Talk 6: The geometric McKay correspondence

Recall $\left\{ \begin{array}{l} \text{fin. subgrps.} \\ \text{of } SL_2(\mathbb{C}) \end{array} \right\} \Big/ \underset{\substack{\uparrow \\ \text{equiv } SU(2)}}{\simeq} \xrightarrow{\simeq} \left\{ \begin{array}{l} \text{Dynkin graphs} \\ A_n, D_n, E_6, E_7, E_8 \end{array} \right\}$

Have seen: two ways to realize "$\xrightarrow{\simeq}$":

- Talk 4: $G \subset SU(2) \curvearrowright \mathbb{A}^2_{\mathbb{C}}$

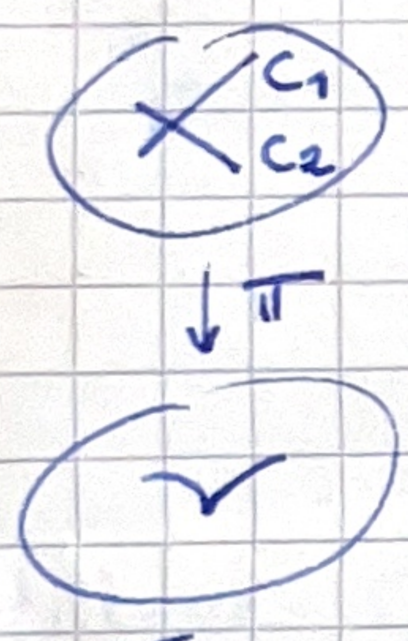
  $\rightsquigarrow \mathbb{A}^2/G$ singular. For $\pi : \tilde{X} \to \mathbb{A}^2/G$ minimal resol'n, $\pi^{-1}(0) = \bigcup_{i \in I} C_i$ w. $C_i \simeq \mathbb{P}^1$

  $\rightsquigarrow$ Graph $D(\pi)$ ← Dynkin graph of G

  – vertices $= \{C_i\}_i$;

  – ~~edges~~ adj. mat. $= (C_i \cdot C_j + 2\delta_{ij})_{i,j}$

E.g. $G = \mathbb{Z}/3\mathbb{Z}$



$\rightsquigarrow$ $A_2 = D(\pi)$:

$\mathbb{A}^2/\mathbb{Z}/3\mathbb{Z} \simeq \{x^3 - yz = 0\} \subseteq \mathbb{A}^3_{\mathbb{C}}$

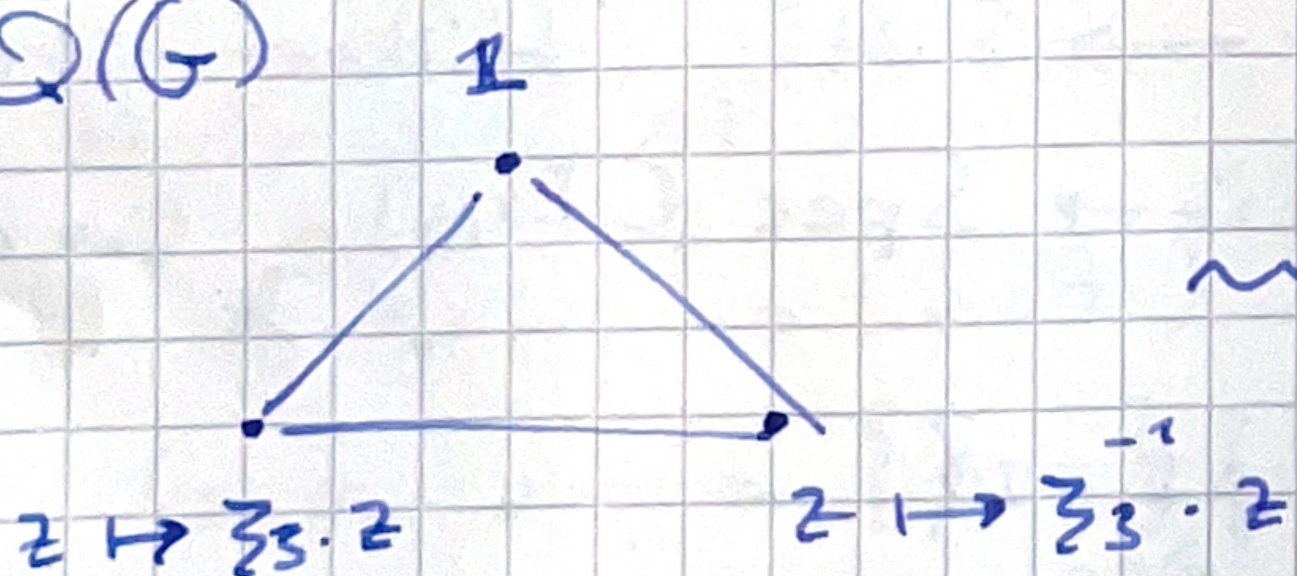
- Talk 5: $Q(G) =$ McKay graph

  $D(G) =$ "$Q(G) -$ triv. repr."

E.g. $G = \mathbb{Z}/3\mathbb{Z} \hookrightarrow SU(2)$

$1 \longmapsto \begin{pmatrix} \zeta_3 & 0 \\ 0 & \zeta_3^{-1} \end{pmatrix}$

$Q(G)$ "=" $\hat{A}_2$



$\rightsquigarrow$ $D(G) = A_2$

Goals

- Know: For $\pi: X \to \mathbb{A}^2/G$ minimal resol'n $\exists$ non-canonical iso $D(\pi) \cong D(G)$

- want: A coordinate-free interpretation of $\pi: \tilde{X} \to \mathbb{A}^2/G$ and an explicit iso
$$D(\pi) \cong D(G).$$

Hilbert scheme of points

For $n \geq 1$, $\mathrm{Hilb}^n(\mathbb{A}^2)$ ~~is~~ represents

$$\mathbb{C}\text{-alg} \longrightarrow \text{Sets}$$
$$(\mathbb{C} \to R) \longmapsto \left\{ J \trianglelefteq R[x,y] \;\middle|\; \begin{array}{l} R \to R[x,y]/J \\ \text{f.g. proj. } R\text{-mod of} \\ \text{rank } n \end{array} \right\}$$

In particular: $\mathrm{Hilb}^n(\mathbb{A}^2)(\mathbb{C}) = \left\{ J \trianglelefteq \mathbb{C}[x,y] \;\middle|\; \dim_{\mathbb{C}} \frac{\mathbb{C}[x,y]}{J} \right.$

Facts

- $\mathrm{Hilb}^n(\mathbb{A}^2)$ smooth q. proj. $\mathbb{C}$-var. ;
- $\dim \mathrm{Hilb}^n(\mathbb{A}^2) = 2 \cdot n$

For $n \geq 1$ also consider
$$S^n \mathbb{A}^2 = (\underbrace{\mathbb{A}^2 \times \dots \times \mathbb{A}^2}_{n \text{ many}}) /\!/ S_n.$$

$$\to S^n \mathbb{A}^2(\mathbb{C}) = \{ \underbrace{\{x_1, \dots, x_n\}}_{\text{multiset}} \mid x_i \in \mathbb{A}^2(\mathbb{C}) \}.$$

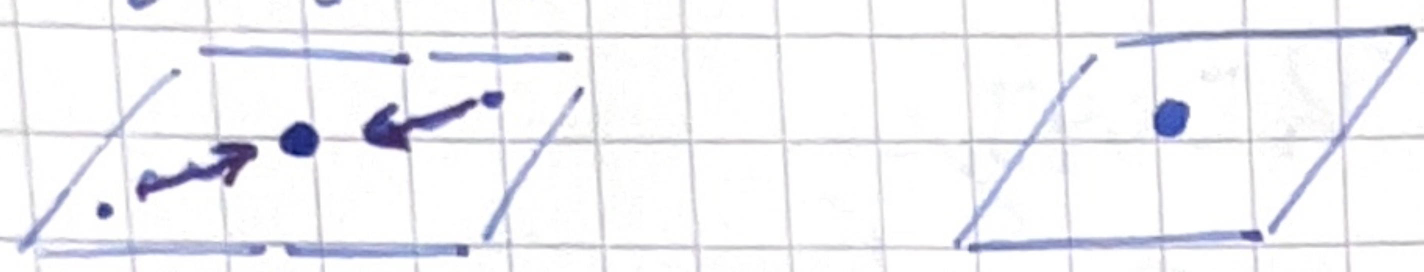
There exists $\mathrm{Hilb}^n(\mathbb{A}^2) \xrightarrow{\pi} S^n \mathbb{A}^2$ (Hilbert-Chow morphism)
$$J \trianglelefteq \mathbb{C}[x,y] \longmapsto \mathrm{Spec}\, \mathbb{C}[x,y]/J \hookrightarrow \mathbb{A}^2_{\mathbb{C}}.$$

E.g.
- $(x^2-1, y) \longmapsto \{1, -1\}$ $\cdot$ $\cdot$
- $(x^2, y) \longmapsto \{0, 0\}$ $\bullet$

Facts:
- π proper
- generally not an iso.

E.g. $\mathrm{Hilb}^2(\mathbb{A}^2) \xrightarrow{\pi} S^2\mathbb{A}^2$

$\pi^{-1}\{0,0\} \cong \mathbb{P}^1$ $\qquad \{0,0\}$ $\qquad \pi^{-1}\{0,0\} = \{(x^2, xy, y^2, ux+vy) \mid (u:v)\in\mathbb{P}^1\}$



G-Hilbert scheme

$G \subset SU(2) \rightsquigarrow G \circlearrowright \mathbb{A}^2_{\mathbb{C}}$

$\to \mathrm{Hilb}^n(\mathbb{A}^2),\ S^n\mathbb{A}^2$ inherit G-action.

Def. For $n = \#G$,

G-fix points

$$\begin{tikzcd}
G\text{-}\mathrm{Hilb}(\mathbb{A}^2) := (\mathrm{Hilb}^n(\mathbb{A}^2))^G \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \cancel{G \times}\, \mathrm{Hilb}^n(\mathbb{A}^2) \arrow[d] \\
\mathrm{Hilb}^n(\mathbb{A}^2) \arrow[r, "\Delta"] & \underbrace{\mathrm{Hilb}^n(\mathbb{A}^2) \times \cancel{\mathrm{Hilb}^n(\mathbb{A}^2)} \times \cdots \times \mathrm{Hilb}^n(\mathbb{A}^2)}_{\#G \text{ many}}
\end{tikzcd}$$

$\cancel{(g,x)} \mapsto \cancel{(x, g\cdot x)}$ $\qquad x \mapsto (g\cdot x)_{g\in G}$

the G-Hilbert scheme. We have

$$\cancel{G}\text{-}\mathrm{Hilb}^G(\mathbb{A}^2)(R) = \{ \underset{\trianglelefteq R[x,y] \circlearrowleft G}{J} \in \mathrm{Hilb}^n(\mathbb{A}^2)(R) \mid g\cdot J = J \ \forall g\in G \}.$$

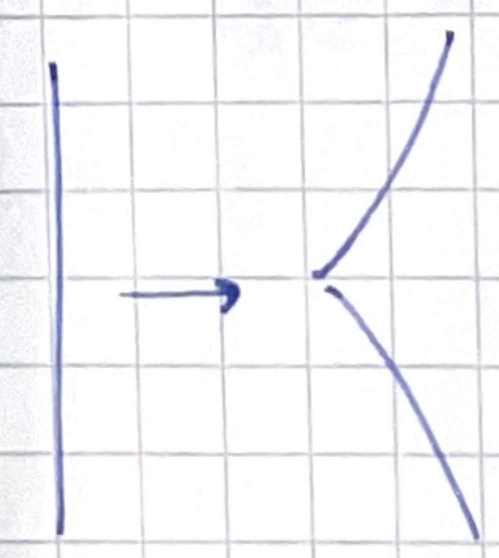
Hilbert-Chow:

$$\begin{tikzcd}
\cancel{G}\text{-}\mathrm{Hilb}^G(\mathbb{A}^2) \arrow[r, hook] \arrow[d, "\pi"] \arrow[dr, phantom, "\circledcirc"] & \mathrm{Hilb}^n(\mathbb{A}^2) \arrow[d, "\pi"] \\
\mathbb{A}^2/G \cong (S^n\mathbb{A}^2)^G \arrow[r, hook] & S^n\mathbb{A}^2
\end{tikzcd}$$

Prop. $\mathbb{A}^2/G \cong (S^n\mathbb{A}^2)^G$

Sketch $\mathbb{A}^2/G \xrightarrow{\cong} (S^n\mathbb{A}^2)^G$

$\qquad G\cdot x \longmapsto \{g\cdot x\}_{g\in G}$



bijection + normal $(S^n\mathbb{A}^2)^G$.

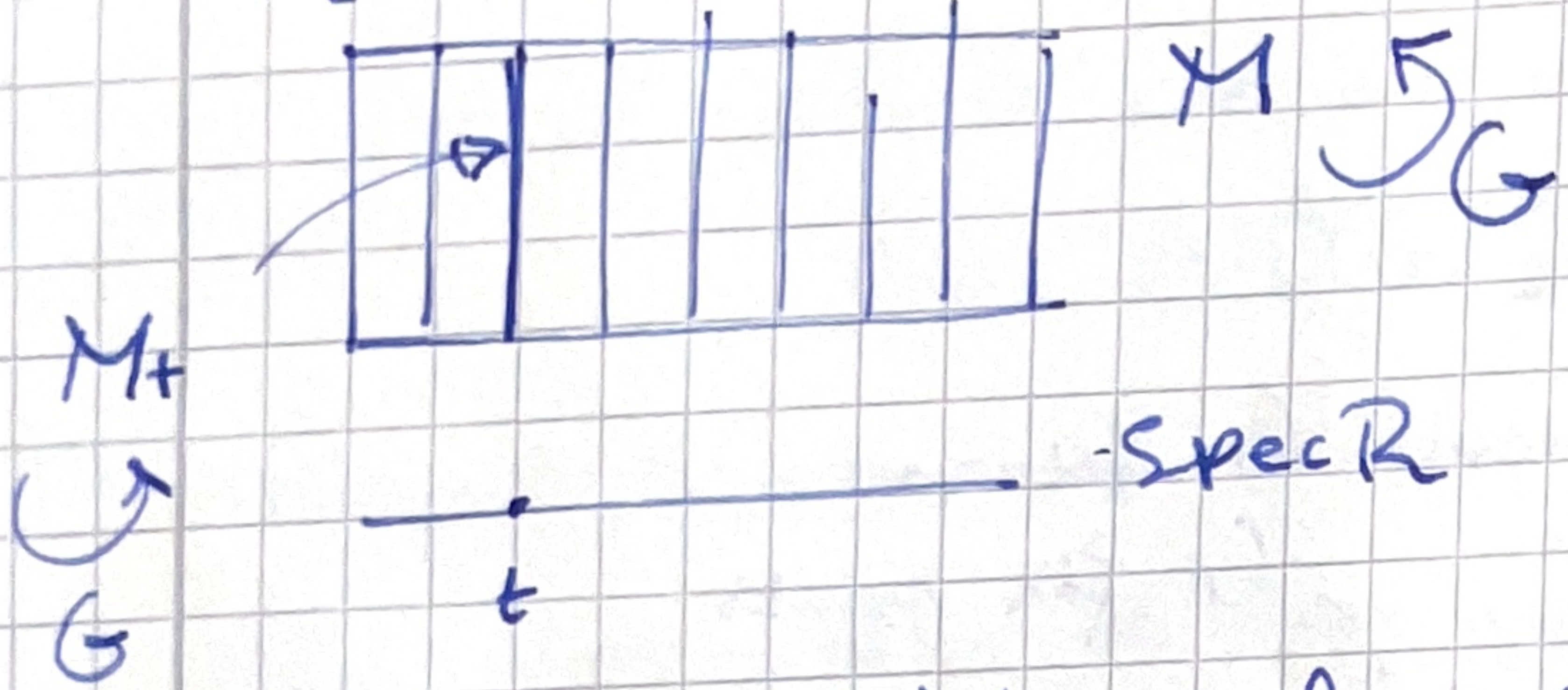
$\Rightarrow$ isomorphism. $\square$

Fact $\mathrm{Hilb}^G(\mathbb{A}^2)$ is smooth, but generally disconnected.

Q.: "What are its components?"

$\mathcal{J} \in \mathrm{Hilb}^G(\mathbb{A}^2)(R)$, $M = R[x,y]/\mathcal{J}$.

$R \xrightarrow{t} \mathbb{C} \rightsquigarrow M_t := M \otimes_R \mathbb{C}$ $\quad \mathbb{C}[G]$-module.



Iso class of M_t is loc. constant on $\mathrm{Spec}\, R$.

For V a rep'n of G, $n = \dim V$, set

$$X(V) = \{\mathcal{J} \in \mathrm{Hilb}^G(\mathbb{A}^2) \mid \mathbb{C}[x,y]/\mathcal{J} \cong_G V\}$$

Prop. $\mathrm{Hilb}^G(\mathbb{A}^2) = \coprod_{\dim V = \# G} X(V)$ and $X(V)$ is connected. $\boxtimes$

Exp. For $x \in \mathbb{A}^2 \setminus \{0\}$, we have

$$\mathcal{J}(G \cdot x) \in X(\mathfrak{z}),$$

where $\mathfrak{z}$ = regular rep'n of G.

Prop. $\pi : X(\mathfrak{z}) \longrightarrow \mathbb{A}^2/G$ is the minimal resol'n.

"Proof"

- above exp. shows: π iso away from 0.
- π projective + $X(\mathfrak{z})$ conn. $\Rightarrow$ π resol'n.
- $X(\mathfrak{z})$ has a symplectic structure

$\rightarrow K_{X(\mathfrak{z})}$ is trivial $\Rightarrow$ minimality. "$\boxtimes$"

Next: Understand $\pi^{-1}(0)$ in terms of rep'ns of G.

Prop. For ρ non-triv. irr. repn. of G, $\exists!\, J_\rho \trianglelefteq \mathbb{C}[x,y]$ s.t.

- $g\cdot J_\rho = J_\rho \quad \forall g\in G$;
- $\mathbb{C}[x,y]/J_\rho \cong_G \zeta - \rho$.

Exp. $G = \mathbb{Z}/3\mathbb{Z} \hookrightarrow SU(2)$, $1 \mapsto \begin{pmatrix}\zeta_3 & 0\\ 0 & \zeta_3^{-1}\end{pmatrix}$, $\zeta_3 \in \mathbb{C}^\times\setminus\{1\}$, $\zeta_3^3 = 1$.

- $J_{\overset{\rho_1}{z\mapsto\zeta_3\cdot z}} = (x^2, xy, y^2, \underset{(\alpha x+\beta y)}{x})$

$$\mathbb{C}[x,y]/J \simeq \mathbb{C}\cdot 1 \oplus \mathbb{C}\cdot y$$
$$\simeq \zeta - \rho_1$$

- $J_{\rho_2: z\mapsto \zeta_3^{-1}\cdot z} = (x^2, xy, y^2, y)$.

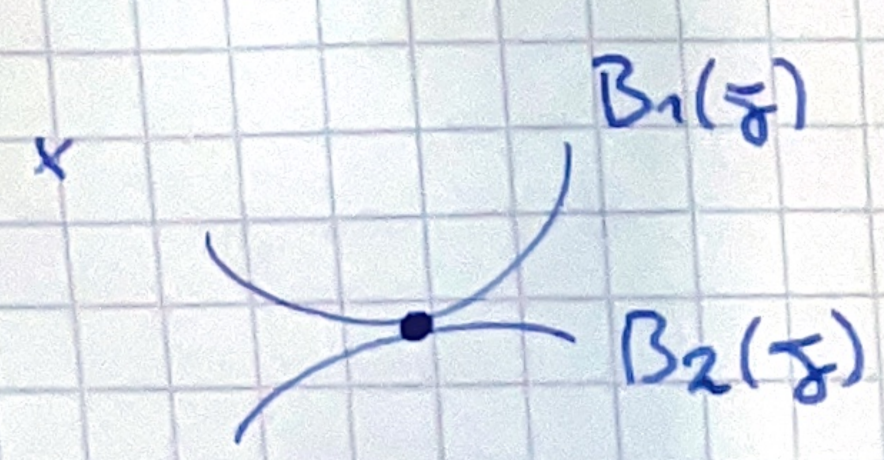
Let $I = \{\text{irr. reps. of } G\}$. Define
$$B_i(\zeta) = \{J\in X(\zeta) \mid J\subset J_i\}, \quad (i\in I\setminus\{1\}).$$
Then $B_i(\zeta)\subset\pi^{-1}(0)$.

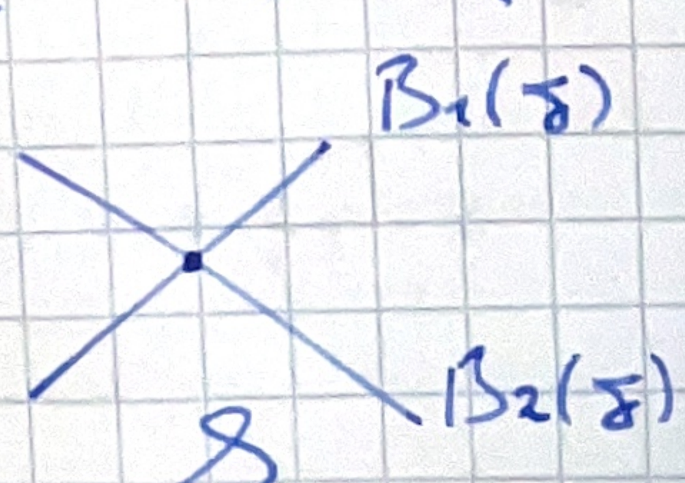
Thm.
- $B_i(\zeta)\cong\mathbb{P}^1$;
- $\pi^{-1}(0) = \bigcup_{i\in I\setminus\{1\}} B_i(\zeta)$;
- $(B_i(\zeta)\cdot B_j(\zeta)) = (a_{ij} - 2\delta_{ij})$

$A(Q) = (a_{ij})$ adj. matrix of McKay graph Q.

Exp. $G = \mathbb{Z}/3\mathbb{Z}$
- $B_1(\zeta) = \{(uy^2+vx, x^2, xy) \mid (u:v)\in\mathbb{P}^1\}$;
- $B_2(\zeta) = \{(sx^2+ty, y^2, xy) \mid (s:t)\in\mathbb{P}^1\}$.

$B_1(\zeta)\cap B_2(\zeta) = \{(x^2, y^2, xy) = \mathfrak{m}^2\}$.





$R = \mathbb{C}[x,y]$

$T_{\underline{m}^2} X(\gamma) \subset T_{\underline{m}^2} \mathrm{Hilb}^3(\mathbb{A}^2) = \mathrm{Hom}_R(\underline{m}^2, R/\underline{m}^2)$

$\Updownarrow$

$0 \to R/\underline{m}^2 \xrightarrow{\cdot \varepsilon} \underline{m}^2 + \varepsilon R / \varepsilon \cdot \underline{m}^2 \rightleftarrows \underline{m}^2 \to 0$

$\Updownarrow$

$\tilde{I} \subset R[\varepsilon]$ s.t. $\tilde{I} \otimes_{\mathbb{C}[\varepsilon]} \mathbb{C} = \underline{m}^2$

$\Updownarrow$

$\mathrm{Hom}_{\underline{m}^2}(\mathrm{Spec}\, \mathbb{C}[\varepsilon], \mathrm{Hilb}) = T_{\underline{m}^2} \mathrm{Hilb}$.

$B_1(\gamma)$.

$\wr$

$\mathrm{Spec}\, \mathbb{C}[\varepsilon] \xrightarrow{(1:0)+\varepsilon} \mathbb{P}^1 \longrightarrow \mathrm{Hilb}^3(\mathbb{A}^2)$

$\Updownarrow$

$\tilde{\Phi}_1 = (y^2 + \varepsilon \cdot x, y^2, xy)$

$\rightsquigarrow \tilde{\varphi}_1 : y^2 \longmapsto x, \; xy, x^2 \longmapsto 0$

$B_2(\gamma) \rightsquigarrow \tilde{\varphi}_2 : x^2 \mapsto y, \; xy, y^2 \mapsto 0$

$\tilde{\varphi}_1, \tilde{\varphi}_2$ are $\mathbb{C}$-lin. indep. in $T_{\underline{m}^2} \mathrm{Hilb}^3(\mathbb{A}^2)$.