $G^{(p')}$ and called the *maximal prime-to-p quotient* of $G$.

2.2. **The specialization homomorphism.** Notation is as before. Let $\eta \in S$ and $s \in S$ be points of $S$ such that there is a specialization of points $\eta \rightsquigarrow s$. Let $\overline{\eta} \to S$, respectively $\overline{s} \to S$, be a geometric point of $S$ lying over $\eta$, respectively $s$. Let $\overline{x} \to U_{\overline{\eta}}$, respectively $\overline{y} \to U_{\overline{s}}$, be a geometric point of $U_{\overline{s}}$, respectively $U_{\overline{\eta}}$. Denote by $A = \mathscr{O}^{\mathrm{hs}}_{S,\overline{s}}$ the strict henselization of $S$ at $\overline{s}$, and write $S_{(\overline{s})} = \operatorname{Spec} A$.

**Proposition 2.1** ([GR71, Exposé XIII, 2.10]). *The homomorphism of tame fundamental groups*

$$\pi_1^t(U_{\overline{s}},\overline{y}) \to \pi_1^t(U_A,\overline{y})$$

*is an isomorphism. Here tameness is with respect to $D_{\overline{s}}$ and $D_A$.*

By the fact that we have a specialization $\eta \rightsquigarrow s$, $\eta$ defines a point of $\operatorname{Spec}\mathscr{O}_{S,s}$. Since $S_{(\overline{s})} \to \operatorname{Spec}\mathscr{O}_{S,s}$ is surjective, by the fact that $\mathscr{O}_{S,s} \to A$ is faithfully flat, we can lift $\eta$ to a point of $S_{(\overline{s})}$ with residue field a separable extension of $\kappa(\eta)$. Because $\kappa(\overline{\eta})$ is separably closed, we can choose a morphism $\overline{\eta} \to S_{(\overline{s})}$ that gives rise to a commutative diagram

$$\begin{tikzcd}
\overline{\eta} \arrow[r] \arrow[dr] & S_{(\overline{s})} \arrow[d] & \overline{s} \arrow[l] \arrow[dl] \\
& S. &
\end{tikzcd}$$

A choice of such a morphism $\overline{\eta} \to S_{(\overline{s})}$ is sometimes called a *specialization of geometric points* and denoted $\overline{\eta} \rightsquigarrow \overline{s}$. Choose also an étale path $\mathrm{Fib}_{\overline{y}} \simeq \mathrm{Fib}_{\overline{x}}$ on $U_A$. We obtain the *specialization map*

$$\mathrm{sp}\colon \pi_1^t(U_{\overline{\eta}}, \overline{x}) \to \pi_1^t(U_A, \overline{x}) \simeq \pi_1^t(U_A, \overline{y}) \xrightarrow{\simeq} \pi_1^t(U_{\overline{s}}, \overline{y}). \tag{2.2}$$

Here the final isomorphism is the inverse of the one from Proposition 2.1. The specialization map of course depends on the choice of map $\overline{\eta} \to S_{(\overline{s})}$ and the choice of étale path from $\overline{x}$ to $\overline{y}$.

**Theorem 2.2** ([GR71, Exposé XIII, Corollaire 2.12])**.** *If $X \to S$ is furthermore smooth, then the specialization map of* (2.2) *induces an isomorphism on maximal prime-to-p quotients*

$$\pi_1^{(p')}(U_{\overline{\eta}}, \overline{x}) \xrightarrow{\simeq} \pi_1^{(p')}(U_{\overline{s}}, \overline{y}),$$

*where $p$ denotes the residue characteristic of $S$ at $s$.*

**Remark 2.3.** We describe a context that frequently occurs in this thesis in which the choices made to construct the specialization map can be washed away. Assume the existence of a section $S \to U$ and fix one such section $\sigma\colon S \to U$. We can then take $\overline{x} = \sigma(\overline{\eta})$ and $\overline{y} = \sigma(\overline{s})$ above. A choice of specialization $\overline{\eta} \rightsquigarrow \overline{s}$ canonically gives rise to an étale path $\mathrm{Fib}_{\overline{s}} \simeq \mathrm{Fib}_{\overline{\eta}}$ on $S_{(\overline{s})}$. This étale path pushes forward to one from $\overline{y}$ to $\overline{x}$ on $U_A$. The specialization map $\mathrm{sp}\colon \pi_1(U_{\overline{\eta}}, \overline{x}) \to \pi_1(U_{\overline{s}}, \overline{y})$ is now even independent of the choice $\overline{\eta} \rightsquigarrow \overline{s}$.

### 2.3. **A homotopy exact sequence.**

Let $S$ be a connected scheme. Let $X \to S$ be a smooth proper $S$-scheme with geometrically connected fibers. Suppose $f\colon U \hookrightarrow Z \to S$ is the complement of an effective cartier divisor $D \rightarrowtail X$ that is smooth over $S$ and has support in the smooth locus of $X \to S$. Assume that $f\colon U \to S$ admits a section. Let $\mathscr{L}$ be a set of primes invertible on $S$.

Let $\overline{x} \to U$ be a geometric point of $U$. Write also $\overline{x}$ for the induced geometric point of $S$. Denote by $K$ the kernel

$$K = \ker \pi_1(U, \overline{x}) \to \pi_1(S, \overline{x}),$$

and let $N$ be the smallest normal subgroup of $K$ such that $K/N$ is a pro-$\mathscr{L}$ group. Then $N$ is also normal in $\pi_1(U, \overline{x})$ and we define

$$\pi_1'(U, \overline{x}) = \pi_1(U, \overline{x})/N.$$

Let $\overline{s} \to S$ be a geometric point of $S$, and let $\overline{x}$ be the geometric point of $U$ obtained as the composition $\overline{s} \to S \to U$. We obtain a sequence of homomorphisms

$$\begin{tikzcd} 1 \arrow[r] & \pi_1^{\mathscr{L}}(U_{\overline{s}}, \overline{x}) \arrow[r] & \pi_1'(U, \overline{x}) \arrow[r] & \pi_1(S, \overline{s}) \arrow[r] \arrow[l, bend right, dashed] & 1. \end{tikzcd} \tag{2.3}$$

It is not hard to check that the composition $\pi_1^{\mathscr{L}}(U_{\overline{s}}, \overline{x}) \to \pi_1'(U, \overline{x}) \to \pi_1(S, \overline{s})$ is the trivial homomorphism.

**Proposition 2.4.** *Under the hypotheses above, the sequence* (2.3) *is a split exact sequence.*

*Proof.* This follows from the combination of [GR71, Exposé XIII, Proposition 4.3] and [GR71, Exposé XIII, Exemple 4.4]. ■

**Remark 2.5.** Throughout, we have restricted ourselves to the case where $D$ is smooth over the base $S$. We can generalize the results in this section by weakening this assumption to the assumption that $D$ is a *normal crossings divisor relative to* $S$, as defined in [GR71, Exposé XIII, 2.1].

## 3. PRELIMINARIES ON $\ell$-ADIC LOCAL SYSTEMS

Throughout, $\ell$ denotes a prime and $X$ denotes a separated noetherian scheme on which $\ell$ is invertible. Most of the material in this section is based on [FK13, Section 1.12], [Fu11, Section 10.1] and [KW01, Appendix A]. $\ell$-adic local systems are the étale analogue of locally constant sheaves of finite dimensional $\mathbb{C}$-vector spaces on complex manifolds, also known as *complex local systems.*

If $\mathbb{L}$ denotes a local system on a complex manifold $M$ and $z \in M$ is a point, then there is an induced $\mathbb{C}$-linear action of the fundamental group $\pi_1(M, z)$ on the stalk $\mathbb{L}_z$, called the *monodromy action.* For a connected complex manifold $M$ this induces an equivalence of categories:

$$\mathrm{Loc}_{\mathbb{C}}(M) \simeq \mathrm{Rep}_{\mathbb{C}}(\pi_1(M, z)), \tag{3.1}$$

where $\mathrm{Loc}_{\mathbb{C}}(M)$ denotes the category of complex local systems on $M$, and $\mathrm{Rep}_{\mathbb{C}}(\pi_1(M, x))$ denotes the category of finite-dimensional $\mathbb{C}$-linear representations of the topological fundamental group $\pi_1(M, z)$ of $M$ based at $z$.

**Example 3.1.** Let $\Delta \subset \mathbb{C}$ denote the unit disc, and let $\Delta^*$ denote the punctured unit disc $\Delta \setminus \{0\}$. We let $f \colon M \to \Delta$ be an elliptic fibration with singular fiber of type $I_1$ in Kodaira's classification of singular fibers: a rational curve with a single node and no other singular points. By Ehresmann's lemma, the map $M^* = M \setminus f^{-1}(0) \to \Delta^*$ is a fiber bundle. It follows that $R^1 f_* \mathbb{C}_{M^*}$ is a complex local system on $\Delta^*$, where $\mathbb{C}_{M^*}$ denotes the constant sheaf with value $\mathbb{C}$ on $M^*$. Let $z \in \Delta^*$ be a point. By the proper base change theorem, the stalk $(R^1 f_* \mathbb{C}_{M^*})_z$ is the first singular cohomology group $H^1(f^{-1}(z), \mathbb{C})$ of the fiber $f^{-1}(z)$. The generator $\gamma \in \pi_1(\Delta^*, z)$ corresponding to a loop going around 0 counterclockwise once acts on $H^1(f^{-1}(z), \mathbb{C})$ via the matrix

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

after picking an appropriate basis of $H^1(f^{-1}(z), \mathbb{C}) \simeq \mathbb{C} \oplus \mathbb{C}$. See [Ach22].

Motivated by the topological situation, we desire an equivalence between $\ell$-adic local systems and finite dimensional continuous $\overline{\mathbb{Q}}_\ell$-linear representations of the étale fundamental group $\pi_1^{\text{ét}}(X, \overline{x})$ based at a geometric point $\overline{x} \to X$ when $X$ is connected. This equivalence is established in Section 3.4.

3.1. **The category of (A-R) $\pi$-adic sheaves.** Let $E/\mathbb{Q}_\ell$ be an $\ell$-adic field with ring of integers $\mathscr{O}_E$, and let $\pi$ be a uniformizer of $\mathscr{O}_E$.

Consider the category of projective systems $\mathscr{F} = (\mathscr{F}_n)_{n \in \mathbb{Z}}$ of torsion $\mathscr{O}_E$-modules on $X$ in the small étale topology for which there exists $n_0 \in \mathbb{Z}$ such that $\mathscr{F}_n = 0$ for all $n \leq n_0$. Given $\mathscr{F}$ and an integer $r$, we write $\mathscr{F}[r]$ for the projective system given by

$$\mathscr{F}[r]_n = \mathscr{F}_{n+r}.$$

For $r \geq 0$ we have a canonical morphism

$$\mathscr{F}[r] \to \mathscr{F}.$$

The collections of morphisms of the form $\mathscr{F}[r] \to \mathscr{F}$ is a multiplicative system, and hence we can localize the category of projective systems considered above at this collection; see [Wei13, Section 10.3]. The resulting category is referred to simply as *the A-R category of projective systems.* It is an abelian category.

3.1.1. *$\pi$-adic sheaves.*

**Definition 3.2.** *A $\pi$-adic, or $\mathscr{O}_E$-, sheaf on $X$ is a projective system $\mathscr{F} = (\mathscr{F}_n)_{n \in \mathbb{Z}}$ such that*

- $\mathscr{F}_n = 0$ *for $n < 0$;*
- $\mathscr{F}_n$ *is a constructible sheaf of $\mathscr{O}_E/\pi^n\mathscr{O}_E$-modules;*
- *the transition map $\mathscr{F}_{n+1} \to \mathscr{F}_n$ induces an isomorphism*

$$\mathscr{F}_{n+1} \otimes_{\mathscr{O}_E/\pi^{n+1}\mathscr{O}_E} \mathscr{O}_E/\pi^n\mathscr{O}_E \xrightarrow{\simeq} \mathscr{F}_n$$

*for all $n$.*

A projective system $\mathscr{F}$ is said to be *A-R $\pi$-adic* if it is isomorphic to a $\pi$-adic sheaf in the A-R category. The category of A-R $\pi$-adic sheaves is the full subcategory of the A-R category of projective systems spanned by the A-R $\pi$-adic sheaves. It is not hard to see that the natural functor from the category of $\pi$-adic sheaves to the category of A-R $\pi$-adic sheaves is an equivalence. Working with A-R $\pi$-adic sheaves, instead of $\pi$-adic sheaves, will offer some extra flexibility that we will need when we define derived pushforwards.

**Example 3.3.** (i) The projective system $\mathcal{O}_{E,X} = ((\mathcal{O}_E/\pi^n\mathcal{O}_E)_X)$ is a $\pi$-adic sheaf, where $(\mathcal{O}_E/\pi^n\mathcal{O}_E)_X$ denotes the constant sheaf with value $\mathcal{O}_E/\pi^n\mathcal{O}_E$ on $X$. By abuse of notation, we will often just denote it by $\mathcal{O}_E$.

(ii) For $n \geq 1$, let

$$\mu_{\ell^n} = \mu_{\ell^n,X} = \ker \mathcal{O}_X^\times \xrightarrow{\cdot \ell^n} \mathcal{O}_X^\times.$$

Then $\mu_{\ell^n}$ is naturally a sheaf of finite $\mathbb{Z}/\ell^n\mathbb{Z}$-modules. It is represented by the finite étale cover $\operatorname{Spec} \mathcal{O}_X[t]/(t^{\ell^n} - 1) \to X$, and hence it is locally constant constructible. We define the $\mathbb{Z}_\ell$-sheaf $\mathbb{Z}_\ell(1)$ to be the projective system $\mathbb{Z}_\ell(1) = (\mu_{\ell^n})$. For $m \in \mathbb{Z}$ we define $\mathbb{Z}_\ell(m) = (\mu_{\ell^n}^{\otimes m})$. If $m < 0$, then we set $\mu_{\ell^n}^{\otimes m} := (\mu_{\ell^n}^\vee)^{\otimes(-m)}$, where $\mu_{\ell^n}^\vee$ denotes the $\mathbb{Z}/\ell^n\mathbb{Z}$-dual of $\mu_{\ell^n}$.

We say that a $\pi$-adic sheaf $\mathcal{F} = (\mathcal{F}_n)$ is *lisse* if each of the $\mathcal{F}_n$ is locally constant. In particular, the sheaves $\mathcal{O}_E$ and $\mathbb{Z}_\ell(m)$ from the examples above are lisse.

**Proposition 3.4** ([Fu11, Proposition 10.1.7(i)]). *The category of A-R $\pi$-adic sheaves is an abelian subcategory of the A-R category of projective systems.*

Given an A-R $\pi$-adic sheaf $\mathcal{F} = (\mathcal{F}_n)$ on $X$, and a geometric point $\overline{x} \to X$ of $X$, we define the *stalk of $\mathcal{F}$ at $\overline{x}$* to be

$$\mathcal{F}_{\overline{x}} = \varprojlim_{n \in \mathbb{Z}} \mathcal{F}_{n,\overline{x}}.$$

It is a well-defined module over $\mathcal{O}_E$. Additionally, it is finitely generated over $\mathcal{O}_E$ [Sta24, Tag 03UQ]. We obtain a functor

$$\mathcal{F} \mapsto \mathcal{F}_{\overline{x}}$$

from the category of A-R $\pi$-adic sheaves to the category of finitely generated $\mathcal{O}_E$-modules. As for ordinary sheaves on the étale site, a sequence

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

of A-R $\pi$-adic sheaves is exact if and only if the sequence of stalks

$$0 \to \mathcal{F}_{\overline{x}} \to \mathcal{G}_{\overline{x}} \to \mathcal{H}_{\overline{x}} \to 0$$

is exact for all geometric points $\overline{x} \to X$[Fu11, Proposition 10.1.17].

An A-R $\pi$-adic sheaf is said to be *torsion* if all its stalks $\mathcal{F}_{\overline{x}}$ are torsion. Equivalently, $\pi^n : \mathcal{F} \to \mathcal{F}$ is zero for some $n \geq 1$, because $X$ is noetherian.

**Example 3.5.** Fix an integer $m \geq 0$ and let $\mathcal{F}_m$ be a constructible $\mathcal{O}_E/\pi^m\mathcal{O}_E$-sheaf. Viewing $\mathcal{F}_m$ as a constructible sheaf of modules over $\mathcal{O}_E$, we can define $\mathcal{F} = (\mathcal{F}_m \otimes_{\mathcal{O}_E} \mathcal{O}_E/\pi^n\mathcal{O}_E)_{n\geq 1}$. Then $\mathcal{F}$ is a torsion $\pi$-adic sheaf, because it is killed by $\pi^m$.

3.1.2. *Functoriality of $\pi$-adic sheaves.* Let $f\colon Y \to X$ be a morphism of schemes with $Y$ separated noetherian. The pullback of a constructible sheaf $\mathscr{F}$ on $X$ to $Y$ along $f$ is again constructible. This allows us to pull back a $\pi$-adic sheaf $\mathscr{F} = (\mathscr{F}_n)$ on $X$ to a $\pi$-adic sheaf $f^{-1}\mathscr{F} = (f^{-1}\mathscr{F}_n)$ on $Y$. Notice also that the inverse image of a locally constant constructible sheaf is again locally constant constructible, so that $f^{-1}$ sends lisse $\pi$-adic sheaves on $X$ to lisse $\pi$-adic sheaves on $Y$.

The assignment $\mathscr{F} \mapsto f^{-1}\mathscr{F}$ gives a functor from the category of A-R $\pi$-adic sheaves on $X$ to those on $Y$. We will often write $\mathscr{F}|_Y$ for the pullback $f^{-1}\mathscr{F}$ if no confusion can arise.

We now concern ourselves with pushforwards of lisse sheaves.

**Proposition 3.6** ([FK13, Theorem 8.9])**.** *If $f\colon Y \to X$ is smooth and proper, then the higher derived image $R^i f_*\mathscr{F}$ of a locally constant constructible sheaf $\mathscr{F}$ on $Y_{\text{ét}}$ with torsion orders invertible on $S$ is locally constant constructible.*

**Corollary 3.7.** *Let $f\colon Y \to X$ be a smooth proper morphism, and let $\mathscr{F}$ be a lisse $\pi$-adic sheaf on $Y$. Then for $i \geq 0$ the projective system $R^i f_*\mathscr{F} = (R^i f_*\mathscr{F}_n)_{n\geq 0}$ of torsion $\mathscr{O}_E$-modules is A-R isomorphic to a lisse $\pi$-adic sheaf.*

*Proof.* The system $R^i f_*\mathscr{F}$ is A-R $\pi$-adic [Fu11, Proposition 10.1.18(ii)]. Then because every sheaf $R^i f_*\mathscr{F}_n$ is locally constant constructible by Proposition 3.6, $R^i f_*\mathscr{F}$ is in fact A-R isomorphic to a lisse $\pi$-adic sheaf [Fu11, Proposition 10.1.1]. ■

3.2. **The category of $E$-sheaves.** By inverting the multiplicative collection of morphisms $\pi^m\colon \mathscr{F} \to \mathscr{F}$ in the category of A-R $\pi$-adic sheaves, we obtain the category $M(X,E)$ of *$E$-sheaves* on $X$. Equivalently, it is the quotient of the category of A-R $\pi$-adic sheaves by the Serre subcategory of torsion A-R $\pi$-adic sheaves [Wei13, Exercise 10.3.2]. In particular, it is abelian. It comes equipped with an exact functor from the category of A-R $\pi$-adic sheaves. If $\mathscr{F}$ is an A-R $\pi$-adic sheaf, then we denote its image in $M(X,E)$ by $\mathscr{F} \otimes_{\mathscr{O}_E} E$, or simply $\mathscr{F} \otimes E$.

We describe the category $M(X,E)$ more concretely. The objects of $M(X,E)$ are just A-R $\pi$-adic sheaves. Morphisms in $M(X,E)$ are defined by

$$\operatorname{Hom}(\mathscr{F} \otimes E, \mathscr{G} \otimes E) = \operatorname{Hom}(\mathscr{F}, \mathscr{G}) \otimes_{\mathscr{O}_E} E$$

with composition defined in the obvious way.

If $\mathscr{F}$ is an A-R $\pi$-adic sheaf, and $\overline{x} \to X$ is a geometric point of $X$, then the stalk of the $E$-sheaf $\mathscr{F} \otimes E$ at $\overline{x}$ is defined to be

$$(\mathscr{F} \otimes E)_{\overline{x}} = \mathscr{F}_{\overline{x}} \otimes E.$$

We obtain a functor

$$\begin{aligned} M(X,E) &\to \mathrm{vect}_E \\ \mathscr{F} &\mapsto \mathscr{F}_{\overline{x}}, \end{aligned}$$

where $\mathrm{vect}_E$ denotes the category of finite dimensional $E$-vector spaces.

**Example 3.8.** We define the constant $E$-sheaf $E_X = \mathscr{O}_{E,X} \otimes E$, where $\mathscr{O}_{E,X}$ denotes the $\pi$-adic sheaf from Example 3.3. By abuse of notation we also denote $E_X$ simply by $E$.

**Example 3.9.** Recall the definition of $\mathbb{Z}_\ell(m)$ from Example 3.3 (ii). We define

$$\mathbb{Q}_\ell(m) = \mathbb{Z}_\ell(m) \otimes \mathbb{Q}_\ell.$$

3.2.1. *Functoriality of $E$-sheaves.* The functor $f^{-1}$ from A-R $\pi$-adic sheaves on $X$ to A-R $\pi$-adic sheaves on $Y$ induces a functor

$$f^{-1}\colon M(X,E) \to M(Y,E). \tag{3.2}$$

An $E$-sheaf is said to be *lisse* if it is isomorphic to $\mathscr{F} \otimes E$ with $\mathscr{F}$ a lisse $\pi$-adic sheaf. Lisse $E$-sheaves on $X$ are also referred to as *local systems on $X$ with coefficients in $E$*. We denote the full subcategory of $M(X,E)$ consisting of local systems on $X$ with coefficients in $E$ by $\mathrm{Loc}_E(X)$. If $f\colon Y \to X$ is smooth and proper, then we also obtain a functor

$$R^i f_*\colon \mathrm{Loc}_E(Y) \to \mathrm{Loc}_E(X). \tag{3.3}$$

Notice that if $\mathscr{F}$ is a lisse $E$-sheaf on $X$, and $X$ is connected, then the (finite) dimension of $\mathscr{F}_{\overline{x}}$ as an $E$-vector space is independent of the chosen geometric point $\overline{x}$. It is referred to as the *rank* of the local system $\mathscr{F}$.

3.3. **The category of $\overline{\mathbb{Q}}_\ell$-sheaves.** Let $E \subset E'$ be an extension of $\ell$-adic sheaves. If $\mathscr{F} = (\mathscr{F}_n)$ denotes an A-R $\mathscr{O}_E$-sheaf, then $\mathscr{F} \otimes \mathscr{O}_{E'} = (\mathscr{F}_n \otimes_{\mathscr{O}_E/\mathfrak{m}_E^n} \mathscr{O}_{E'}/\mathfrak{m}_{E'}^n)$ is an A-R $\mathscr{O}_{E'}$-sheaf. This gives a functor from the category of A-R $\mathscr{O}_E$-sheaves to the category of A-R $\mathscr{O}_{E'}$-sheaves. Hence, we obtain an induced functor

$$M(X,E) \to M(X,E')$$

on localizations. Let $\overline{\mathbb{Q}}_\ell$ be an algebraic closure of $\mathbb{Q}_\ell$. We define the category of $\overline{\mathbb{Q}}_\ell$-sheaves as the direct limit[2]

$$M(X,\overline{\mathbb{Q}}_\ell) = \varinjlim M(X,E)$$

taken over all $\ell$-adic fields $E \subset \overline{\mathbb{Q}}_\ell$. An object in $M(X,\overline{\mathbb{Q}}_\ell)$ is an $E$-sheaf $\mathscr{F}$ over some $\ell$-adic field $E \subset \overline{\mathbb{Q}}_\ell$. We write $\mathscr{F} \otimes \overline{\mathbb{Q}}_\ell$ for its image in $M(X,\overline{\mathbb{Q}}_\ell)$. If $\mathscr{F}$, respectively $\mathscr{G}$, is an $E$-sheaf, respectively an $E'$-sheaf, then we can find $F \subset \mathbb{Q}_\ell$ an $\ell$-adic field containing $E$ and $E'$, and we have

$$\mathrm{Hom}_{\overline{\mathbb{Q}}_\ell}(\mathscr{F} \otimes \overline{\mathbb{Q}}_\ell, \mathscr{G} \otimes \overline{\mathbb{Q}}_\ell) = \mathrm{Hom}_F(\mathscr{F} \otimes_E F, \mathscr{G} \otimes_{E'} F) \otimes_F \overline{\mathbb{Q}}_\ell.$$

Given a $\overline{\mathbb{Q}}_\ell$-sheaf on $X$, represented by an $E$-sheaf $\mathscr{F}$, its stalk at a geometric point $\overline{x} \to X$ is defined to be

$$(\mathscr{F} \otimes \overline{\mathbb{Q}}_\ell)_{\overline{x}} = \mathscr{F}_{\overline{x}} \otimes_E \overline{\mathbb{Q}}_\ell.$$

---

[2] Strictly speaking, this is a *2-colimit* of categories.

We obtain a functor

$$\begin{aligned} M(X,\overline{\mathbb{Q}}_\ell) &\to \mathrm{vect}_{\overline{\mathbb{Q}}_\ell} \\ \mathscr{F} &\mapsto \mathscr{F}_{\overline{x}}. \end{aligned}$$

3.3.1. *Functoriality of $\overline{\mathbb{Q}}_\ell$-sheaves.* By taking a direct limit, the functor from (3.2) yields a functor

$$f^{-1}\colon M(X,\overline{\mathbb{Q}}_\ell) \to M(Y,\overline{\mathbb{Q}}_\ell). \tag{3.4}$$

A $\overline{\mathbb{Q}}_\ell$-sheaf is said to be lisse if it is isomorphic to $\mathscr{F} \otimes \overline{\mathbb{Q}}_\ell$ with $\mathscr{F}$ a lisse $E$-sheaf. Lisse $\overline{\mathbb{Q}}_\ell$-sheaves are also referred to as *local systems on $X$ with coefficients in $\overline{\mathbb{Q}}_\ell$*, or simply *($\ell$-adic) local systems*. The full subcategory of $M(X,\overline{\mathbb{Q}}_\ell)$ consisting of local systems is denoted by $\mathrm{Loc}_{\overline{\mathbb{Q}}_\ell}(X)$, or simply $\mathrm{Loc}(X)$. Besides $\mathscr{F}$, we will also often use the symbol $\mathbb{L}$ to denote a local system.

If $f\colon Y \to X$ is smooth and proper, then from (3.3) we obtain a functor

$$R^i f_*\colon \mathrm{Loc}(Y) \to \mathrm{Loc}(X). \tag{3.5}$$

On a connected scheme, as for local systems with coefficients in $E$, the *rank* of an $\ell$-adic local system is well-defined.

**Example 3.10** (Local systems coming from geometry)**.** Let $f\colon Y \to X$ be a smooth and proper morphism. By abuse of notation, we denote by $\overline{\mathbb{Q}}_\ell = \overline{\mathbb{Q}}_{\ell,X}$ the $\ell$-adic local system $\mathbb{Q}_{\ell,X} \otimes \overline{\mathbb{Q}}_\ell$ on $X$. Then $R^i f_* \overline{\mathbb{Q}}_\ell$ is a local system on $X$ by the discussion above with, by proper base change, stalks

$$(R^i f_* \overline{\mathbb{Q}}_\ell)_{\overline{x}} = H^i(Y \times_X \overline{x}; \overline{\mathbb{Q}}_\ell).$$

The local system $R^i f_* \overline{\mathbb{Q}}_\ell$ is an example of a local system *coming from geometry*.

More generally, we say that a local system $\mathscr{F}$ on $X$ normal of finite type over a separably closed field *comes from geometry* if there exists a dense open subscheme $U \subset X$ and a smooth proper morphism $f\colon Y \to U$ such that $\mathscr{F}|_U$ is a subquotient of $\bigoplus_{i\geq 0} R^i f_* \overline{\mathbb{Q}}_\ell$.

3.4. **Monodromy representations.** Recall that a locally constant constructible (i.e., with finite stalks) sheaf of sets $\mathscr{F}$ on $X$ is represented by a finite étale cover $Y \to X$ by fpqc descent. For a given geometric point $\overline{x} \to X$, the fiber $Y \times_X \overline{x} \to \overline{x}$ is precisely the stalk $\mathscr{F}_{\overline{x}}$, and so by the general theory of the étale fundamental group we obtain a continuous action of $\pi_1(X,\overline{x})$ on $\mathscr{F}_{\overline{x}}$. In case $X$ is connected, this gives us an equivalence of categories between locally constant constructible sheaves of sets on $X$ and finite sets with a continuous $\pi_1(X,\overline{x})$-action. Suppose now that $\mathscr{F}$ additionally caries the structure of a sheaf of $R$-modules over some ring $R$; then we even obtain a continuous action

$$\pi_1(X,\overline{x}) \to \mathrm{Aut}_R(\mathscr{F}_{\overline{x}})$$

of $\pi_1(X,\overline{x})$ on $\mathscr{F}_{\overline{x}}$ compatible with the structure of $\mathscr{F}_{\overline{x}}$ as an $R$-module.

Now let $\mathscr{F} = (\mathscr{F}_n)$ be a lisse $\mathscr{O}_E$-sheaf, and let $\overline{x} \to X$ denote a geometric point of $X$. For every $n \geq 1$, we obtain a continous homomorphism

$$\pi_1(X, \overline{x}) \to \mathrm{Aut}_{\mathscr{O}_E/\pi^n\mathscr{O}_E}(\mathscr{F}_{n,\overline{x}})$$

by the paragraph above. These are compatible since $\mathscr{F}_n = \mathscr{F}_{n+1}/\pi^n\mathscr{F}_{n+1}$. So we obtain a continuous homomorphism

$$\pi_1(X, \overline{x}) \to \varprojlim_n \mathrm{Aut}_{\mathscr{O}_E/\pi^n\mathscr{O}_E}(\mathscr{F}_{n,\overline{x}}) \simeq \mathrm{Aut}_{\mathscr{O}_E}(\mathscr{F}_{\overline{x}}),$$

where $\mathrm{Aut}_{\mathscr{O}_E}(\mathscr{F}_{\overline{x}})$ is equipped with the inverse limit topology. The proof of [Fu11, Theorem 10.23] shows that then also the induced map

$$\pi_1(X, \overline{x}) \to \mathrm{GL}(\mathscr{F}_{\overline{x}} \otimes E)$$

is continuous. We refer to it as the *monodromy representation* associated to $\mathscr{F} \otimes E$. We obtain a functor

$$\begin{aligned} \mathrm{Loc}_E(X) &\to \mathrm{Rep}_E(\pi_1(X, \overline{x})) \\ \mathscr{F} &\mapsto \mathscr{F}_{\overline{x}}, \end{aligned} \tag{3.6}$$

from $\mathrm{Loc}_E(X)$ to $\mathrm{Rep}_E(\pi_1(X, \overline{x}))$, the category of continuous finite-dimensional $E$-linear representations of $\pi_1(X, \overline{x})$.

**Proposition 3.11** ([Fu11, Theorem 10.23]). *If $X$ is in addition connected, then the functor* (3.6) *is an equivalence of categories.*

Suppose that $G$ denotes a profinite group and $\rho\colon G \to \mathrm{GL}(V)$ is a representation of $G$, where $V$ is a finite dimensional $\overline{\mathbb{Q}}_\ell$-vector space. Then $\rho$ is said to be *continuous* if there exists an $E$-linear subspace $W \subset V$ for some $\ell$-adic field $E \subset \overline{\mathbb{Q}}_\ell$, with a continuous $E$-linear action of $G$ on $W$, such that $V \simeq W \otimes_E \overline{\mathbb{Q}}_\ell$ as $G$-representations. Often we will just say that $\rho$ is an *$\ell$-adic representation*, leaving continuity and finite dimensionality implicit.

Taking the limit over the functors of (3.6), we obtain a functor

$$\mathrm{Loc}(X) \to \mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(\pi_1(X, \overline{x})), \tag{3.7}$$

where $\mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(\pi_1(X, \overline{x}))$ denotes the category of *$\ell$-adic representations* of $\pi_1(X, \overline{x})$.

**Terminology 3.12.** In this thesis, an *$\ell$-adic representation* of a profinite group $G$ is always supposed to be a $\overline{\mathbb{Q}}_\ell$-linear continuous finite-dimensional representation of $G$.

**Corollary 3.13.** *If $X$ is in addition connected, then the functor of* (3.7) *is an equivalence of categories.* ■

**Example 3.14.** We consider again the setting of Example 3.10. We obtain a monodromy representation

$$\pi_1(X, \overline{x}) \to \mathrm{GL}(H^i_{\text{ét}}(Y \times_X \overline{x}; \overline{\mathbb{Q}}_\ell)).$$

Let $\varphi\colon Y \to X$ be a morphism of schemes with $Y$ noetherian. If $\mathbb{L}$ denotes a local system on $X$, then we obtain a local system $\varphi^{-1}\mathbb{L}$ on $Y$ from (3.4). If $\rho\colon \pi_1(Y,\overline{y}) \to \mathrm{GL}(\mathbb{L}_{\overline{x}})$ denotes the monodromy representation of $\mathbb{L}$, where $\overline{y} \to Y$ is a geometric point that is mapped to $\overline{x} \to X$, then the monodromy representation of $\varphi^{-1}\mathbb{L}$ is given by the composition

$$\pi_1(Y,\overline{y}) \xrightarrow{\varphi_*} \pi_1(X,\overline{x}) \xrightarrow{\rho} \mathrm{GL}(\mathbb{L}_{\overline{x}}),$$

where we identify the fibers $(\varphi^{-1}\mathbb{L})_{\overline{y}} = \mathbb{L}_{\overline{x}}$.

**Remark 3.15.** If we only want to consider a local system up to isomorphism on $X$ connected, we will sometimes simply write $\rho\colon \pi_1(X) \to \mathrm{GL}(V)$, omitting the choice of base point. This is justified by the fact that for any two base points $\overline{x}, \overline{x}'$ there is a canonical isomorphism $\pi_1(X,\overline{x}) \simeq \pi_1(X,\overline{x}')$ *up to an inner automorphism* of $\pi_1(X,\overline{x})$. Precomposing $\rho$ with an inner automorphism does not alter its isomorphism class, and so this ambiguity can be ignored.

3.4.1. *(Semi-)simplicity.* If $\mathbb{L}$ is a local system on $X$, then $\mathbb{L}$ is said to be *simple*, or *irreducible*, if $\mathbb{L}$ has no proper lisse subobjects in the category of $\overline{\mathbb{Q}}_\ell$-sheaves. If $\mathbb{L}$ is a finite direct sum of simple lisse sheaves, then $\mathbb{L}$ is said to be *semisimple*. *Crucially*, these notions make sense even when $X$ is not connected. If $X$ does happen to be connected, then by identifying lisse sheaves with their monodromy representations via the equivalence of Corollary 3.13, these notions coincide with the usual notions from representation theory.

**Example 3.16.**
- (i) The constant local system $\overline{\mathbb{Q}}_\ell$ on $X$ is semisimple. It is irreducible if $X$ is additionally connected.
- (ii) Assume that $X$ is normal and of finite type over a separably closed field. Let $f\colon Y \to X$ be smooth and proper. We then know $\mathbb{L} = R^i f_* \overline{\mathbb{Q}}_\ell$ to be a local system by Example 3.10. It is furthermore semisimple [Del80, Corollaire 3.4.13]. Hence, we also see that local systems coming from geometry are exactly those local systems $\mathbb{L}$ such that $\mathbb{L}|_U$ is a *direct summand* of $\bigoplus_{i\geq 0} R^i f_* \overline{\mathbb{Q}}_\ell$, for some dense open subscheme $U$ and some smooth and proper morphism $f\colon Y \to U$.

**Lemma 3.17.** *Let $G$ be a profinite group and let $U \subset G$ be an open subgroup of $G$. For a given $\ell$-adic representation $\rho\colon G \to \mathrm{GL}_r(\overline{\mathbb{Q}}_\ell)$, $\rho$ is semisimple if and only if $\rho\,|_U$ is semisimple.*

*Proof.* The "if" statement follows from an averaging argument akin to the one in the proof of Maschke's Theorem. For the "only if" statement, let $V \subset U$ be an open normal subgroup of $G$. Then $\rho\,|_V$ is semisimple by Clifford's Theorem [Cli37]. The fact that $\rho\,|_U$ is semisimple then follows from the "if part" of the lemma. ∎

**Proposition 3.18.** *Let $Y \to X$ be a surjective finite étale cover of $X$, and let $\mathbb{L}$ be a local system on $X$. Then $\mathbb{L}$ is semisimple if and only if the pullback $\mathbb{L}|_Y$ of $\mathbb{L}$ to $Y$ is semisimple.*

*Proof.* If $X$ and $Y$ are connected, this follows from the previous lemma.

It is clear that a local system is semisimple if and only if its restriction to every connected component is semisimple, and so the general case reduces to the case where both $X$ and $Y$ are connected. ■

**Remark 3.19.** The above proposition will prove to be particularly useful when considering questions about the semisimplicity of local systems, because it will allow us in practice to restrict our attention to local systems whose monodromy representations factor over the more manageable pro-$\ell$ quotient of $\pi_1$. Indeed, given a representation $\rho\colon G \to \mathrm{GL}_r(E)$, $\rho$ lands in $\mathrm{GL}_r(\mathscr{O}_E)$ after conjugating and the group

$$H = \ker(\mathrm{GL}_r(\mathscr{O}_E) \to \mathrm{GL}_r(\mathscr{O}_E/\mathfrak{m}))$$

is an open pro-$\ell$ subgroup of $\mathrm{GL}_r(\mathscr{O}_E)$.

3.5. **Weights.** In this section we assume that $X$ is normal of finite type over $\mathbb{F}_q$. Write $k$ for an algebraic closure of $\mathbb{F}_q$. Let $\mathscr{F}$ be a $\overline{\mathbb{Q}}_\ell$-sheaf on $X$. Given a closed point $x \in X$ and a $k$-valued geometric point $\overline{x} \to X$ over it, the *geometric Frobenius* $F_x \in \mathrm{Gal}(k/\kappa(x))$ at $x$ is defined to be the inverse of the usual Frobenius $\alpha \mapsto \alpha^{N(x)}$. Here $N(x)$ denotes the cardinality of the residue field $\kappa(x)$. The geometric Frobenius $F_x$ gives rise to a $\overline{\mathbb{Q}}_\ell$-linear operator

$$F_x\colon \mathscr{F}_{\overline{x}} \to \mathscr{F}_{\overline{x}}.$$

**Definition 3.20** ([Del80, Définition 1.2.2]). *Let $w \in \mathbb{Z}$ be a integer. The sheaf $\mathscr{F}$ is said to be* pure of weight $w$ *if for every closed point $x \in X$, all the eigenvalues $\alpha$ of*

$$F_x\colon \mathscr{F}_{\overline{x}} \to \mathscr{F}_{\overline{x}}$$

*are algebraic numbers, and such that for every embedding $\iota\colon \mathbb{Q}(\alpha) \hookrightarrow \mathbb{C}$ we have*

$$|\iota(\alpha)|^2 = N(x)^w.$$

*If $\mathscr{F}$ is pure of some weight $w$, then we say that $\mathscr{F}$ is pure.*

**Lemma 3.21** ([Laf02, Proposition VII.7]). *Let $\mathbb{L}$ be a simple local system with finite determinant. Then $\mathbb{L}$ is pure of weight* 0.

Lafforgue proves the above lemma by first reducing to the case that $\mathscr{X}$ is a curve. This reduction step is, however, flawed. See [Del12, Section 0.7 and Sections 1.5-1.9] for the corrected proof.

**Lemma 3.22** ([Del80, Theorem 3.4.1(iii)]). *Let $\mathbb{L}$ be a pure local system on $X$. Then $\overline{\mathbb{L}}$, the pullback of $\mathbb{L}$ to $\overline{X} := X \times_{\mathbb{F}_q} \operatorname{Spec} k$, is semisimple.*

3.6. **Purity of the branch locus.** Assume in this section that $X$ is connected and regular. Let $U \subset X$ be a dense Zariski open. Denote by $\overline{\eta} \to U$ a geometric point over the generic point of $U$. Write $\overline{K} = \kappa(\overline{\eta})$. We give a description of the kernel of the surjective map $\pi_1(U,\overline{\eta}) \twoheadrightarrow \pi_1(X,\overline{\eta})$.

For a given point $\zeta \in X \setminus U$ that is of codimension 1 in $X$, pick a geometric point $\overline{\zeta} \to X$ over $\zeta$. We choose a specialization $\overline{\eta} \rightsquigarrow \overline{\zeta}$ of geometric points (see Section 2.2 for the terminology). Write $K^{\mathrm{hs}}_{\overline{\zeta}} = \operatorname{Frac} \mathscr{O}^{\mathrm{hs}}_{X,\overline{\zeta}}$. We set

$$I_\zeta = \operatorname{im}(\operatorname{Gal}(\overline{K}/K^{\mathrm{hs}}_{\overline{\zeta}}) \to \pi_1(U,\overline{\eta})),$$

and call it *the inertia group of* $\zeta$. We should point out the abuse of notation/terminology occuring here: the inertia group $I_\zeta$ depends on the choice of geometric point $\overline{\zeta}$, and on the choice of specialization $\overline{\eta} \rightsquigarrow \overline{\zeta}$. Different choices would give rise to subgroups that equally deserve to be called the inertia group of $\zeta$. However, all of the resulting subgroups are conjugate, and for our purposes we do not need to distinguish between them.

**Proposition 3.23.** *Let* $\zeta_1, \dots, \zeta_n$ *be the codimension-*1 *points of* $X$ *contained in* $X \setminus U$. *Write* $I_i \subset \pi_1(U,\overline{\eta})$ *for the inertia group of* $\zeta_i$. *Then* $\ker \pi_1(U,\overline{\eta}) \to \pi_1(X,\overline{\eta})$ *is the smallest closed normal subgroup of* $\pi_1(U,\overline{\eta})$ *containing* $I_1, \dots, I_n$.

*Proof.* This is a rephrasing of the usual statements on purity of the branch locus [Sta24, Tags 0BTD, 0BTE]. ■

**Remark 3.24.** The above statement is applied in particular in the following way. Say $\rho_U \colon \pi_1(U,\overline{\eta}) \to \mathrm{GL}(V)$ is an $\ell$-adic representation. Then $\rho_U$ can be extended to a representation of $\pi_1(X,\overline{\eta})$ if and only if $\rho(I_\zeta) = \{\mathrm{id}\}$ for all codimension-1 points $\zeta \in X$ contained in $X \setminus U$.

## 4. A SPREADING ARGUMENT FOR $\ell$-ADIC REPRESENTATIONS

Let $G$ and $H$ be profinite groups and let $G$ act continuously on $H$. Write $\widetilde{G} = H \rtimes G$ for the semi-direct product of $G$ and $H$. It is again a profinite group, because the underlying topology of $\widetilde{G}$ is that of $G \times H$; hence, it is compact and totally disconnected. We think of $G$ and $H$ as living inside this semi-direct product. In particular, we denote the action of $G$ on $H$ by conjugation.

One scenario in which this situation arises naturally is described by the following standard result.

**Proposition 4.1.** *Suppose we have an exact sequence of profinite groups*

$$1 \to H \to K \to G \to 1$$

*such that* $K \to G$ *admits a continuous section* $G \to K$. *Then* $G$ *acts continuously on* $H$ *by conjugation, and we have an isomorphism* $K \simeq H \rtimes G$ *fitting into the commutative diagram*

$$\begin{tikzcd}
1 \arrow[r] & H \arrow[r] \arrow[d, "="] & K \arrow[r] \arrow[d, "\simeq"] & G \arrow[r] \arrow[d, "="] & 1 \\
1 \arrow[r] & H \arrow[r] & H \rtimes G \arrow[r] & G \arrow[r] & 1.
\end{tikzcd}$$

■

Given a representation $\rho\colon H \to \mathrm{GL}(V)$ of $H$ and an element $g \in G$, we obtain a representation $\rho^g\colon H \to \mathrm{GL}(V)$ by precomposing $\rho$ with the map $H \to H$ obtained from the $G$-action on $H$. We consider the set of isomorphism classes of representations derived from $\rho$ in this way:

$$[\rho] \cdot G = \{\rho^g : g \in G\}/\simeq .$$

If there exists an open subgroup $U \subset G$ such that $\rho$ is the restriction of a representation $\widetilde{\rho}\colon H \rtimes U \to \mathrm{GL}(V)$, then clearly $[\rho] \cdot G$ must be finite: for all $g \in U$ we have

$$\rho^g = \widetilde{\rho}(g) \cdot \rho \cdot \widetilde{\rho}(g)^{-1}.$$

The first part of the main result of this section, Theorem 4.8(i), states that finiteness of $[\rho] \cdot G$ is actually a sufficient condition for the existence of $\widetilde{\rho}$ if $\rho$ is an irreducible continuous $\overline{\mathbb{Q}}_\ell$-representation of $H$. This is a well-known result whose proof can also be found in [Lit21, Propostion 3.1.1]. Furthermore, in Theorem 4.8(ii), we prove that if the determinant of $\rho$ is finite, then its spreading to an open subgroup $H \rtimes U$ can be chosen to have finite determinant as well. In corollary 4.9 this additional condition is used to show that in this case spreadings are *unique* up to a diminution of $U$; although we do not make use of this fact anywhere else in the thesis.

In Appendix B we collect some basic results from the theory of non-abelian Galois cohomology with non-discrete coefficients, which are needed in this section. These are likely well known, but for lack of a complete reference, we spell them out.

4.1. **The space of $\ell$-adic representations.** Notation is as in the introduction. Throughout, we fix a prime $\ell$. By an *$\ell$-adic representation* of a profinite group $H$ we mean a continuous finite-dimensional $\overline{\mathbb{Q}}_\ell$-representation. We define the set $\mathscr{R}$ to be the set of isomorphism classes of semisimple $\ell$-adic representations of $H$:

$$\mathscr{R} = \{\text{semisimple } \ell\text{-adic representations } H \to \mathrm{GL}(V)\}/\simeq .$$

For $\rho\colon H \to \mathrm{GL}(V)$ a semisimple $\ell$-adic representation, we denote by $[\rho] \in \mathscr{R}$ its isomorphism class. We have a natural right action of $G$ on $\mathscr{R}$ defined by

$$[\rho]^g = [\rho^g] \quad (g \in G, [\rho] \in \mathscr{R}),$$

where

$$\rho^g(h) = \rho(ghg^{-1}) \quad (h \in H).$$

We intend to equip $\mathscr{R}$ with a topology such that the action described above is continuous. Denote by $\mathrm{Map}(H, \overline{\mathbb{Q}}_\ell)$ the set of continuous (set-)maps $H \to \overline{\mathbb{Q}}_\ell$. We obtain an injection

$$\begin{aligned} \mathscr{R} &\hookrightarrow \mathrm{Map}(H, \overline{\mathbb{Q}}_\ell) \\ [\rho] &\mapsto \mathrm{Tr}\rho := \mathrm{Tr} \circ \rho, \end{aligned}$$

where $\mathrm{Tr}\colon \mathrm{GL}(V) \to \overline{\mathbb{Q}}_\ell$ denotes the (continuous) trace map, by the following lemma.

**Lemma 4.2** ([Wie12, Proposition 2.4.3])**.** *Let $k$ be a field of characteristic* $0$*, $A$ a $k$-algebra, and $V$ and $V'$ two* semisimple *$A$-modules of finite $k$-dimension. If the class maps* $\mathrm{Tr}_V\colon G \to k$ *and* $\mathrm{Tr}_{V'}\colon G \to k$ *obtained by sending $g \in G$ to* $\mathrm{Tr}(g|_V)$*, respectively* $\mathrm{Tr}(g|_{V'})$*, are equal, then $V$ and $V'$ are isomorphic as $A$-modules.* ■

We equip $\mathrm{Map}(H, \overline{\mathbb{Q}}_\ell)$ with the compact-open topology. Then $\mathscr{R}$ is equipped with the subspace topology inherited from $\mathrm{Map}(H, \overline{\mathbb{Q}}_\ell)$.

**Proposition 4.3.** *The action of $G$ on $\mathscr{R}$ is continuous.*

*Proof.* By assumption, the map $G \times H \to H$ is continuous. The induced map

$$\varphi\colon G \to \mathrm{Map}(H, H)$$

is continuous if we equip $\mathrm{Map}(H, H)$ with the compact-open topology [Mun14, Theorem 46.11]. The space $H$ is locally compact and Hausdorff ($H$ is profinite), and so we find that the composition map

$$c\colon \mathrm{Map}(H, H) \times \mathrm{Map}(H, \overline{\mathbb{Q}}_\ell) \to \mathrm{Map}(H, \overline{\mathbb{Q}}_\ell)$$

is continuous [Mun14, Exercise 7, §46]. As a result, the map

$$G \times \mathrm{Map}(H, \overline{\mathbb{Q}}_\ell) \xrightarrow{\varphi\times\mathrm{id}} \mathrm{Map}(H, H) \times \mathrm{Map}(H, \overline{\mathbb{Q}}_\ell) \xrightarrow{c} \mathrm{Map}(H, \overline{\mathbb{Q}}_\ell)$$

is continuous, and hence so is

$$G \times \mathscr{R} \to \mathscr{R}.$$

■

**Corollary 4.4.** *Let $[\rho] \in \mathscr{R}$. Then the stabilizer* $\mathrm{Stab}_{[\rho]}$ *of $[\rho]$ is a closed subgroup of $G$. If the orbit of $[\rho]$ is finite, then* $\mathrm{Stab}_{[\rho]}$ *is also open.*

*Proof.* Write

$$\Phi\colon G \times \mathscr{R} \to \mathscr{R}$$

for the action map. We have

$$\mathrm{Stab}_{[\rho]} = \Phi^{-1}([\rho]) \cap G \times \{[\rho]\},$$

and so we only have to argue that $[\rho] \in \mathscr{R}$ is a closed point. This follows from the fact that $\mathrm{Map}(H, \overline{\mathbb{Q}}_\ell)$ is Hausdorff as $\overline{\mathbb{Q}}_\ell$ is Hausdorff [Mun14, Exercise 6, §46]. The second part of the corollary follows from the fact that $\mathrm{Stab}_{[\rho]}$ has finite index if the orbit of $[\rho]$ is finite. ■

**Corollary 4.5.** *Let $[\rho] \in \mathscr{R}$ such that orbit of $[\rho]$ in $\mathscr{R}$ is finite. Then the orbit of each of the irreducible constituents of $\rho$ is finite.*

*Proof.* Let $U$ be the stabilizer of $[\rho]$. By Corollary 4.4 it is open. The subgroup $U$ permutes the irreducible constituents of $\rho$, and so there exists an open subgroup $V \subset U$ fixing all of them. ■

4.1.1. Let $H' \subset H$ be an open subgroup that is stable under the action of $G$. Write $\mathscr{R}_{H'}$ for the space of conjugacy classes of semisimple $\ell$-adic representations $H' \to \mathrm{GL}_r(\overline{\mathbb{Q}}_\ell)$. By Lemma 3.17 there is a natural map

$$\mathscr{R}_H \to \mathscr{R}_{H'}. \tag{4.1}$$

**Lemma 4.6.** *The map of* (4.1) *has finite fibers.*

*Proof.* Given a semisimple representation $\rho\colon H' \to \mathrm{GL}_r(\overline{\mathbb{Q}}_\ell)$, there are only finitely many ways to extend it to a representation of $H$ up to conjugacy. ■

**Proposition 4.7.** *Let $\rho\colon H \to \mathrm{GL}_r(\overline{\mathbb{Q}}_\ell)$ be a semisimple $\ell$-adic representation. Write $\rho' = \rho|H'$ for its restriction to $H'$. Then $[\rho]\cdot G \subset \mathscr{R}_H$ is finite, if and only $[\rho']\cdot G \subset \mathscr{R}_{H'}$ is finite.*

*Proof.* If $[\rho]\cdot G$ is finite, then clearly also $[\rho']\cdot G$ is finite.

If $[\rho']\cdot G$ is finite, then by shrinking $G$ we can assume without loss of generality that it is trivial. Now the set $\{[\rho]^\sigma\}_{\sigma\in G}$ lies in the fiber over $[\rho']$ under the map of (4.1); hence, it is finite by the above lemma. ■

## 4.2. **Spreading $\ell$-adic representations with finite determinant.** Notation is as before.

**Theorem 4.8.** *(i) Let $\rho\colon H \to \mathrm{GL}_r(\overline{\mathbb{Q}}_\ell)$ be an irreducible $\ell$-adic representation and assume that the orbit $[\rho]\cdot G \subset \mathscr{R}$ is finite. Then there exists an open subgroup $U \subset G$ such that $\rho$ extends to a continuous representation*

$$\widetilde{\rho}\colon \widetilde{U} := H \rtimes U \to \mathrm{GL}_r(\overline{\mathbb{Q}}_\ell).$$

*(ii) Furthermore, if* $\det\rho$ *is finite (i.e., the determinant character of $\rho$ has finite image), then $\widetilde{\rho}$ can be chosen such that* $\det\widetilde{\rho}$ *is finite.*

*Proof.* We can find $E$ a finite extension of $\mathbb{Q}_\ell$ such that $\rho$ factors as $H \to \mathrm{GL}_r(E) \to \mathrm{GL}_r(\overline{\mathbb{Q}}_\ell)$. By the assumption that $[\rho]\cdot G$ is finite, and Corollary 4.4, $\mathrm{Stab}_{[\rho]}$ is open. By potentially replacing $G$ by the open subgroup $\mathrm{Stab}_{[\rho]}$, we can assume that $G$ acts trivially on $[\rho]$. As a result, for every $g \in G$ there is an isomorphism $\rho^g \simeq \rho$ so that $\rho$ and $\rho^g$ are conjugate to each other by some $A_g \in \mathrm{GL}_r(E)$:

$$\rho^g = A_g \cdot \rho \cdot A_g^{-1}.$$

We can indeed take the $A_g$ to be defined over $E$, because $G$ acts trivially on the trace character of $\rho$, and hence trivially on the $E$-isomorphism class of $\rho$ by Lemma 4.2. We define

$$\begin{aligned} \overline{A}\colon G &\to \mathrm{PGL}_r(E) \\ g &\mapsto \overline{A}_g, \end{aligned}$$

where $\overline{A}_g$ denotes the class of $A_g$ in $\mathrm{PGL}_r(E)$. It is easily seen that $\overline{A}$ is a homomorphism by Schur's Lemma. We argue that $\overline{A}$ is additionally continuous by applying Lemma 4.11

below. We employ the notation introduced in that lemma. Notice first that, since $\rho$ is irreducible over $\overline{\mathbb{Q}}_\ell$, we have

$$E[\rho(h)\colon h \in H] = \mathrm{Mat}(r \times r; E)$$

[EG11, Theorem 3.2.2]. For $1 \leq i, j \leq r$ we can therefore write

$$e_{i,j} = \sum_{h \in H} \alpha_h^{i,j} \rho(h)$$

with $\alpha_h^{i,j} \in E$ zero for all but finitely many $h \in H$. Then for $g \in G$ we compute

$$\begin{aligned}(\mathrm{ev}_{i,j} \circ \overline{A})(g) &= A_g e_{i,j} A_g^{-1} \\ &= \sum_{h \in H} \alpha_h^{i,j} \rho^g(h) \\ &= \sum_{h \in H} \alpha_h^{i,j} \rho(ghg^{-1}).\end{aligned}$$

We see that $\mathrm{ev}_{i,j} \circ \overline{A}$ is a linear combination of continuous functions and hence is continuous. It follows that $\overline{A}$ is continuous by Lemma 4.11.

Our goal is to lift $\overline{A}\,|_U$ to a continuous homomorphism $A\colon U \to \mathrm{GL}_r(E)$ for some open subgroup $U \subset G$. To this end, we apply the theory of continuous non-abelian cohomology, which is recalled in Appendix B. Consider the strict exact sequence of topological $G$-groups (each with trivial $G$-action)

$$1 \to \mu_r \to \mathrm{SL}_r(E) \to \mathrm{PSL}_r(E) \to 1$$

from Corollary 4.14 below. By Theorem B.3, we obtain an exact sequence of pointed sets

$$H^1_{\mathrm{cont}}(G; \mathrm{SL}_r(E)) \to H^1_{\mathrm{cont}}(G; \mathrm{PSL}_r(E)) \xrightarrow{\delta} H^2_{\mathrm{cont}}(G; \mu_r).$$

By potentially shrinking $G$ to an open subgroup we can assume that the image of $\overline{A}$ lies in $\mathrm{PSL}_r(E)$ by Lemma 4.13. Let $U \subset G$ be an open subgroup such that $\mathrm{res}^G_U(\delta(\overline{A})) = 0$. This is possible by the fact that $\mu_r$ is discrete. Then, since restriction is compatible with connecting homomorphisms, we find

$$\delta(\mathrm{res}^G_U(\overline{A})) = 0 \in H^2_{\mathrm{cont}}(U; \mu_r),$$

where now $\delta$ denotes the connecting homomorphism in the sequence

$$H^1_{\mathrm{cont}}(U; \mathrm{SL}_r(E)) \to H^1_{\mathrm{cont}}(U; \mathrm{PSL}_r(E)) \to H^2_{\mathrm{cont}}(U; \mu_r).$$

It follows that there exists $A \in H^1_{\mathrm{cont}}(U; \mathrm{SL}_r(E))$ lifting $\overline{A}\,|_U$[3]. We now set

$$\widetilde{\rho} = \rho \rtimes A\colon H \rtimes U \to \mathrm{GL}_r(E).$$

The last part of the proposition follows by construction. ■

---

[3] Notice that by surjectivity of $\mathrm{SL}_r(E) \to \mathrm{PSL}_r(E)$ we can find an actual lift of $\overline{A}$ and not just of its conjugacy class in $H^1_{\mathrm{cont}}(U; \mathrm{SL}_r(E))$.

**Corollary 4.9.** *With hypotheses as in the theorem above, suppose that* $\rho\colon H \to \mathrm{GL}_r(\overline{\mathbb{Q}}_\ell)$ *is irreducible with finite determinant. Then an extension* $\widetilde{\rho}\colon H \rtimes U \to \mathrm{GL}_r(\overline{\mathbb{Q}}_\ell)$ *of* $\rho$ *with finite determinant is* unique *up to a diminution of* $U$.

*Proof.* Let $\widetilde{\rho}, \widetilde{\rho}'\colon H \rtimes U \rightrightarrows \mathrm{GL}_r(\overline{\mathbb{Q}}_\ell)$ be two extensions of $\rho$ with finite determinant. Then their "projectivizations" $H \rtimes U \rightrightarrows \mathrm{GL}_r(\overline{\mathbb{Q}}_\ell) \to \mathrm{PGL}_r(\overline{\mathbb{Q}}_\ell)$ must both equal $\rho \rtimes \overline{A}$, where $\overline{A}\colon U \to \mathrm{PGL}_r(\overline{\mathbb{Q}}_\ell)$ is the unique homomorphism such that $\rho^g = \overline{A}_g \rho \overline{A}_g^{-1}$ for all $g \in U$. Therefore, $\widetilde{\rho}$ and $\widetilde{\rho}'$ differ by a character $\chi\colon H \rtimes U \to \overline{\mathbb{Q}}_\ell^\times$. By finiteness of the determinants this must be a finite character. Since $\chi$ is trivial on $H$, $\chi$ will vanish after shrinking $U$. Hence, $\widetilde{\rho}$ and $\widetilde{\rho}'$ will coincide after shrinking $U$. ■

**Corollary 4.10.** *Let* $\rho\colon H \to \mathrm{GL}_r(\overline{\mathbb{Q}}_\ell)$ *be a semisimple representation such that the orbit* $[\rho] \cdot G \subset \mathscr{R}$ *is finite. Then there exists an open subgroup* $U \subset G$ *such that* $\rho$ *extends to a representation*

$$\widetilde{\rho}\colon H \rtimes U \to \mathrm{GL}_r(\overline{\mathbb{Q}}_\ell).$$

*Proof.* Each of the irreducible constituents of $\rho$ has finite $G$-orbit by Lemma 4.5. Then we apply Theorem 4.8 to spread each of the irreducible constituents. After taking an appropriate direct sum, we find a spreading of $\rho$. ■

4.3. **Some auxiliary results.** Let $E$ be a finite extension of $\mathbb{Q}_\ell$. The projective general linear group $\mathrm{PGL}_r(E)$ is equipped with the quotient topology from $\mathrm{GL}_r(E) \twoheadrightarrow \mathrm{PGL}_r(E)$. Denote by $M_r(E)$ the algebra of $r \times r$-matrices over $E$.

**Lemma 4.11** (Topology of $\mathrm{PGL}_r$)**.** *The space* $\mathrm{PGL}_r(E)$ *has the coarsest topology making each of the evaluation maps*

$$\begin{aligned} \mathrm{ev}_{i,j}\colon \mathrm{PGL}_r(E) &\to M_r(E) \\ M &\mapsto M e_{i,j} M^{-1} \end{aligned}$$

*continuous. Here* $e_{i,j} \in M_r(E)$ *denotes the matrix with a* $1$ *in the* $(i,j)$*-th entry and zeroes everywhere else.*

*Proof.* By the Skölem-Noether Theorem [GS17, Theorem 2.7.2], we obtain a continuous bijection

$$\begin{aligned} \mathrm{PGL}_r(E) &\to \mathrm{Aut}_E(M_r(E)), \\ M &\mapsto (\varphi_M\colon N \mapsto MNM^{-1}), \end{aligned} \tag{4.2}$$

where $\mathrm{Aut}_E(M_r(E))$ denotes the set of $E$-algebra automorphisms of $M_r(E)$ with the subspace topology inherited from $\mathfrak{gl}(M_r(E))$, the set of $E$-linear endomorphisms of $M_r(E)$. As a topological space, it is homeomorphic to $E^{r^4}$. The space $\mathrm{Aut}_E(M_r(E))$ is a locally compact Hausdorff space, because it is a subspace of $\mathfrak{gl}(M_r(E))$. As a result, it is a Baire space. We conclude that the map from (4.2) is a homeomorphism [Ser92, Part II, Chapter

IV, Section 4, Lemma 1]. It is clear that the topology on $\mathfrak{gl}(M_r(E))$ is the coarsest one for which each of the maps

$$\begin{aligned} \mathrm{ev}_{i,j}\colon\ \mathfrak{gl}(M_r(E)) &\to M_r(E) \\ \varphi &\mapsto \varphi(e_{i,j}) \end{aligned}$$

is continuous. The result follows. ■

Denote by $\mathrm{PSL}_r(E)$ the image of $\mathrm{SL}_r(E)$ in $\mathrm{PGL}_r(E)$ equipped with the subspace topology.

**Lemma 4.12.** *The map* $\mathrm{SL}_r(E) \twoheadrightarrow \mathrm{PSL}_r(E)$ *admits a continuous (set-theoretic) section* $\mathrm{PSL}_r(E) \to \mathrm{SL}_r(E)$.

*Proof.* We show that the map $\mathrm{SL}_r(E) \to \mathrm{PGL}_r(E)$ of $\ell$-adic Lie groups induces an isomorphism on Lie-algebras. The Lie algebra of $\mathrm{SL}_r(E)$ is $\mathfrak{sl}_r(E)$, the $r \times r$ matrices over $E$ with trace 0. By construction of the quotient Lie group [Ser92, Part II, Chapter IV, Section 5], the Lie algebra of $\mathrm{PGL}_r(E)$ is given by

$$\mathrm{Lie}(\mathrm{PGL}_r(E)) = M_r(E)/EI_r,$$

where $I_r$ denotes the identity matrix. The induced map

$$\mathfrak{sl}_r(E) \to \mathrm{Lie}(\mathrm{PGL}_r(E))$$

is an isomorphism of Lie algebras. By the Inverse Function Theorem [Ser92, Part II, Chapter 2, Section 9], $\mathrm{SL}_r(E) \to \mathrm{PGL}_r(E)$ is a local isomorphism. Certainly then, $\mathrm{SL}_r(E) \to \mathrm{PSL}_r(E)$ admits sections locally. Since $\mathrm{PSL}_r(E)$ has a basis consisting of open subgroups, we can find a local sections that extends to the whole of $\mathrm{PSL}_r(E)$. ■

**Lemma 4.13.** *The subspace* $\mathrm{PSL}_r(E) \subset \mathrm{PGL}_r(E)$ *is open.*

*Proof.* The proof of Lemma 4.12 shows that $\mathrm{SL}_r(E) \to \mathrm{PGL}_r(E)$ is a local isomorphism, from which it follows that the image of $\mathrm{SL}_r(E)$ in $\mathrm{PGL}_r(E)$ is open. ■

**Corollary 4.14.** *We have a strict exact sequence of topological groups*

$$1 \to \mu_r \to \mathrm{SL}_r(E) \to \mathrm{PSL}_r(E) \to 1, \tag{4.3}$$

*where $\mu_r \subset E$ denotes the set of $r$-th roots of unity in $E$. The sequence* (4.3) *satisfies properties (i) and (ii) of Section B.2.*

## 5. ARITHMETIC LOCAL SYSTEMS

We develop the basic theory of *arithmetic $\ell$-adic local systems* in this section. See also [Lit21]. Let $K$ be a separably closed field of characteristic $p \geq 0$. Throughout $X \to \operatorname{Spec} K$ is always a normal separated $K$-scheme of finite type.

**Definition 5.1.** *A local system $\mathbb{L}$ on $X$ is said to be* arithmetic *if there exists a finitely generated field $F \subset K$, an $F$-scheme $X_F \to \operatorname{Spec} F$ such that $X_F \times_F \operatorname{Spec} K = X$, and a local system $\mathbb{L}_F$ on $X_F$ such that $\mathbb{L}_F|X \simeq \mathbb{L}$.*

If $X$ is connected, and $\rho\colon \pi_1(X) \to \mathrm{GL}_r(\overline{\mathbb{Q}}_\ell)$ is an $\ell$-adic representation. Then $\rho$ is said to be *arithmetic* if the associated local system is.

5.1. **Basic properties.** Let $k \subset K$ be a finitely generated subfield of $K$ and denote by $k \subset \overline{k} \subset K$ its separable closure in $K$. Assume that $X$ descends to a geometrically connected scheme $X_k \to \operatorname{Spec} k$. There is now a natural outer action

$$\mathrm{Gal}(\overline{k}/k) \to \mathrm{Out}(\pi_1(X_{\overline{k}})) \tag{5.1}$$

given by choosing a lift of $\sigma \in \mathrm{Gal}(\overline{k}/k)$ to $\pi_1(X_k)$ and acting on $\pi_1(X_{\overline{k}})$ by conjugation with this lift. For $\rho\colon \pi_1(X_{\overline{k}}) \to \mathrm{GL}_r(\overline{\mathbb{Q}}_\ell)$ an $\ell$-adic representation of $\pi_1(X_{\overline{k}})$, we define $\rho^\sigma$ by precomposing $\rho$ with the outer action of $\sigma$ on $\pi_1(X)$. This only defines $\rho^\sigma$ up to isomorphism, and so we obtain an action of $\mathrm{Gal}(\overline{k}/k)$ on the set of conjugacy classes $[\rho]$ of $\ell$-adic representations $\rho\colon \pi_1(X_{\overline{k}}) \to \mathrm{GL}_r(\overline{\mathbb{Q}}_\ell)$.

After enlarging $k$ to a finite extension (in $\overline{k}$) we are free to assume that $X_k$ admits a $k$-rational point $x\colon \operatorname{Spec} k \to X_k$. Write $\overline{x}$ for the induced $\overline{k}$-point. Now $x$ splits the exact sequence

$$\begin{tikzcd} 1 \arrow[r] & \pi_1(X_{\overline{k}}, \overline{x}) \arrow[r] & \pi_1(X_k, \overline{x}) \arrow[r] & \mathrm{Gal}(\overline{k}/k) \arrow[r] \arrow[l, bend right, dashed] & 1. \end{tikzcd}$$

As a result, the outer action of (5.1) becomes a proper continuous action of $\mathrm{Gal}(\overline{k}/k)$ on $\pi_1(X_{\overline{k}}, \overline{x})$. This puts us in a position to apply the theory of Section 4 to characterize arithmetic representations.

**Proposition 5.2.** *Let* $\rho_{\overline{k}}\colon \pi_1(X_{\overline{k}}) \to \mathrm{GL}_r(\overline{\mathbb{Q}}_\ell)$ *be a semisimple local system on* $X_{\overline{k}}$ *and write* $\rho = \rho_{\overline{k}}|\pi_1(X)$. *Then the following are equivalent.*

*(i)* $\rho$ *is arithmetic;*
*(ii)* $\rho_{\overline{k}}$ *is arithmetic;*
*(iii) The orbit of* $[\rho_{\overline{k}}]$ *under* $\mathrm{Gal}(\overline{k}/k)$ *is finite.*

**Remark 5.3.** The above Proposition generalizes [JE24, Remark 3.2], where it is only proved in characteristic 0.

*Proof of Proposition 5.2.* Clearly (ii) ⇒ (i).

The implication (iii) ⇒ (ii) is by Theorem 4.8: we find an open subgroup $U = \mathrm{Gal}(\overline{k}/k')$ corresponding to a finite extension $k \subset_f k' \subset \overline{k}$ such that $\rho_{\overline{k}}$ spreads to a representation of $\pi_1(X_{\overline{k}}, \overline{x}) \rtimes \mathrm{Gal}(\overline{k}/k') \simeq \pi_1(X_{k'}, \overline{x})$.

We prove (i) ⇒ (iii). By assumption, we can find a finitely generated subfield $F \subset K$ such that $\rho$ spreads out to $\rho_F$ on $X_F$. The compositum $kF \subset K$ is also finitely generated, and so we are free to assume that $k \subset F$. Consider the commutative diagram with (compatibly) split exact rows

$$\begin{tikzcd}
1 \arrow[r] & \pi_1(X_{\overline{F}},\overline{x}) \arrow[r] \arrow[d] & \pi_1(X_F,\overline{x}) \arrow[r] \arrow[d] & \mathrm{Gal}(\overline{F}/F) \arrow[r] \arrow[d] \arrow[l, bend right, dashed] & 1 \\
1 \arrow[r] & \pi_1(X_{\overline{k}},\overline{x}) \arrow[r] & \pi_1(X_k,\overline{x}) \arrow[r] & \mathrm{Gal}(\overline{k}/k) \arrow[r] \arrow[l, bend left, dashed] & 1.
\end{tikzcd}$$

This diagram shows that the action of $\mathrm{Gal}(\overline{k}/k)$ on the set of conjugacy classes of representations $\pi_1(X_{\overline{k}},\overline{x}) \to \mathrm{GL}_r(\overline{\mathbb{Q}}_\ell)$ is, in the appropriate way, compatible with the action of $\mathrm{Gal}(\overline{F}/F)$ on the set of conjugacy classes of represenations $\pi_1(X_{\overline{F}},\overline{x}) \to \mathrm{GL}_r(\overline{\mathbb{Q}}_\ell)$. The conjugacy class $[\rho_{\overline{F}}]$ is invariant under the action of $\mathrm{Gal}(\overline{F}/F)$ because of the spreading $\rho_F$. It follows by surjecivity of $\pi_1(X_{\overline{F}},\overline{x}) \to \pi_1(X_{\overline{k}},\overline{x})$ that $[\rho_{\overline{k}}]$ is invariant under the image of $\mathrm{Gal}(\overline{F}/F)$ in $\mathrm{Gal}(\overline{k}/k)$. This image is open, and so we have proved (iii). ■

**Corollary 5.4.** *In the situation of Proposition 5.2, given a connected finite étale cover* $f\colon Y \to X$, *the representation* $\rho$ *is arithmetic if and only if the restriction* $\rho|\pi_1(Y)$ *is.*

*Proof.* After potentially extending $k$, we can assume that $f$ spreads to a finite étale cover $f_{\overline{k}}\colon Y_{\overline{k}} \to X_{\overline{k}}$. By the equivalence (i) ⇔ (ii) of Proposition 5.2, we are then free to assume $K = \overline{k}$. Now the corollary follows from Proposition 4.7 and part (iii). ■

**Corollary 5.5.** *In the situation of Proposition 5.2, assume that* $\rho'$ *is a subquotient of* $\rho$. *Then if* $\rho$ *is arithmetic, so is* $\rho'$.

*Proof.* The representation $\rho'$ descends to a subquotient (equivalently, a direct summand) of the semisimple representation $\rho_{\overline{k}}$. Now by the equivalence (ii) ⇔ (iii) of Proposition 5.2 and Corollary 4.5, we conclude that also $\rho'$ is arithmetic. ■

**Example 5.6.** Assume $K = \overline{k}$. We argue that local systems of geometric origin on $X$ (see Examples 3.10 and 3.16(ii)) are arithmetic. Consider first the local system $\mathbb{L} = R^i f_* \overline{\mathbb{Q}}_\ell$ for $f\colon Y \to X$ a smooth proper morphism. We can assume that $f$ spreads to a smooth proper morphism $f_k\colon Y_k \to X_k$ over $k$. Then $\mathbb{L} = (R^i f_{k,*}\overline{\mathbb{Q}}_\ell)|X$ by proper base change, and so $\mathbb{L}$ is arithmetic. Any subquotient of $\mathbb{L}$ is arithmetic by the above corollary. For a general local system $\mathbb{L}$ on $X$ of geometric origin, we can find a dense open $U \subset X$ on which it is arithmetic by the arguments above. Now arithmeticity of $\mathbb{L}$ follows from surjectivity of $\pi_1(U) \twoheadrightarrow \pi_1(X)$ and part (iii) of Proposition 5.2.

## 5.2. **Finiteness of arithmetic characters.**

**Proposition 5.7.** *Let* $X \to \operatorname{Spec} K$ *be a smooth connected curve and let* $\chi\colon \pi_1(X) \to \overline{\mathbb{Q}}_\ell^\times$ *be an arithmetic character. Then* $\chi$ *is a finite character.*

The proof goes by reducing to the case where $K$ is the algebraic closure of a finite field. We spell it out in some detail here, because the argument comes in handy in Section 7.

*Proof.* Replacing $X$ by an étale cover, we are free to assume that $\chi$ factors over $\pi_1^{(\ell)}(X)$. Let $\overline{X} \supset X$ be the compactification of $X$. Write

$$x_1, \dots, x_n \colon \operatorname{Spec} K \to \overline{X}$$

for the points on the boundary $\overline{X} \setminus X$. Fix furthermore a point $x \colon \operatorname{Spec} K \to X$. By spreading out, we find a finite type $\mathbb{Z}$-algebra $R \subset K$, a smooth proper scheme $\overline{X}_R \to \operatorname{Spec} R =: S$ with geometrically connected fibers, and sections $x_R, x_{i,R} \colon S \to \overline{X}_R$ with disjoint images such that

$$\overline{X}_R \times_R \operatorname{Spec} K = \overline{X}, \quad x_R \times_R \operatorname{Spec} K = x, \quad x_{i,R} \times_R \operatorname{Spec} K = x_i.$$

Set

$$X_R = \overline{X}_R \setminus (x_1(S) \cup \ldots \cup x_n(S)).$$

Then $X_R \times_R \operatorname{Spec} K = X$ and $x_R$ is a section of $X_R \to S$. Now Proposition 2.3 gives rise to the exact sequence

$$\begin{tikzcd} 1 \arrow[r] & \pi_1^{(\ell)}(X,x) \arrow[r] & \pi'_1(X_R,x) \arrow[r] & \pi_1(S,x) \arrow[r] \arrow[l, bend right, dashed] & 1 \end{tikzcd} \tag{5.2}$$

split by $x_R$.

After further enlarging $R$, we can assume it is regular, and so in particular normal. We can also assume that $\ell$ is invertible in $R$. Write $k$ for the fraction field of $R$, and denote by $\overline{k} \subset K$ the separable closure. We obtain a commutative diagram

$$\begin{tikzcd} 1 \arrow[r] & \pi_1(X_{\overline{k}},x) \arrow[r] \arrow[d, two heads] & \pi_1(X_k,x) \arrow[r] \arrow[d] & \operatorname{Gal}(\overline{k}/k) \arrow[r] \arrow[d, two heads] \arrow[l, bend right, dashed] & 1 \\ 1 \arrow[r] & \pi_1^{(\ell)}(X,x) \arrow[r] & \pi'_1(X_R,x) \arrow[r] & \pi_1(S,x) \arrow[r] \arrow[l, bend left, dashed] & 1 \end{tikzcd}$$

with (compatibly) split exact rows. Here the rightmost vertical map is surjective by normality of $S$, and the leftmost vertical map is surjective due to the fact that it arises as the composition $\pi_1(X_{\overline{k}},x) \twoheadrightarrow \pi_1^{(\ell)}(X_{\overline{k}},x) \simeq \pi_1^{(\ell)}(X,x)$.

The general theory of Section 4 gives rise to an action of $\pi_1(S,x)$ on the set of characters $\pi_1^{(\ell)}(X,x) \to \overline{\mathbb{Q}}_\ell^\times$, which by the above diagram is compatible with the action of $\operatorname{Gal}(\overline{k}/k)$ on the set of characters $\pi_1(X_{\overline{k}},x) \to \overline{\mathbb{Q}}_\ell^\times$. Propostion 5.2 and arithmeticity of $\chi$ implies that it has finite orbit under $\operatorname{Gal}(\overline{k}/k)$ when viewed as a character of $\pi_1(X_{\overline{k}},x)$. Surjectivity of $\operatorname{Gal}(\overline{k}/k) \to \pi_1(S,x)$ and of $\pi_1(X_{\overline{k}},x) \to \pi_1^{(\ell)}(X,x)$ now implies that $\chi$ also has finite orbit under $\pi_1(S,x)$. By Theorem 4.8, $\chi$ spreads to $\chi_R \colon \pi'_1(X_R,x) \to \overline{\mathbb{Q}}_\ell^\times$ after replacing $S$ by an étale cover.

Pick a geometric point $\overline{s} \to S$ over a closed point $s$ of $S$ of residue characteristic different from $\ell$. Notice that $\kappa(s)$ is a finite field. Pick a specialization $x \rightsquigarrow \overline{s}$ on $S$. It gives rise to a canonical étale path on $S$, and hence via the section $S \to X_R$ to one on $X_R$. The character

$\chi_R$ now restricts to $\chi_{\overline{s}}\colon \pi_1^{(\ell)}(X_{\overline{s}},\overline{s}) \to \overline{\mathbb{Q}}_\ell^\times$ along $\pi_1^{(\ell)}(X_{\overline{s}},\overline{s}) \to \pi_1'(X_R,\overline{s}) \simeq \pi_1'(X_R,x)$, which is again arithmetic: it comes from a character of $\pi_1(X_s,\overline{s})$. Theorem 2.2 in combination with Remark 2.3 gives rise to an isomorphism $\pi_1^{(\ell)}(X,x) \xrightarrow{\simeq} \pi_1^{(\ell)}(X_{\overline{s}},\overline{s})$ which fits into a commutative square

$$\begin{tikzcd}
\pi_1^{(\ell)}(X,x) \arrow[r, "\simeq"] \arrow[d] & \pi_1^{(\ell)}(X_{\overline{s}},\overline{s}) \arrow[d] \\
\pi_1'(X_R,x) \arrow[r, "\simeq"] & \pi_1'(X_R,\overline{s}).
\end{tikzcd}$$

We now conclude by the fact that $\chi_{\overline{s}}$ is finite [Del80, Proposition 1.3.4]. ■

**Remark 5.8.** If $\rho\colon \pi_1(X) \to \mathrm{GL}_r(\overline{\mathbb{Q}}_\ell)$ is arithmetic, then so is its determinant character $\det\rho\colon \pi_1(X) \to \overline{\mathbb{Q}}_\ell^\times$. By the above proposition it is a finite character.

**Remark 5.9.** We can generalize the proposition above for $X \to \operatorname{Spec} K$ only connected normal of finite type. Since we don't need this more general statement in this thesis, we only outline the proof. By the theory of alterations à la De Jong, we can reduce to proving the statement for $X \to \operatorname{Spec} K$ the complement of a strict normal crossings divisor on a smooth connected projective scheme $\overline{X} \to \operatorname{Spec} K$. Then, by applying the weak Lefschetz Theorem for tame fundamental groups [EK15], we can further reduce to the case where $X$ is a curve, which is proved above.

## 6. A COUNTEREXAMPLE TO THE NAIVE LOCAL KASHIWARA CONJECTURE

Let $R$ be a strictly henselian discrete valuation ring with algebraically closed residue field and write $S = \operatorname{Spec} R$. Denote the closed point of $S$ by $s$ and the generic point of $S$ by $\eta$. Denote the residue field of $R$ by $k$ and let $p \geq 0$ be its characteristic. Throughout, $\ell$ denotes a prime different from $p$. The goal of this section is to give a counterexample to the naive version of Conjecture 1.5, where the arithmeticity condition is omitted. Specifically, we will construct $\mathscr{X} \to S$ a smooth quasi-compact separated morphism of schemes and a semi-simple $\ell$-adic local system $\mathbb{L}$ on $\mathscr{X}$ such that the pullback of $\mathbb{L}$ to a local system on $\mathscr{X}_s$ is not semi-simple. We mention that our example works for $R$ of mixed or equal characteristic and for any residue characteristic.

Our counterexample is based on a suggestion by T. Mochizuki; see Appendix A. It involves an elliptic fibration whose special fiber is a rational nodal curve. We proceed by first working out some facts regarding the fundamental group of this nodal curve. The counterexample is then constructed in Section 6.2.

### 6.1. The (tame) fundamental group of a nodal curve.

Let $C$ denote an integral curve over $k$. We suppose that there is a simple node $n \in C(k)$, i.e., the (necessarily strict) henselization $\mathscr{O}_{C,n}^{\mathrm{h}}$ of the local ring $\mathscr{O}_{C,n}$ is isomorphic to $(k[x,y]/(xy))^{\mathrm{h}}_{(x,y)}$ as a local $k$-algebra.

Denote by $\pi\colon \widetilde{C} \to C$ the morphism obtained by normalizing $c$; i.e., we take the normalization of an open neighborhood of $n$ in which $n$ is the only singular point and we then glue the resulting curve to $C \setminus \{n\}$. The preimage of $n$ under $\pi$ consists of exactly two points. This can be seen by first pulling back $\pi$ along $\operatorname{Spec}\mathscr{O}^{\mathrm{h}}_{C,n} \to C$ and the fact that this pullback is the normalization of $\operatorname{Spec}\mathscr{O}^{\mathrm{h}}_{C,n}$ [Sta24, Tag 0CBM]; then notice that the normalization of $\operatorname{Spec}(k[x,y]/(xy))^{\mathrm{h}}_{(x,y)} \simeq \operatorname{Spec}\mathscr{O}^{\mathrm{h}}_{C,n}$ is

$$\operatorname{Spec} k[x]^{\mathrm{h}}_{(x)} \sqcup \operatorname{Spec} k[y]^{\mathrm{h}}_{(y)} \to \operatorname{Spec}(k[x,y]/(xy))^{\mathrm{h}}_{(x,y)} \tag{6.1}$$

and that the fiber of this morphism over the closed point consists of exactly two points. Denote by $a$ and $b$ the points of $\widetilde{C}$ lying over $n$. Consider the following "normalization sequence" of $\mathscr{O}_C$-modules

$$0 \to \mathscr{O}_C \to \pi_*\mathscr{O}_{\widetilde{C}} \to k_n \to 0, \tag{6.2}$$

where $k_n$ is the skyscraper sheaf supported only at the node $n$ with stalk $k$. Here the map $\pi_*\mathscr{O}_{\widetilde{C}} \to k_n$ is defined by $f \mapsto f(a) - f(b)$.

**Proposition 6.1.** *The sequence* (6.2) *is exact.*

The above proposition is a commonly known fact from the theory of algebraic curves [Har10, p. IV.1.8]. We prove it here also directly.

*Proof.* It is clear that for every point $c \in C$, the sequence of stalks at $c$ is exact, except possibly at the node $n$. Consider the sequence of stalks at $n$

$$0 \to \mathscr{O}_{C,n} \to (\pi_*\mathscr{O}_{\widetilde{C}})_n \to k = \mathscr{O}_{C,n}/\mathfrak{m}_n \to 0.$$

Recall that the henselization $\mathscr{O}_{C,n} \to \mathscr{O}^{\mathrm{h}}_{C,n}$ is faithfully flat, and so we need only prove that the sequence we obtain after tensoring with $\mathscr{O}^{\mathrm{h}}_{C,n}$,

$$0 \to \mathscr{O}^{\mathrm{h}}_{C,n} \to (\pi_*\mathscr{O}_{\widetilde{C}})_n \otimes_{\mathscr{O}_{C,n}} \mathscr{O}^{\mathrm{h}}_{C,n} \to \mathscr{O}^{\mathrm{h}}_{C,n}/\mathfrak{m}_n\mathscr{O}^{\mathrm{h}}_{C,n} = k \to 0,$$

is exact. This sequence identifies with the sequence

$$0 \to (k[x,y]/(xy))^{\mathrm{h}}_{(x,y)} \to k[x]^{\mathrm{h}}_{(x)} \times k[y]^{\mathrm{h}}_{(y)} \to k \to 0 \tag{6.3}$$

by (6.1) and the argument preceeding it. The sequence (6.3) is clearly exact. ■

6.1.1. *The category of finite étale covers.* Denote by $\mathscr{C}$ the category whose objects are finite étale covers $X \xrightarrow{f} \widetilde{C}$ equipped with an isomorphism $\varphi\colon f^{-1}(a) \xrightarrow{\simeq} f^{-1}(b)$ and whose morphisms $(f,\varphi) \to (f',\varphi')$ are morphisms of finite étale covers such that the square

$$\begin{tikzcd}
f^{-1}(a) \arrow[r] \arrow[d, "\varphi"] & f'^{-1}(b) \arrow[d, "\varphi'"] \\
f^{-1}(a) \arrow[r] & f'^{-1}(b)
\end{tikzcd}$$

commutes. The isomorphism $\varphi$ is referred to as *a glueing datum for $f$*. Given a finite étale cover $g\colon Y \to C$, we obtain a finite étale cover

$$\pi^* g\colon \pi^* Y := Y \times_C \widetilde{C} \to C$$

by pulling the morphism $g$ back along $\pi$. We have a canonical isomorphism

$$(\pi^* g)^{-1}(a) \simeq (\pi^* g)^{-1}(b)$$

and hence a canonical descent datum for $\pi^* g$. This gives us a functor

$$\pi^*\colon \mathrm{Fét}_C \to \mathscr{C}. \tag{6.4}$$

The rest of this section is devoted to constructing a pseudo-inverses to these functors.

6.1.2. *A glueing construction.* Given an object $(X \xrightarrow{f} \widetilde{C}, \varphi)$ of $\mathscr{C}$, we intend to construct a finite étale cover of $C$ by "glueing the fibers $f^{-1}(a)$ and $f^{-1}(b)$ together along the isomorphism $\varphi$". Let $U$ be an affine open neighborhood of $n$, denote by $\widetilde{U} \subset \widetilde{C}$ the pullback of $U$ under $\pi$, and denote by $V \subset X$ the pullback of $\widetilde{U}$ under $f$. The open $V$ is again affine. Define the ring $\mathscr{O}(\overline{V})$ by

$$\mathscr{O}(\overline{V}) = \{f \in \mathscr{O}(V) : f(x) = f(\varphi(x)) \text{ for all } x \in f^{-1}(a)\}.$$

It defines the coordinate ring of an affine scheme $\overline{V}$.

**Proposition 6.2.** *The square*

$$\begin{tikzcd}
f^{-1}(a) \sqcup f^{-1}(b) \arrow[r] \arrow[d, "\varphi\sqcup\mathrm{id}"] & V \arrow[d, "\rho"] \\
f^{-1}(b) \arrow[r] & \overline{V}
\end{tikzcd}$$

*defines a pushout square in the category of affine $k$-schemes.*

*Proof.* The corresponding map on rings defines a pullback square in the category of $k$-algebras. ■

We glue $\overline{V}$ together with $X \setminus (f^{-1}(a) \cup f^{-1}(b))$ to obtain a scheme $\overline{X}$, and we let $\rho\colon X \to \overline{X}$ denote the obvious map. Clearly, the scheme $\overline{X}$ does not depend on the affine open $U$ we started with.

By construction of the scheme $\overline{X}$, we also have an exact sequence of $\mathscr{O}_{\overline{X}}$-modules on $\overline{X}$

$$0 \to \mathscr{O}_{\overline{X}} \to \rho_* \mathscr{O}_X \to \bigoplus_{x \in f^{-1}(a)} k_{\rho(x)} \to 0, \tag{6.5}$$

where $\bigoplus_{x \in f^{-1}(a)} k_{\rho(x)}$ is the skyscraper sheaf on $\overline{X}$ supported at the points $\rho(x) = \rho(\varphi(x))$ with stalk $k$ for $x \in f^{-1}(a)$. Letting $y = \varphi(x)$ and $z = \rho(x) = \rho(y)$, we obtain the exact sequence of stalks

$$0 \to \mathscr{O}_{\overline{X},z} \to (\rho_* \mathscr{O}_X)_z \to k \to 0.$$

Set $\mathscr{O}_{X,x\cup y} = (\rho_*\mathscr{O}_X)_z$. This is a semilocal ring with maximal ideals $\mathfrak{m}_x$ and $\mathfrak{m}_y$, corresponding to the points $x$ and $y$. The ideal $\mathfrak{m}_z\mathscr{O}_{X,x\cup y}$ is precisely the Jacobson radical of $\mathscr{O}_{X,x\cup y}$. As a result, the $\mathfrak{m}_z$-completion of $\mathscr{O}_{X,x\cup y}$ is isomorphic to $\widehat{\mathscr{O}}_{X,x} \times \widehat{\mathscr{O}}_{Y,y}$ by [MR86, Theorem 8.15]. Taking $\mathfrak{m}_z$-completions of $\mathscr{O}_{\overline{X},z}$-modules now yields the exact sequence

$$0 \to \widehat{\mathscr{O}}_{\overline{X},z} \to \widehat{\mathscr{O}}_{X,x} \times \widehat{\mathscr{O}}_{X,y} \to k \to 0. \tag{6.6}$$

6.1.3. *An equivalence of categories.* Start with an object $(X \xrightarrow{f} \widetilde{C}, \varphi)$ of $\mathscr{C}$ and let $U, \widetilde{U}$ and $V$ be as before. We obtain by Proposition 6.2 a commutative diagram

(6.7)
$$\begin{tikzcd} V \arrow[r, "\rho"] \arrow[d, "f"] & \overline{V} \arrow[d, dashed, "\overline{f}"] \\ \widetilde{U} \arrow[r, "\pi"] & U. \end{tikzcd}$$

Glueing the morphism $\overline{f}$ with the morphism $X \setminus (f^{-1}(a) \cup f^{-1}(b)) \to X \setminus \{n\}$, we obtain $\overline{f} \colon \overline{X} \to C$. Again, it is clear that this morphism does not depend on our choice of open neighborhood $U$.

**Proposition 6.3.** *The morpism $\overline{f} \colon \overline{X} \to C$ constructed in* (6.7) *is finite étale.*

*Proof.* It is affine by construction, and finite by finiteness of $f$ and $\pi$ and the fact that $C$ is noetherian [Sta24, Tag 00FP]. The fact that it is étale at all points outside the fiber over $n$ is clear. To prove that it is étale at the point $z = \rho(x) = \rho(y)$ in the fiber over $n$, for some $x \in f^{-1}(a)$ and $y = \varphi(x)$, we show that the induced map on completed local rings $\widehat{\mathscr{O}}_{C,n} \to \widehat{\mathscr{O}}_{\overline{X},z}$ is an isomorphism. We conclude that $\overline{f}$ is étale at $z$ [Har10, Chapter 2, Exercise 10.4]. Recall the exact sequence from (6.6). We can derive a similar such sequence for the completed local ring of $C$ at $n$ from (6.2). We obtain a commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & \widehat{\mathscr{O}}_{\overline{X},z} \arrow[r] & \widehat{\mathscr{O}}_{X,x} \times \widehat{\mathscr{O}}_{X,y} \arrow[r] & k \arrow[r] & 0 \\ 0 \arrow[r] & \widehat{\mathscr{O}}_{C,n} \arrow[r] \arrow[u] & \widehat{\mathscr{O}}_{\widetilde{C},a} \times \widehat{\mathscr{O}}_{\widetilde{C}_m,b} \arrow[r] \arrow[u, "\simeq"] & k \arrow[r] \arrow[u, "="] & 0, \end{tikzcd}$$

where the middle vertical map is an isomorphism by the fact that $f$ is étale. It follows that $\widehat{\mathscr{O}}_{C,n} \to \widehat{\mathscr{O}}_{\overline{X},z}$ is an isomorphism. ∎

This construction is clearly functorial, and so we obtain the functor

$$\begin{gathered} G \colon \mathscr{C} \to \mathrm{Fét}_{C^\circ} \\ (X \xrightarrow{f} \widetilde{C}, \varphi) \mapsto (\overline{X} \xrightarrow{\overline{f}} C). \end{gathered} \tag{6.8}$$

**Theorem 6.4.** *The functors from* (6.4) *and* (6.8) *are pseudo-inverse to each other. As a result, we obtain an equivalence of categories*

$$\mathrm{F\acute{e}t}_C \simeq \mathscr{C}.$$

*Proof.* Let $Y \to C$ be a finite étale morphism. We naturally obtain a commutative triangle

$$\begin{tikzcd}
\overline{\pi^* Y} \arrow[rr, dashed] \arrow[dr] & & Y \arrow[dl] \\
& C. &
\end{tikzcd}$$

Since $\overline{\pi^* Y}$ and $Y$ are finite étale of the same degree over $C$, the dashed arrow above is finite étale of degree 1 because it is surjective. It follows that the dashed arrow is an isomorphism. So we have an isomorphism of functors $G \circ \pi^* \simeq \mathrm{id}$.

Conversely, if $(X, \varphi)$ is a finite étale cover of $\widetilde{C}$ with a glueing datum, then we naturally obtain a commutative triangle

$$\begin{tikzcd}
X \arrow[rr, dashed] \arrow[dr] & & \pi^* \overline{X} \arrow[dl] \\
& \widetilde{C}. &
\end{tikzcd}$$

By the same argument as before, the dashed arrow is an isomorphism. Hence, we obtain an isomorphism of functors $\pi^* \circ G \simeq \mathrm{id}$. ■

**Remark 6.5.** We have chosen to work very explicitly in this section. To prove the above theorem, one could alternatively show that $\pi\colon \widetilde{C} \to C$ is an *effective descent morphism* for finite étale covers [GR71, Exp IX, Thm 4.2]. This implies that $\mathrm{F\acute{e}t}_C$ is isomorphic to the category of finite étale covers of $\widetilde{C}$ with a descent datum relative to $\pi$ in the sense of [Gro65]. One can then write down an explicit equivalence between this category and the category $\mathscr{C}$ of this section.

We will need also the "tame version" of the above theorem. Denote by $\mathscr{C}^t$ the full subcategory consisting of pairs $(f, \varphi)$ with $f$ a tamely ramified [4] covering.

**Theorem 6.6.** *The equivalence of Theorem 6.4 restricts to an equivalence*

$$\mathrm{F\acute{e}t}_C^t \simeq \mathscr{C}^t.$$

---

[4] In general, one needs to specify a compactification to even define what it means for an étale covering to be tame (see Section 2). For curves there is a unique compactification, and so this ambiguity does not arise.

6.1.4. *Computing the (tame) fundamental group.* We fix, once and for all, an étale path $\mathrm{Fib}_a \simeq \mathrm{Fib}_b$ on $\widetilde{C}$. Via this étale path, we obtain the equivalences of categories

$$\begin{aligned}
\mathrm{Fét}_C &\simeq \mathscr{C} \\
&\simeq \{\text{pairs } (f\colon X \to \widetilde{C}, \varphi \in \mathrm{Aut}(f^{-1}(a))) \text{ with } f \text{ finite étale}\} \\
&\simeq \{\text{finite sets } F \text{ with a continuous action of } \pi_1(\widetilde{C}, a) \text{ and of } \hat{\mathbb{Z}}\} \\
&\simeq \pi_1(\widetilde{C}, a) * \hat{\mathbb{Z}}\text{-sets},
\end{aligned} \tag{6.9}$$

where the last category consists of finite sets with a continuous action of $\pi_1(\widetilde{C}, a) * \hat{\mathbb{Z}}$, the coproduct of $\pi_1(\widetilde{C}, a)$ and $\hat{\mathbb{Z}}$ as profinite groups. Here the third equivalence is induced by the functor $\mathrm{Fib}_a$. We have a commutative triangle

$$\begin{tikzcd}
\mathrm{Fét}_C \arrow[rr, "\simeq"] \arrow[dr, "\mathrm{Fib}_n"] & & \pi_1(\widetilde{C}, a) * \hat{\mathbb{Z}}\text{-sets} \arrow[dl] \\
& \text{sets,} &
\end{tikzcd} \tag{6.10}$$

where the arrow $\pi_1(\widetilde{C}, a) * \hat{\mathbb{Z}}\text{-sets} \to \text{sets}$ is the evident forgetful functor.

**Theorem 6.7.** *(i) The diagram* (6.10) *induces a canonical isomorphism*

$$\pi_1(C, n) \simeq \pi_1(\widetilde{C}, a) * \hat{\mathbb{Z}}.$$

*(ii) The map $\pi_*\colon \pi_1(\widetilde{C}, a) \to \pi_1(C, n)$ is identified with the canonical inclusion*

$$\pi_1(\widetilde{C}, a) \to \pi_1(\widetilde{C}, a) * \hat{\mathbb{Z}}$$

*under the isomorphism of (i).*

*Proof.* Part (i) is clear by the general machinery of Galois categories. Part (iii) follows from the commutative square

$$\begin{tikzcd}
\mathrm{Fét}_C \arrow[r, "\pi^*"] \arrow[d, "\simeq"] & \mathrm{Fét}_{\widetilde{C}} \arrow[d, "\simeq"] \\
\pi_1(C, a) * \hat{\mathbb{Z}}\text{-sets} \arrow[r] & \pi_1(\widetilde{C}, a)\text{-sets,}
\end{tikzcd}$$

where the horizontal arrow on the bottom is given by precomposing the action on a finite set by the canonical inclusion $\pi_1(\widetilde{C}, a) \to \pi_1(\widetilde{C}, a) * \hat{\mathbb{Z}}$. ■

**Remark 6.8.** Of course, the theorem above holds for tame fundamental groups by the same arguments, applying instead Theorem 6.6.

We will want to apply the above theorem (or rather its tame version) in particular in the case where $C$ is a rational curve with a single node minus two smooth points. To this end, we state the following lemma.

**Lemma 6.9.** *Denote by $\mathbb{G}_m$ the curve $\mathbb{P}^1_k \setminus \{0, \infty\}$. There is an isomorphism $\pi_1^t(\mathbb{G}_m) \simeq \widehat{\mathbb{Z}}^{(p')}$.*

*Proof.* Let $\varphi\colon Y \to \mathbb{G}_m$ be a degree $d$ connected tame étale cover of $\mathbb{G}_m$. Denote by $\overline{Y}$ the unique smooth compactifcation of $Y$. Then $\varphi$ extends to a degree-$d$ morphism $\overline{\varphi}\colon \overline{Y} \to \mathbb{P}^1$ of curves. Denote by $e_1,\dots,e_r$, respectively $f_1,\dots,f_s$, the ramification indices of $\overline{\varphi}$ over 0, respectively $\infty$. Since $\overline{\varphi}$ is tame, we can apply the Riemann Hurwitz formula [Har10, Chapter IV, Corollary 2.4]. We obtain

$$2g_Y - 2 = -2d + \sum_{i=1}^{r}(e_i - 1) + \sum_{j=1}^{s}(f_j - 1),$$

where $g_Y$ denotes the genus of $Y$. After some consideration, this shows that $g_Y = 0$ and $e_1 = f_1 = d$. This leaves exactly one option for $Y$ and $\varphi$ up to isomorphism, namely $Y \simeq \mathbb{G}_m$ and $\varphi\colon y \mapsto y^d$ with $d$ different from $p$. It has automorphism group cyclic of order $d$. We find

$$\pi_1^t(\mathbb{G}_m) \simeq \varprojlim_{d\neq p} \mathbb{Z}/d\mathbb{Z} \simeq \widehat{\mathbb{Z}}^{(p')}.$$

■

6.2. **A counterexample (after Mochizuki).** Denote by $\overline{\eta} \to S$ a geometric point over the generic point of $S$. Let $\overline{\mathscr{X}} \to S$ be an elliptic fibration with special fiber of type $I_1$ in Kodaira's classification of singular fibers[5]. Specifically,

- $\overline{\mathscr{X}} \to S$ is a proper morphism with $\overline{\mathscr{X}}$ a regular scheme;
- the generic fiber $\overline{\mathscr{X}}_{\overline{\eta}} \to \overline{\eta}$ is a smooth curve of genus 1;
- the special fiber $\overline{\mathscr{X}}_s \to s$ is a rational curve with a single node. Denote by $n \in \overline{\mathscr{X}}_s(k)$ this node.

Fix pairwise distinct non-singular points $x_0, x_1, x_2 \in \overline{\mathscr{X}}_s(k)$. By the above Lemma, we can now apply Hensel's Lemma [Mil16, Chapter I, Exercise 4.13] to obtain $R$-points

$$s_i\colon S \to \overline{\mathscr{X}},$$

for $i = 1,2$, such that $x_i$ is equal to the composition

$$\operatorname{Spec} k \to S \to \overline{\mathscr{X}}.$$

Since $\overline{\mathscr{X}} \to S$ is separated, the $s_i$ are closed immersions [Har10, Chapter II, Exercise 4.8]. We obtain an effective Cartier divisor $D = s_1(S) + s_2(S)$ on $\overline{\mathscr{X}}$. Set $D := \sigma_1(S) \cup \sigma_2(S)$. We define the $S$-scheme $\mathscr{X}' \to S$ by $\mathscr{X}' := \overline{\mathscr{X}} \setminus D$. It makes sense to now consider the tame fundamental groups $\pi_1^t(\mathscr{X}'_s)$ and $\pi_1^t(\mathscr{X}')$, where tameness is measured with respect to the boundary divisors $D$ and $D_s$. Proposition 2.1 now gives

**Proposition 6.10.** *The canonical homomorphism* $\pi_1^t(\mathscr{X}'_s, x_0) \xrightarrow{\simeq} \pi_1^t(\mathscr{X}', x_0)$ *is an isomorphism.*

---

[5] See [Sil86, Appendix C.15] for the terminology.

Consider the normalization $\pi\colon \mathbb{P}^1_k \to \overline{\mathscr{X}}_s$. Without loss of generality, we assume that the points lying over $x_1$ and $x_2$ are 0 and $\infty$, respectively. Denote the point over $x_0$ by $\tilde{x}_0$. We obtain the normalization $\mathbb{G}_m \to \mathscr{X}'_s$ of $\mathscr{X}'_s$ by restricting to $\mathbb{G}_m = \mathbb{P}^1 \setminus \{0,\infty\}$. In accordance with the previous section, we let $a$ and $b$ be the points over the node $n$. We fix, once and for all, an étale path $\mathrm{Fib}_{\tilde{x}_0} \simeq \mathrm{Fib}_a$. This induces an étale path $\mathrm{Fib}_{x_0} \simeq \mathrm{Fib}_n$. We obtain a commutative diagram

$$\begin{tikzcd}
\pi_1^t(\mathbb{G}_m, \tilde{x}_0) \arrow[r, "\simeq"] \arrow[d, "\pi_*"] & \pi_1^t(\mathbb{G}_m, a) \arrow[r, "="] \arrow[d, "\pi_*"] & \pi_1^t(\mathbb{G}_m, a) \arrow[d] \\
\pi_1^t(\mathscr{X}'_s, x_0) \arrow[r, "\simeq"] & \pi_1^t(\mathscr{X}'_s, n) \arrow[r, "\simeq"] & \pi_1^t(\mathbb{G}_m, a) * \hat{\mathbb{Z}}.
\end{tikzcd} \tag{6.11}$$

Here the isomorphisms of the left square come from the fixed étale paths, and the right hand square is by the tame version of Theorem 6.7. It is by the isomorphisms in (6.11) that we identify $\pi_1^t(\mathbb{G}_m, \tilde{x}_0)$ with $\pi_1^t(\mathbb{G}_m, a)$ and $\pi_1(\mathscr{X}'_s, x_0)$ with $\pi_1^t(\mathbb{G}_m, a) * \hat{\mathbb{Z}}$. In particular, by Lemma 6.9, we obtain isomorphisms

$$\pi_1^t(\mathbb{G}_m, \tilde{x}_0) \simeq \hat{\mathbb{Z}}^{(p')} \quad \text{and} \quad \pi_1^t(\mathscr{X}'_s, x_0) \simeq \hat{\mathbb{Z}}^{(p')} * \hat{\mathbb{Z}},$$

and under these identifications the map $\pi_1^t(\mathbb{G}_m, \tilde{x}_0) \to \pi_1^t(\mathscr{X}'_s, x_0)$ corresponds to the inclusion $\hat{\mathbb{Z}}^{(p')} \hookrightarrow \hat{\mathbb{Z}}^{(p')} * \hat{\mathbb{Z}}$.

Recall that $\mathrm{GL}_2(\mathbb{Z}_\ell)$ is a profinite group and that $\ker \mathrm{GL}_2(\mathbb{Z}_\ell) \to \mathrm{GL}_2(\mathbb{F}_\ell)$ is a pro-$\ell$ group. Consider the homomorphism

$$\hat{\mathbb{Z}}^{(p')} * \hat{\mathbb{Z}} \to \mathrm{GL}_2(\mathbb{Z}_\ell)$$

defined by sending the first generator to $\left(\begin{smallmatrix} 1 & \ell \\ 0 & 1 \end{smallmatrix}\right)$ and the second generator to $\left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$. We obtain a continuous representation

$$\rho'_s\colon \pi_1^t(\mathscr{X}'_s, x_0) \simeq \hat{\mathbb{Z}}^{(p')} * \hat{\mathbb{Z}} \to \mathrm{GL}_2(\mathbb{Z}_\ell) \hookrightarrow \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell). \tag{6.12}$$

**Lemma 6.11.** *The representation $\rho'_s$ from* (6.12) *is irreducible.*

*Proof.* The image of $\rho'_s$ contains the subgroup of $\mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$ spanned by the matrices $\left(\begin{smallmatrix} 1 & \ell \\ 0 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$. It is easy to check that this subgroup does not fix any proper subspace of $\overline{\mathbb{Q}}_\ell^{\oplus 2}$. ■

Consider now the representation

$$\rho'_s \circ \pi_* : \pi_1^t(\mathbb{G}_m, \tilde{x}_0) \to \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell). \tag{6.13}$$

**Lemma 6.12.** *The representation $\rho'_s \circ \pi_*$ from* (6.13) *is not semi-simple.*

*Proof.* Identifying $\pi_1^t(\mathbb{G}_m, \tilde{x}_0)$ with $\hat{\mathbb{Z}}^{(p')}$ as before and restricting $\rho'_s \circ \pi_*$ to $\mathbb{Z}$, we obtain the representation $\mathbb{Z} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$ sending 1 to $\left(\begin{smallmatrix} 1 & \ell \\ 0 & 1 \end{smallmatrix}\right)$. Clearly, this representation has a single non-trivial subrepresentation, and hence is not semi-simple. The same is then true for the represenation $\rho'_s \circ \pi_*$ by continuity. ■

6.2.1. *Conclusion.* Consider the $S$-scheme $\mathscr{X} := \mathscr{X}' \setminus \{n\} \to S$. Its special fiber is $\mathscr{X}_s = \mathscr{X}'_s \setminus \{n\} \to \operatorname{Spec} k$. Define the representation $\rho\colon \pi_1(\mathscr{X}, x_0) \to \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$ to be the composition

$$\pi_1(\mathscr{X}, x_0) \xrightarrow{\simeq} \pi_1(\mathscr{X}', x_0) \twoheadrightarrow \pi_1^t(\mathscr{X}', x_0) \simeq \pi_1^t(\mathscr{X}'_s, x_0) \xrightarrow{\rho'_s} \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell). \tag{6.14}$$

Here the first isomorphism is by Proposition 3.23 and the second isomorphism comes from Proposition 6.10.

**Corollary 6.13.** *The representation $\rho$ from* (6.14) *is irreducible.*

*Proof.* This follows from Lemma 6.11 and the fact that $\rho$ and $\rho'_s$ share the same image. ■

To finish the construction, we only have to argue that

$$\rho \circ i_*\colon \pi_1(\mathscr{X}_s, x_0) \to \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell) \tag{6.15}$$

is *not* semi-simple. To this end, consider the diagram

$$\begin{tikzcd}
\pi_1(\mathbb{G}_m \setminus \{a,b\}, \tilde{x}_0) \arrow[r, "\simeq"] \arrow[d, two heads] & \pi_1(\mathscr{X}_s, x_0) \arrow[r, "i_*"] \arrow[d] & \pi_1(\mathscr{X}, x_0) \arrow[d, "\simeq"'] \arrow[ddr, bend left, "\rho"] & \\
\pi_1(\mathbb{G}_m, \tilde{x}_0) \arrow[r] \arrow[d, two heads] & \pi_1(\mathscr{X}'_s, x_0) \arrow[r] \arrow[d, two heads] & \pi_1(\mathscr{X}', x_0) \arrow[d, two heads] & \\
\pi_1^t(\mathbb{G}_m, \tilde{x}_0) \arrow[r, "\pi_*"] & \pi_1^t(\mathscr{X}'_s, x_0) \arrow[r, "\simeq"] \arrow[rr, bend right, "\rho'_s"'] & \pi_1^t(\mathscr{X}', x_0) & \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell).
\end{tikzcd} \tag{6.16}$$

**Corollary 6.14.** *The representation $\rho \circ i_*$ from* (6.15) *is not semi-simple.*

*Proof.* This follows from the fact that $\rho \circ i_*$ has the same image as $\rho'_s \circ \pi_*$ by (6.16), and the fact that $\rho'_s \circ \pi_*$ is not semi-simple by Lemma 6.12. ■

## 7. THE ARITHMETIC LOCAL KASHIWARA CONJECTURE IN EQUAL CHARACTERISTIC

Let $C/F$ be a connected normal curve, where $F$ is the complex numbers or a finite field, and denote by $\overline{c} \to C$ a geometric point of $C$ whose image is a closed point $c$. Write $\mathscr{O} = \mathscr{O}^{\mathrm{hs}}_{C,\overline{c}}$ for the strict henselization of $C$ at $\overline{c}$ and set $S = \operatorname{Spec} \mathscr{O}$. Let $\overline{\eta} \to S$ be a geometric point over the generic point $\eta$ of $S$. Let $\mathscr{X} \to S$ be a smooth quasi-compact separated morphism such that $\mathscr{X}_{\overline{\eta}}$ is of pure dimension 1.

**Theorem 7.1.** *Let $\mathbb{L}$ be a local system on $\mathscr{X}$ and suppose that $\mathbb{L}_{\overline{\eta}} := \mathbb{L}|\mathscr{X}_{\overline{\eta}}$ is semisimple and arithmetic. Then the local system $\mathbb{L}_{\overline{c}}$ is semisimple.*

**Remark 7.2.** The previous section demonstrates that the arithmeticity condition in the above theorem cannot be removed. Specifically, if $\mathscr{X} \to S$ is the scheme of Section 6.2.1, and $\rho\colon \pi_1(\mathscr{X}) \to \mathrm{GL}_r(\overline{\mathbb{Q}}_\ell)$ is the representation produced there, then by virtue of the above theorem $\rho_{\overline{\eta}}\colon \pi_1(\mathscr{X}_{\overline{\eta}}) \to \mathrm{GL}_r(\overline{\mathbb{Q}}_\ell)$ *cannot* be arithmetic. There does not seem to be another obvious way to rule out arithmeticity; e.g., it has finite determinant, as do arithmetic local systems by Proposition 5.7.

7.1. **Proof of Theorem 7.1.** We immediately reduce to the case where $\mathscr{X}$ is connected. We also assume that $\mathscr{X} \to S$ is surjective, because otherwise there is nothing to prove.

We intend to spread $\mathscr{X} \to S$ out to a smooth quasi-compact separated $C$-scheme $\mathscr{X}_C \to C$, and to spread the local system $\mathbb{L}$ on $\mathscr{X}$ out to a local system $\mathbb{L}_C$ on $\mathscr{X}_C$ such that $\mathbb{L}_C$ has finite determinant.

**Claim 7.3.** *If $\mathbb{L}_C$ as above exsits, then $\mathbb{L}_{\overline{c}}$ is a semisimple local system on $\mathscr{X}_{\overline{c}}$.*

*Proof.*

- Suppose $F$ is the field of complex numbers. We invoke Corollary 1.3 and the fact that on a complex variety $X$ an $\ell$-adic local system $\rho\colon \pi_1^{\text{ét}}(X) \to \mathrm{GL}_r(\overline{\mathbb{Q}}_\ell)$ is semisimple if and only if $\pi_1(X(\mathbb{C})) \to \pi_1^{\text{ét}}(X) \xrightarrow{\rho} \mathrm{GL}_r(\overline{\mathbb{Q}}_\ell) \simeq \mathrm{GL}_r(\mathbb{C})$ is semisimple for some isomorphism $\mathbb{C} \simeq \overline{\mathbb{Q}}_\ell$.
- Suppose $F$ is a finite field. By Lemma 3.21, $\mathbb{L}_C$ is pure of weight 0. Then also $\mathbb{L}_c = \mathbb{L}_C|\mathscr{X}_c$ is pure of weight 0. By Lemma 3.22, $\mathbb{L}_{\overline{c}}$ is semisimple.

■

In the remainder of the proof we argue that such an $\mathbb{L}_C$ can be assumed to exist. We first make a few remarks.

(i) Since $\mathscr{X}_\eta \hookrightarrow \mathscr{X}$ is an open subscheme of an integral scheme, also $\mathscr{X}_\eta$ is connected. The curve $\mathscr{X}$ admits a rational point over $\kappa(\eta)$ because $X \to S$ has a section by [Mil16, Chapter I, Exercise 4.13]. It follows that $\mathscr{X}_{\overline{\eta}}$ is also connected. In fact, all schemes we consider below are. Hence, we are free to identify local systems with their monodromy representations under the equivalence of Corollary 3.13.

(ii) Notice that we are free to replace $\mathscr{X}$ by an étale cover, because semisimplicity can be checked after pulling back along an étale cover by Proposition 3.18. In particular, we can assume that $\mathbb{L}_{\overline{x}}$, as a representation of $\pi_1(X, \overline{x})$, factors over the pro-$\ell$ quotient $\pi_1^{(\ell)}(\mathscr{X}, \overline{x})$ for a given geometric point $\overline{x} \to \mathscr{X}$.

(iii) After potentially replacing $C$ be an étale neighborhood of $\overline{c}$, we can assume that $\mathscr{X} \to S$ spreads out to a smooth quasi-compact separated $C$-scheme $\mathscr{X}_C \to C$. Furthermore, if we have a triangle

$$\begin{tikzcd}
 & \overline{c} \arrow[dl] \arrow[dr] & \\
\widetilde{C} \arrow[rr] & & C
\end{tikzcd}$$

with $\widetilde{C}$ another normal connected $F$-curve, then we are also free to replace $C$ by $\widetilde{C}$ after pulling back along $\widetilde{C} \to C$. This is in particular used in the following way. If $\kappa(\eta) \subset \kappa(\widetilde{\eta}) \subset \kappa(\overline{\eta})$ is any intermediate extension finite over $\kappa(\eta)$, and if $\widetilde{\mathscr{O}}$ denotes the normalization of $\mathscr{O}$ in $\kappa(\overline{\eta})$, then, after replacing $C$ by an étale neighborhood of $\overline{c}$, we can spread out to find $\widetilde{C} \to C$ such that we have a diagram

$$\begin{tikzcd}
\overline{c} \arrow[r] \arrow[dr] & \widetilde{S} \arrow[r] \arrow[d] & \widetilde{C} \arrow[d] \\
& S \arrow[r] & C,
\end{tikzcd}$$

where $\widetilde{S} = \operatorname{Spec} \widetilde{\mathscr{O}}$ and the square on the right is a pullback. This implies that we have an isomorphism $\mathscr{O}^{\mathrm{hs}}_{\widetilde{C},\overline{c}} \xrightarrow{\simeq} \widetilde{\mathscr{O}}$. So we are free to always enlarge $\kappa(\eta)$ to a finite extension, since $\operatorname{Frac} \widetilde{\mathscr{O}} = \kappa(\overline{\eta})$.

**Claim 7.4.** *The canonical homomorphism $\pi_1(\mathscr{X}_{\overline{\eta}}) \to \pi_1(\mathscr{X})$ is surjective.*

*Proof.* We are to prove that any connected étale cover of $\mathscr{X}$ pulls back to a connected étale cover of $\mathscr{X}_{\overline{\eta}}$. But this is true by precisely the same argument as given in (i) to prove connectedness of $\mathscr{X}_{\overline{\eta}}$. ■

As a result of the above claim, the local system $\mathbb{L}$ on $\mathscr{X}$ is also semisimple. Every simple summand of $\mathscr{X}$ restricts to a simple summand of $\mathbb{L}_{\overline{\eta}}$, which is again arithmetic by Corollary 5.5. Hence, it suffices to prove the theorem for all simple summands of $\mathbb{L}$. Hence, we can reduce to the case where $\mathbb{L}$ *is simple.* The proof now proceeds in a few steps.

**Step 1.** *We can assume that $\mathscr{X}_{C\setminus\{c\}} \to C \setminus \{c\}$ admits an open $C \setminus \{c\}$-immersion into a smooth proper scheme $\overline{\mathscr{X}}_{C\setminus\{c\}} \to C \setminus \{c\}$ with geometrically connected fibers such that $\mathscr{X}_{C\setminus\{c\}}$ is the complement of a divisor étale over $C \setminus \{c\}$. We can furthermore assume that $\mathscr{X}_{C\setminus\{c\}} \to C \setminus \{c\}$ admits a section.*

The curve $\mathscr{X}_\eta \to \operatorname{Spec} \kappa(\eta)$ has a unique compactification $\overline{\mathscr{X}}_\eta$. By (iii), we are free to assume that the points in the boundary $\overline{\mathscr{X}}_\eta \setminus \mathscr{X}_\eta$ are $\kappa(\eta)$-rational, and hence étale over $\operatorname{Spec} \kappa(\eta)$. Writing $\eta$ as $\varprojlim U \times_C (C \setminus \{c\})$, where the limit ranges over the connected étale neighborhoods $\overline{c} \to U \to C$ of $\overline{c}$, we produce $\overline{\mathscr{X}}_{C\setminus\{c\}}$ by a spreading argument as in the proof of Proposition 5.7, in the process replacing $C$ by some $U \to C$. To obtain a section $C \setminus \{c\} \to \mathscr{X}_{C\setminus\{c\}}$, spread out a rational point $x \colon \eta \to \mathscr{X}_\eta$.

Write $\overline{x} \colon \overline{\eta} \to \eta \xrightarrow{x} \mathscr{X}_\eta$ for the induced geometric point over $x$. We gain from step 1 the split exact homotopy sequence

$$\begin{tikzcd}
1 \arrow[r] & \pi_1^{(\ell)}(\mathscr{X}_{\overline{\eta}}, \overline{x}) \arrow[r] & \pi_1'(\mathscr{X}_{C\setminus\{c\}}, \overline{x}) \arrow[r] & \pi_1(C \setminus \{c\}, \overline{x}) \arrow[r] \arrow[l, bend right, dashed] & 1
\end{tikzcd} \tag{7.1}$$

of Proposition 2.4. The sequence gives rise to an action of $\pi_1(C \setminus \{c\}, \overline{x})$ on $\pi_1^{(\ell)}(\mathscr{X}_{\overline{\eta}}, \overline{x})$ by conjugation.

**Step 2.** *We can assume that $\mathbb{L}_{\overline{\eta}}$ spreads to a local system $\mathbb{L}_{C\setminus\{c\}}$ on $\mathscr{X}_{C\setminus\{c\}}$ with finite determinant.*

Recall from (ii) that $\mathbb{L}_{\overline{\eta},\overline{x}}$ is assumed to factor through a representation of $\pi_1^{(\ell)}(\mathscr{X}_{\overline{\eta}},\overline{x})$. The orbit of $[\mathbb{L}_{\overline{\eta},\overline{x}}]$ under $\pi_1(C\setminus\{c\},\overline{x})$, as defined in Section 5.1, is finite by arithmeticity (cf. the proof of Proposition 5.7). By Theorem 4.8 we can now assume (after again replacing $C$) that $\mathbb{L}_{\overline{\eta}}$ comes from a local system on $\mathscr{X}_{C\setminus\{c\}}$, which we can additionally assume to have finite determinant because $\mathbb{L}_{\overline{\eta}}$ does by Proposition 5.7.

**Step 3.** *We can assume that* $\mathbb{L}_{C\setminus\{c\}}$ *is unramified over* $\mathscr{X}_C$.

By Proposition 3.23 we have to show that $\mathbb{L}_{C\setminus\{c\}}$ is trivial on the intertia group $I_\zeta \subset \pi_1(X_C\setminus\{c\},\overline{x})$ for every point $\zeta\in\mathscr{X}_c$ that has codimension 1 in $\mathscr{X}_C$. This inertia group already lies in $\pi_1(\mathscr{X}_\eta,\overline{x})$, and so it is sufficient to show that $\mathbb{L}'_\eta := \mathbb{L}_{C\setminus\{c\}}|\mathscr{X}_\eta$ is unramified over $\mathscr{X}$. Let

$$V = \mathrm{Hom}_{\pi_1(\mathscr{X}_{\overline{\eta}},\overline{x})}(\mathbb{L}'_{\eta,\overline{x}},\mathbb{L}_{\eta,\overline{x}}),$$

and let $\chi\colon \pi_1(\mathscr{X}_\eta,\overline{x})\to \mathrm{GL}(V)$ be the rank 1 character of $\pi_1(\mathscr{X}_\eta,\overline{x})$ given by

$$\chi(\sigma) = (\varphi\mapsto \sigma\varphi\sigma^{-1}).$$

There is now a natural isomorphism

$$\mathbb{L}'_{\eta,\overline{x}}\otimes\chi \simeq \mathbb{L}_{\eta,\overline{x}}$$

of $\pi_1(\mathscr{X}_\eta,\overline{x})$-representations. The character $\chi$ factors over $\pi_1(\mathscr{X}_\eta,\overline{x})/\pi_1(\mathscr{X}_{\overline{\eta}},\overline{x}) \simeq \mathrm{Gal}(\overline{\eta}/\eta)$. We argue that $\chi$ is in fact a *finite* character of $\mathrm{Gal}(\overline{\eta}/\eta)$. By Claim 7.4 and finiteness of $\det\mathbb{L}_{\overline{\eta},\overline{x}}$, we see that $\det\mathbb{L}_{\eta,\overline{x}}$ is finite. The character $\det\mathbb{L}'_{\eta,\overline{x}}$ is finite by step 2. We conclude that also $\chi$ must be finite. It becomes trivial after a finite extension of $\kappa(\eta)$ and hence we are free to assume that it *is* trivial by (iii). It follows that $\mathbb{L}'_\eta$ is unramified over $\mathscr{X}$.

Invoking Claim 7.3, we are done.

## APPENDIX A. MOCHIZUKI'S COUNTEREXAMPLE

Let $\Delta$ denote the complex unit disc, and let $\Delta^* = \Delta\setminus\{0\}$ be the punctured unit disc. In this appendix, we give a counterexample to the complex-geometric version of Conjecture 1.5 with the arithmeticity condition removed. Specifically, we construct a smooth quasi-projective morphism $f\colon\mathscr{X}\to\Delta$ of complex manifolds, and a semi-simple complex local system $\mathbb{L}$ on $\mathscr{X}$ such that the pullback $\mathbb{L}_0$ of $\mathbb{L}$ to a local system on $\mathscr{X}_0$ is not semisimple. Here $\mathscr{X}_0$ denotes the fiber of $f$ over 0.

The counterexample constructed in this section is originally due to Takurō Mochizuki. Recall the correspondence between complex local systems and complex monodromy representations from (3.1).

A.1. **Topological aspects of the construction.** Let $f\colon X\to\Delta$ be an elliptic fibration such that the fibre $X_0$ over 0 is reduced and of type $I_1$ in the Kodaira classification of singular fibers of elliptic fibrations [Kod63]. We take this to mean that $f$ is a proper holomorphic map, $f$ is smooth over $\Delta^*$, the fiber over any point of $\Delta^*$ is a smooth connected genus one curve, and the special fiber $X_0$ is a rational curve with a single node (or ordinary double point). We denote the singular point of $X_0$ by $p_0$. Specifically, $p_0$ has a neighborhood that is complex-analytically isomorphic to a neighborhood of the origin in the zero locus cut out by $xy = 0$ in $\mathbb{C}^2$.

**Example A.1.** Consider

$$X = \{((x : y : z), \lambda) \in \mathbb{P}^2(\mathbb{C}) \times \Delta : y^2 z = 4x^3 + (\lambda - 3)xz^2 + (\lambda - 1)z^3\}$$

with the obvious projection map $X \to \Delta$. It is an elliptic fibration. Its fiber over 0 is the nodal cubic

$$X_0 \colon y^2 z = 4x^3 - 3xz^2 - z^3 = (2x + z)^2 (x - z).$$

Denote by $\varphi \colon \mathbb{P}^1 \to X_0$ the normalization. We can assume that $\varphi^{-1}(p_0) = \{0, \infty\}$. Define points in $\mathbb{P}^1$ by

$$\tilde{z}_0 = 1, \tilde{z}_1 = 1 + \sqrt{-1} \text{ and } \tilde{z}_2 = 1 - \sqrt{-1},$$

and set

$$z_i = \varphi(\tilde{z}_i) \quad (i = 0, 1, 2).$$

Notice that these are all smooth points. By potentially shrinking $\Delta$, we can assume that there exist holomorphic sections $s_1 \colon \Delta \to X$ and $s_2 \colon \Delta \to X$ of $f$ such that $z_1 \in s_1(\Delta), z_2 \in s_2(\Delta)$ and $s_1(\Delta) \cap s_2(\Delta) = \emptyset$. Indeed, $X \setminus \{p_0\} \to \Delta$ is smooth, and smooth maps between complex manifolds always admit local sections at all points.

Consider the loops $\gamma_i : ([0,1], \{0,1\}) \to (\mathbb{P}^1 \setminus \{\tilde{z}_1, \tilde{z}_2\}, \tilde{z}_0)$ for $i = 1, 2$ given by

$$\gamma_1(t) = 1 + \sqrt{-1} - \sqrt{-1} \exp(2\pi\sqrt{-1}t) \quad (0 \leq t \leq 1),$$
$$\gamma_2(t) = 1 - \sqrt{-1} + \sqrt{-1} \exp(2\pi\sqrt{-1}t) \quad (0 \leq t \leq 1).$$

We denote the induced elements in $\pi_1(\mathbb{P}^1 \setminus \{\tilde{z}_1, \tilde{z}_2\}, \tilde{z}_0)$ by $[\gamma_1]$ and $[\gamma_2]$ repectively. They both generate $\pi_1(\mathbb{P}^1 \setminus \{\tilde{z}_1, \tilde{z}_2\}, \tilde{z}_0)$. Furthermore, $[\gamma_1]$ and $[\gamma_2]$ and are inverse to each other. Define paths $\rho_1$ and $\rho_2$ by

$$\begin{aligned} \rho_1 \colon [0, 1/2] &\to \mathbb{P}^1 \\ t &\mapsto 1 - 2u, \end{aligned}$$

and

$$\begin{aligned} \rho_2 \colon [1/2, 1] &\to \mathbb{P}^1 \\ t &\mapsto \frac{1-u}{u - \frac{1}{2}} + 1. \end{aligned}$$

Here $\rho_2(\frac{1}{2})$ is sent to $\infty = (1:0) \in \mathbb{P}^1$. Composing both paths with $\varphi$ and glueing them together we obtain a loop $\rho \colon ([0,1], \{0,1\}) \to (X_0 \setminus D, z_0)$.

**Lemma A.2.** *The group $\pi_1(X_0 \setminus D, z_0)$ is freely generated by $\varphi_*[\gamma_1]$ and $[\rho]$.*

*Proof.* Topologically, $\varphi \colon \mathbb{P}^1 \setminus \{\tilde{z}_1, \tilde{z}_2\} \to X \setminus D$ is the quotient map for the equivalence relation on $\mathbb{P}^1 \setminus \{\tilde{z}_1, \tilde{z}_2\}$ glueing 0 and $\infty$ together. The result then follows by the Van Kampen theorem. ■

**Lemma A.3.** *The inclusion of the fiber* $i_0\colon X_0 \setminus D \hookrightarrow X \setminus D$ *is a homotopy equivalence.*

*Proof.* This uses a slightly more general version of [Cle77, Theorem 5.7]. ■

A.2. **Aspects of the construction from representation theory.** We define the representation $\kappa_0\colon \pi_1(X_0 \setminus D, z_0) \to \mathrm{GL}_2(\mathbb{C})$ by

$$\kappa_0(\varphi_*[\gamma_1]) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \kappa_0([\rho]) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

**Lemma A.4.** *(i) The representation* $\kappa_0$ *is irreducible.*

*(ii) The representation* $\kappa_0 \circ \varphi_*\colon \pi_1(\mathbb{P}^1 \setminus \{\widetilde{z}_1, \widetilde{z}_2\}, \widetilde{z}_0) \to \mathrm{GL}_2(\mathbb{C})$ *is not semi-simple.*

*Proof.* (i) This is clear by the fact that the image of $\kappa_0$ does not fix any non-trivial subspaces of $\mathbb{C}^2$.

(ii) The image $\kappa_0 \circ \varphi_*$ fixes only the subspace $\mathbb{C} \cdot (1,0)$ of $\mathbb{C}^2$.

■

Applying Lemma A.3, we obtain an irreducible representation

$$\pi_1(X \setminus D, z_0) \to \mathrm{GL}_2(\mathbb{C}).$$

To construct our counterexample we now proceed as follows. Let $\mathscr{X} := (X \setminus D)^{\mathrm{sm}} \to \Delta$ be the smooth locus of the relative curve $X \setminus D \to \Delta$. It is precisely $X \setminus D$ with the nodal point $p_0$ of the singular fiber removed. Its fiber over 0 is $\mathscr{X}_0 = (X_0 \setminus D)^{\mathrm{sm}} := X_0 \setminus (D \cup \{p_0\})$. Let $i_0\colon \mathscr{X}_0 \hookrightarrow \mathscr{X}$ denote the inclusion map. The fundamental group $\pi_1(\mathscr{X}, z_0)$ is isomorphic to $\pi_1(X \setminus D, z_0)$, and so we obtain from $\kappa_0$ an irreducible representation

$$\kappa\colon \pi_1(\mathscr{X}, z_0) \to \mathrm{GL}_2(\mathbb{C}).$$

We argue that the restriction of this representation along the fiber over 0 is *not* semisimple. We have the commutative diagram of groups

$$\begin{tikzcd}
\pi_1(\mathbb{P}^1 \setminus \{\widetilde{z}_1, \widetilde{z}_2, 0, \infty\}, \widetilde{z}_0)\}) \arrow[r, "\simeq"] \arrow[d, two heads] & \pi_1(\mathscr{X}_0, z_0) \arrow[r, "i_{0,*}"] \arrow[d] & \pi_1(\mathscr{X}, z_0) \arrow[d, "\simeq"] \\
\pi_1(\mathbb{P}^1 \setminus \{\widetilde{z}_1, \widetilde{z}_2\}, \widetilde{z}_0) \arrow[r, "\varphi_*"] & \pi_1(X_0 \setminus D, z_0) \arrow[r, "\simeq"] & \pi_1(X \setminus D, z_0).
\end{tikzcd}$$

This diagram shows that the representation $\kappa \circ i_{0,*}\colon \pi_1((X_0 \setminus D)^{\mathrm{sm}}, z_0) \to \mathrm{GL}_2(\mathbb{C})$ has the same image as $\kappa_0 \circ \varphi_*$. By Lemma A.4 we see that $\kappa \circ i_{0,*}$ is not semi-simple, since $\kappa_0 \circ \varphi_*$ is not.

## APPENDIX B. CONTINUOUS NON-ABELIAN GALOIS COHOMOLOGY

Throughout, $G$ is a profinite group. We collect a few facts regarding continuous non-abelian cohomology used in Section 4. Although there are many references on both continuous cohomology and non-abelian cohomology, the author was unable to find any references regarding the cohomology of profinite groups with non-discrete and non-abelian coefficients. Many of the statements in this appendix are straightforward generalizations of well known results. In particular, we mimic [Ser79, Appendix to Chapter VII].

### B.1. **The cohomology groups.**

**Definition B.1.** *A $G$-group $T$ is a topological group $T$ (perhaps non-abelian) with a continuous action of $G$. A morphism of topological $G$-groups is a continuous homomorphism compatible with the actions of $G$.*

If $T$ is a $G$-group, $t \in T$ is an element of $T$ and $\sigma \in G$, is an element of $G$, then we will denote the image of $t$ under the action of $\sigma$ by ${}^{\sigma}t$.

**Definition B.2.** *Let $T$ be a $G$-group. We define a continuous one-cocycle of $G$ with coefficients in $T$ to be a continuous map of spaces*

$$c\colon G \to T \quad \sigma \mapsto c_\sigma$$

*such that for all $\sigma, \tau \in G$ we have*

$$c_{\sigma\tau} = c_\sigma {}^{\sigma}c_\tau.$$

*Two one-cocyles $c$ and $b$ are said to be cohomologous if there is $t \in T$ such that for all $\sigma \in G$ we have*

$$c_\sigma = t^{-1} b_\sigma {}^{\sigma}t.$$

For a $G$-group $T$, "being cohomologous" defines an equivalence relation $\sim$ on the set of continuous one-cocyles of $G$ with coefficients in $T$. We define

$$H^1_{\text{cont}}(G;T) = \{\text{continuous one-cocycles } c\colon G \to T\}/\sim.$$

Notice that $H^1_{\text{cont}}(G,T)$ is equipped with a canonical basepoint: the class of the trivial one-cocycle $\sigma \mapsto 1$.

If $T$ happens to be an abelian $G$-group, then $H^2_{\text{cont}}(G;T)$ is defined in the usual way as continuous 2-cocyles modulo continuous 2-coboundaries[Tat76].

If $H \subset G$ is a closed subgroup, then we can define restriction

$$\text{res}^G_U\colon H^i_{\text{cont}}(G,T) \to H^i_{\text{cont}}(U,T)$$

as usual.

### B.2. **An analogue of the long exact sequence.**

Suppose now that we have a *strict exact sequence* of topological $G$-groups

$$1 \to T' \to T \to T'' \to 1.$$

This means in particular that $T'$ carries the subspace topology inherited from $T$ and $T''$ carries the quotient topology inherited from $T$. Assume also that

(i) $T'$ lands in the center of $T$;
(ii) we have a continuous set-theoretic section $s\colon T'' \to T$.

Notice that $T'$ is abelian so that $H^2_{\text{cont}}(G;T')$ is defined. We construct a boundary map

$$\delta\colon H^1_{\text{cont}}(G;T'') \to H^2_{\text{cont}}(G;T') \tag{B.1}$$

as follows: for the class of a continous one-cocyle $c\colon G \to T''$, we define

$$\delta(c)_{\sigma,\tau} = b_\sigma {}^\sigma b_\tau b_{\sigma\tau}^{-1} \in T' \quad (\sigma,\tau \in G), \tag{B.2}$$

where $b\colon G \to T$ is a continuous lift of $c$. Such a lift always exists, since we can compose $c$ with the section $s$ from (ii). This is a 2-cocycle [Ser79]. Furthermore, $\delta(c)$ is continuous by continuity of $b$. If we pick a different lift $\sigma \mapsto a'_\sigma b_\sigma$, with $a'_\sigma \in T'$, then $\sigma \mapsto a'_\sigma$ is continuous. Now, the two-cocycle $\delta(c)$ is replaced by $(\sigma,\tau) \mapsto a_{\sigma,\tau}\delta(c)_{\sigma,\tau}$, where

$$a_{\sigma,\tau} = (\partial a')_{\sigma,\tau} = a'_\sigma {}^\sigma a'_\tau a'^{-1}_{\sigma\tau}.$$

It follows that the class of $\delta(c)$ in $H^2_{\text{cont}}(G;T')$ is independent of the choice of the lift.

We show that $\delta$ does not depend on the choice of representative for the cocycle class of $c$. Indeed, if $c'$ is a continuous one-cocycle cohomologous to $c$, then there is $t'' \in T''$ such that

$$c'_\sigma = t''^{-1} c_\sigma {}^\sigma t'' \quad (\sigma \in G).$$

Let $t \in T$ such that $t \mapsto t''$. We can lift $c'$ to $\sigma \mapsto t^{-1} b_\sigma {}^\sigma t$. Clearly, this is again a continuous lift. The resulting continuous two-cocycles of $G$ with coefficients in $T'$ are the same [Ser79]. We conclude that the map $\delta$ is well-defined.

**Theorem B.3.** *The sequence*

$$H^1_{\text{cont}}(G;T) \to H^1_{\text{cont}}(G;T'') \xrightarrow{\delta} H^2_{\text{cont}}(G;T'),$$

*with $\delta$ the map from* (B.1)*, is an exact sequence of pointed sets.*

*Proof.* The fact that the composition of the two maps is the trivial map is exactly as in the classical discrete case. Suppose we have a one-cocycle $c \in H^1_{\text{cont}}(G;T'')$ such that $\delta(c) = 0 \in H^2_{\text{cont}}(G;T')$. Then there is $a \in C^1_{\text{cont}}(G;T')$ a continuous map such that

$$\delta(c)_{\sigma,\tau} = a_\sigma {}^\sigma a_\tau a_{\sigma\tau}^{-1} \quad (\sigma,\tau \in G).$$

By property (i) above we get

$$(b_\sigma a_\sigma^{-1})^\sigma (b_\tau a_\tau^{-1})(b_{\sigma\tau} a_{\sigma\tau}^{-1})^{-1} = 1 \in T' \quad (\sigma \in G). \tag{B.3}$$

Define now $b'\colon G \to T$ by $\sigma \mapsto b_\sigma a_\sigma^{-1}$. Then $b'$ is continuous by the fact that $a$ and $b$ are. By (B.3), $b'$ is a cocycle. We clearly have $b' \mapsto c$ and so we win. ■

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