ÉTALE FUNDAMENTAL GROUPS OF NODAL CURVES IN POSITIVE CHARACTERISTIC

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ABSTRACT. We relate the étale fundamental group of a nodal curve to the étale fundamental group of its normalization. Combined with known results on étale fundamental groups of smooth curves in positive characteristic, this gives us a good grip on the fundamental groups of these nodal curves.

1. INTRODUCTION

Given a variety over a field of characteristic zero (perhaps singular), its étale fundamental group can be understood in terms of a topological fundamental group by using the generalized Riemann existence theorem; see [GR06, Théorème XII.5.1]. For a smooth variety in positive characteristic, one can sometimes compute the prime-to-p part of the fundamental group from the fundamental group of a lift to characteristic zero, using a specialization argument; see [GR06, Exposé X]. This in particular works for curves. In some cases, for instance for curves, this technique even works when the variety is not proper by the specialization techniques for tame fundamental groups developed in [GR06, Exposé XIII]. Such specialization arguments do not apply when the variety under consideration is singular. In this paper we will relate the fundamental group of (perhaps non-proper) nodal curves over a base field of any characteristic to the fundamental group of its normalization. We will do this by explicitly examining the category of finite étale covers of such a curve. The main result is Theorem 2.7 below.

2. FUNDAMENTAL GROUPS OF NODAL CURVES

k will always denote an algebraically closed field of characteristic $p \geqslant 0$. Let C denote an integral curve over k. We suppose that there is a simple node $n \in C(k)$, i.e., the (necessarily strict) henselization $\mathscr{O}^{\mathrm{h}}_{C,n}$ of the local ring $\mathscr{O}_{C,n}$ is isomorphic to $(k[x,y]/(xy))^{\mathrm{h}}_{(x,y)}$ as a local k-algebra.

Denote by $\pi\colon \widetilde{C} \to C$ the morphism obtained by normalizing c; i.e., we take the normalization of an open neighborhood of n in which n is the only singular point and we then glue the resulting curve to $C\backslash\{n\}$. The preimage of n under π consists of exactly two points. This can be seen by first pulling back π along $\operatorname{Spec}\mathscr{O}^{\mathrm{h}}_{C,n} \to C$ and the fact that this pullback is the normalization of $\operatorname{Spec}\mathscr{O}_{C,n}^{\mathrm{h}}$ by [Sta24, Tag 0CBM]; then notice that the normalization of $\operatorname{Spec}(k[x,y]/(xy))_{(x,y)}^{\mathrm{h}} \simeq \operatorname{Spec}\mathscr{O}_{C,n}^{\mathrm{h}}$ is

$$\operatorname{Spec} k[x]_{(x)}^{\mathrm{h}} \sqcup \operatorname{Spec} k[y]_{(y)}^{\mathrm{h}} \to \operatorname{Spec}(k[x,y]/(xy))_{(x,y)}^{\mathrm{h}} \tag{2.1}$$

and that the fiber of this morphism over the closed point consists of exactly two points. Denote by a and b the points of $\widetilde{C}$ lying over n. Consider the following "normalization sequence" of $\mathscr{O}_C$-modules

$$0 \to \mathscr{O}_C \to \pi_*\mathscr{O}_{\widetilde{C}} \to k_n \to 0, \tag{2.2}$$

where k_n is the skyscraper sheaf supported only at the node n with stalk k. Here the map $\pi_*\mathscr{O}_{\widetilde{C}} \to k_n$ is defined by $f \mapsto f(a) - f(b)$.

Proposition 2.1. *The sequence* (2.2) *is exact.*

The above proposition is a commonly known fact from the theory of algebraic curves; see, for instance, [Har10, p. IV.1.8]. We prove it here also directly.

Proof. It is clear that for every point $c \in C$, the sequence of stalks at c is exact, except possibly at one of the nodes n. Consider the sequence of stalks at n

$$0 \to \mathscr{O}_{C,n} \to (\pi_*\mathscr{O}_{\widetilde{C}})_n \to k = \mathscr{O}_{C,n}/\mathfrak{m}_n \to 0.$$

Recall that the henselization $\mathscr{O}_{C,n} \to \mathscr{O}_{C,n}^{\mathrm{h}}$ is faithfully flat, and so we need only prove that the sequence we obtain after tensoring with $\mathscr{O}_{C,n}^{\mathrm{h}}$,

$$0 \to \mathscr{O}_{C,n}^{\mathrm{h}} \to (\pi_*\mathscr{O}_{\widetilde{C}})_n \otimes_{\mathscr{O}_{C,n}} \mathscr{O}_{C,n}^{\mathrm{h}} \to \mathscr{O}_{C,n}^{\mathrm{h}}/\mathfrak{m}_n\mathscr{O}_{C,n}^{\mathrm{h}} = k \to 0,$$

is exact. This sequence identifies with the sequence

$$0 \to (k[x,y]/(xy))_{(x,y)}^{\mathrm{h}} \to k[x]_{(x)}^{\mathrm{h}} \times k[y]_{(y)}^{\mathrm{h}} \to k \to 0 \tag{2.3}$$

by (2.1) and the argument preceeding it. The sequence (2.3) is clearly exact. ■

2.1. **The category of finite étale covers.** Denote the category of finite étale covers of C by $\mathrm{Fét}_C$. Denote by $\mathscr{C}$ the category whose objects are finite étale covers $X \xrightarrow{f} \widetilde{C}$ equipped with an isomorphism $\varphi\colon f^{-1}(a) \xrightarrow{\simeq} f^{-1}(b)$ and whose morphisms $(f,\varphi) \to (f',\varphi')$ are morphisms of finite étale covers such that the square

$$\begin{array}{ccc} f^{-1}(a) & \longrightarrow & f'^{-1}(b) \\ \big\downarrow{\scriptstyle\varphi} & & \big\downarrow{\scriptstyle\varphi'} \\ f^{-1}(a) & \longrightarrow & f'^{-1}(b) \end{array}$$

commutes. The isomorphism φ is referred to as *a glueing datum for f*. Given a finite étale cover $g\colon Y \to C$, we obtain a finite étale cover

$$\pi^* g\colon \pi^* Y := Y \times_C \widetilde{C} \to C$$

by pulling the morphism g back along π. We have a canonical isomorphism

$$(\pi^* g)^{-1}(a) \simeq (\pi^* g)^{-1}(b)$$

and hence a canonical descent datum for $\pi^* g$. This gives us a functor

$$\pi^*\colon \mathrm{Fét}_C \to \mathscr{C}. \tag{2.4}$$

The rest of this section is devoted to constructing a pseudo-inverse to this functor.

2.1.1. *A glueing construction.* Given an object $(X \xrightarrow{f} \widetilde{C}, \varphi)$ of $\mathscr{C}$, we intend to construct a finite étale cover of C by "glueing the fibers $f^{-1}(a)$ and $f^{-1}(b)$ together along the isomorphism φ". Let U be an affine open neighborhood of n, denote by $\widetilde{U} \subset \widetilde{C}$ the pullback of U under π, and denote by $V \subset X$ the pullback of $\widetilde{U}$ under f. The open V is again affine. Define the ring $\mathscr{O}(\overline{V})$ by

$$\mathscr{O}(\overline{V}) = \{f \in \mathscr{O}(V) : f(x) = f(\varphi(x)) \text{ for all } x \in f^{-1}(a)\}.$$

It defines the coordinate ring of an affine scheme $\overline{V}$.

Proposition 2.2. *The square*

$$\begin{array}{ccc} f^{-1}(a) \sqcup f^{-1}(b) & \longrightarrow & V \\ \big\downarrow{\scriptstyle \varphi \sqcup \mathrm{id}} & & \big\downarrow{\scriptstyle \rho} \\ f^{-1}(b) & \longrightarrow & \overline{V} \end{array}$$

defines a pushout square in the category of affine k-schemes.

Proof. The corresponding map on rings defines a pullback square in the category of k-algebras. ∎

We glue $\overline{V}$ together with $X \backslash (f^{-1}(a) \cup f^{-1}(b))$ to obtain a scheme $\overline{X}$, and we let $\rho\colon X \to \overline{X}$ denote the obvious map. Clearly, the scheme $\overline{X}$ does not depend on the affine open U we started with.

By construction of the scheme $\overline{X}$, we also have an exact sequence of $\mathscr{O}_{\overline{X}}$-modules on $\overline{X}$

$$0 \to \mathscr{O}_{\overline{X}} \to \rho_* \mathscr{O}_X \to \bigoplus_{x \in f^{-1}(a)} k_{\rho(x)} \to 0, \tag{2.5}$$

where $\bigoplus_{x\in f^{-1}(a)} k_{\rho(x)}$ is the skyscraper sheaf on $\overline{X}$ supported at the points $\rho(x) = \rho(\varphi(x))$ with stalk k for $x \in f^{-1}(a)$. Letting $y = \varphi(x)$ and $z = \rho(x) = \rho(y)$, we obtain the exact sequence of stalks

$$0 \to \mathscr{O}_{\overline{X},z} \to (\rho_*\mathscr{O}_X)_z \to k \to 0.$$

Set $\mathscr{O}_{X,x\cup y} = (\rho_*\mathscr{O}_X)_z$. This is a semilocal ring with maximal ideals $\mathfrak{m}_x$ and $\mathfrak{m}_y$, corresponding to the points x and y. The ideal $\mathfrak{m}_z\mathscr{O}_{X,x\cup y}$ is precisely the Jacobson radical of $\mathscr{O}_{X,x\cup y}$. As a result, the $\mathscr{O}_{\overline{X},z}$-completion of $\mathscr{O}_{X,x\cup y}$ is isomorphic to $\widehat{\mathscr{O}}_{X,x}\times\widehat{\mathscr{O}}_{Y,y}$ by [MR86, Theorem 8.15]. Taking completions of $\mathscr{O}_{\overline{X},z}$-modules now yields the exact sequence

$$0 \to \widehat{\mathscr{O}}_{\overline{X},z} \to \widehat{\mathscr{O}}_{X,x}\times\widehat{\mathscr{O}}_{X,y} \to k \to 0. \tag{2.6}$$

2.1.2. *An equivalence of categories.* Start with an object $(X \xrightarrow{f} \widetilde{C}, \varphi)$ of $\mathscr{C}$ and let $U, \widetilde{U}$ and V be as before. We obtain by Proposition 2.2 a commutative diagram

$$\begin{tikzcd} V \arrow[r, "\rho"] \arrow[d, "f"] & \overline{V} \arrow[d, dashed, "\overline{f}"] \\ \widetilde{U} \arrow[r, "\pi"] & U. \end{tikzcd} \tag{2.7}$$

Glueing the morphism $\overline{f}$ with the morphism $X\backslash(f^{-1}(a)\cup f^{-1}(b)) \to X\backslash\{n\}$, we obtain $\overline{f}\colon \overline{X} \to C$. Again, it is clear that this morphism does not depend on our choice of open neighborhood U.

Proposition 2.3. *The morpism $\overline{f}\colon \overline{X} \to C$ constructed in* (2.7) *is finite étale.*

Proof. It is affine by construction, and finite by finiteness of f and π and the fact that C is noetherian (see [Sta24, Tag 00FP]). The fact that it is étale at all points outside the fiber over n is clear. To prove that it is étale at the point $z = \rho(x) = \rho(y)$ in the fiber over n, for some $x \in f^{-1}(a)$ and $y = \varphi(x)$, we show that the induced map on completed local rings $\widehat{\mathscr{O}}_{C,n} \to \widehat{\mathscr{O}}_{\overline{X},z}$ is an isomorphism. Then we can apply [Har10, Chapter 2, Exercise 10.4] and conclude that $\overline{f}$ is étale at z. Recall the exact sequence from (2.6). We can derive a similar such sequence for the completed local ring of C at n from (2.2). We obtain a commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & \widehat{\mathscr{O}}_{\overline{X},z} \arrow[r] & \widehat{\mathscr{O}}_{X,x}\times\widehat{\mathscr{O}}_{X,y} \arrow[r] & k \arrow[r] & 0 \\ 0 \arrow[r] & \widehat{\mathscr{O}}_{C,n} \arrow[r] \arrow[u] & \widehat{\mathscr{O}}_{\mathbb{G}_m,a}\times\widehat{\mathscr{O}}_{\mathbb{G}_m,b} \arrow[r] \arrow[u, "\simeq"] & k \arrow[r] \arrow[u, "="] & 0, \end{tikzcd}$$

where the middle vertical map is an isomorphism by the fact that f is étale. It follows that $\widehat{\mathscr{O}}_{C,n} \to \widehat{\mathscr{O}}_{\overline{X},z}$ is an isomorphism. ■

This construction is clearly functorial, and so we obtain the functor

$$\begin{aligned} G\colon \mathscr{C} &\to \mathrm{Fét}_{C^\circ} \\ (X \xrightarrow{f} \widetilde{C}, \varphi) &\mapsto (\overline{X} \xrightarrow{\overline{f}} C). \end{aligned} \tag{2.8}$$

Theorem 2.4. *The functors from* (2.4) *and* (2.8) *are pseudo-inverse to each other. As a result, we obtain an equivalence of categories*

$$\mathrm{Fét}_C \simeq \mathscr{C}.$$

Proof. Let $Y \to C$ be a finite étale morphism. We naturally obtain a commutative triangle

$$\begin{tikzcd} \overline{\pi^* Y} \arrow[rr, dashed] \arrow[dr] & & Y \arrow[dl] \\ & C. & \end{tikzcd}$$

Since $\overline{\pi^* Y}$ and Y are finite étale of the same degree over C, the dashed arrow above is finite étale of degree 1 because it is surjective. It follows that the dashed arrow is an isomorphism. So we have an isomorphism of functors $G \circ \pi^* \simeq \mathrm{id}$.

Conversely, if (X, φ) is a finite étale cover of $\widetilde{C}$ with a descent datum, then we naturally obtain a commutative triangle

$$\begin{tikzcd} X \arrow[rr, dashed] \arrow[dr] & & \pi^* \overline{X} \arrow[dl] \\ & \widetilde{C}. & \end{tikzcd}$$

By the same argument as before, the dashed arrow is an isomorphism. Hence, we obtain an isomorphism of functors $\pi^* \circ G \simeq \mathrm{id}$. ■

2.2. **Computing the fundamental group.**

2.2.1. *Some generalities about profinite groups.* We first recall a few elementary facts about free products of profinite groups. Denote by Grp the category of groups, by PrGrp the category of profinite groups, and by $\mathrm{PrGrp}^{(p')}$ the category of profinite groups G such that (the supernatural number) $\#G = [G:1]$ is prime to p. We will call such groups *pro-prime-to-p*. Let $\widehat{(-)}\colon$ Grp $\to$ PrGrp be the functor sending a group G to its profinite completion $\widehat{G}$. Let $(-)^{(p')}\colon$ PrGrp $\to \mathrm{PrGrp}^{(p')}$ denote the functor sending a profinite group G to its maximal prime-to-p quotient $G^{(p')}$. Recall that it is defined as

$$G^{(p')} = \varprojlim_{([G:U],p)=1} G/U,$$

where the projective limit ranges over the open normal subgroups U of G of index prime-to-p. It is often much easier to get a grip on the prime-to-p quotient of the fundamental group of a scheme over a field of characteristic p. This is because the prime-to-p quotient filters out *wildly ramified covers*, of which there are usually many. The following lemma illustrates this.

Lemma 2.5. *Denote by $\mathbb{G}_m$ the curve $\mathbb{P}^1_k\backslash\{0,\infty\}$. There is an isomorphism $\pi_1^{(p')}(\mathbb{G}_m) \simeq \widehat{\mathbb{Z}}^{(p')}$.*

The lemma follows immediately from [GR06], but we also give the following elementary proof.

Proof. Let $\varphi\colon Y \to \mathbb{G}_m$ be a degree d connected étale cover of $\mathbb{G}_m$, where d is prime-to-p. Denote by $\overline{Y}$ the unique smooth compactifcation of Y. Then φ extends to a degree-d morphism $\overline{\varphi}\colon \overline{Y} \to \mathbb{P}^1$ of curves. Denote by $e_1,\dots,e_r$, respectively $f_1,\dots,f_s$, the ramification indices of $\overline{\varphi}$ over 0, respectively ∞. Since $\overline{\varphi}$ is of degree prime-to-p, we can apply the Riemann Hurwitz formula ([Har10, Chapter IV, Corollary 2.4]). We obtain

$$2g_Y - 2 = -2d + \sum_{i=1}^{r}(e_i - 1) + \sum_{j=1}^{s}(f_j - 1),$$

where g_Y denotes the genus of Y. After some consideration, this shows that $g_Y = 0$ and $e_1 = f_1 = d$. This leaves exactly one option for Y and φ up to isomorphism, namely $Y \simeq \mathbb{G}_m$ and $\varphi\colon y \mapsto y^d$. It has automorphism group cyclic of order d. We find

$$\pi_1^{(p')}(\mathbb{G}_m) \simeq \varprojlim_{(d,p)=1} \mathbb{Z}/d\mathbb{Z} \simeq \widehat{\mathbb{Z}}^{(p')}.$$

■

We have the adjunctions of functors

$$\begin{tikzcd} \mathrm{Grp} \arrow[r, bend left, "\widehat{(-)}"] \arrow[r, phantom, "\perp"] & \mathrm{PrGrp} \arrow[l, bend left] \arrow[r, bend left, "(-)^{(p')}"] \arrow[r, phantom, "\perp"] & \mathrm{PrGrp}^{(p')}, \arrow[l, bend left] \end{tikzcd} \tag{2.9}$$

where the unnamed arrows are the evident forgetful functors.

Given groups G and G', we denote by $G * G'$ the free product of G and G'. If G and G' are profinite groups, then we also denote by $G * G'$ the *free profinite product* of G and G'. If G and G' are pro-prime-to-p groups, then we denote by $G *^{(p')} G'$ the *free pro-prime-to-p product* of G and G'. In all three cases, this defines the coproduct of G and G' in their respective category. For a construction of the free profinite product and the free pro-prime-to-p product, see [NSW, Chapter IV, Section 1].

Lemma 2.6. *(i) For groups G and G' we have a canonical isomorphism*

$$\widehat{G*G'} \simeq \hat{G} * \hat{G}'.$$

(ii) For profinite groups G and G' we have a canonical isomorphism

$$(G*G')^{(p')} \simeq G^{(p')} *^{(p')} G'^{(p')}.$$

Proof. This is immediate by (2.9) and the fact that left adjoints preserve colimits. ■

2.2.2. *Computing the fundamental group.* We fix, once and for all, an étale path $\mathrm{Fib}_a \simeq \mathrm{Fib}_b$, where $\mathrm{Fib}_a, \mathrm{Fib}_b \colon \mathrm{Fét}_{\widetilde{C}} \rightrightarrows$ sets are the fiber functor over a and b. Via this étale path, we obtain the equivalences of categories

$$\begin{aligned}
\mathrm{Fét}_C &\simeq \mathscr{C} \\
&\simeq \{\text{pairs } (f\colon X \to \widetilde{C}, \varphi \in \mathrm{Aut}(f^{-1}(a))) \text{ with } f \text{ finite étale}\} \\
&\simeq \{\text{finite sets } F \text{ with a continuous action of } \pi_1(\widetilde{C},a) \text{ and of } \hat{\mathbb{Z}}\} \\
&\simeq \pi_1(\widetilde{C},a) * \hat{\mathbb{Z}}\text{-sets},
\end{aligned} \tag{2.10}$$

where the last category consists of finite sets with a continuous action of $\pi_1(\widetilde{C},a) * \hat{\mathbb{Z}}$. Here the third equivalence is induced by the functor Fib_a. We have a commutative triangle

$$\begin{tikzcd}
\mathrm{Fét}_C \arrow[rr, "\simeq"] \arrow[dr, "\mathrm{Fib}_n"'] & & \pi_1(\widetilde{C},a) * \hat{\mathbb{Z}}\text{-sets} \arrow[dl] \\
& \text{sets,} &
\end{tikzcd} \tag{2.11}$$

where the arrow $\pi_1(\widetilde{C},a) * \hat{\mathbb{Z}}\text{-sets} \to \text{sets}$ is the evident forgetful functor.

Theorem 2.7. *(i) The diagram* (2.11) *induces a canonical isomorphism*

$$\pi_1(C,n) \simeq \pi_1(\widetilde{C},a) * \hat{\mathbb{Z}}.$$

(ii) The map $\pi_* \colon \pi_1(\widetilde{C},a) \to \pi_1(C,n)$ is identified with the canonical inclusion

$$\pi_1(\widetilde{C},a) \to \pi_1(\widetilde{C},a) * \hat{\mathbb{Z}}$$

under the isomorphism of (i).

Proof. Part (i) is clear by the general machinery of Galois categories. Part (iii) follows from the commutative square

$$\begin{tikzcd}
\mathrm{Fét}_C \arrow[r, "\pi^*"] \arrow[d, "\simeq"] & \mathrm{Fét}_{\widetilde{C}} \arrow[d, "\simeq"] \\
\pi_1(C,a) * \hat{\mathbb{Z}}\text{-sets} \arrow[r] & \pi_1(\widetilde{C},a)\text{-sets,}
\end{tikzcd}$$

where the horizontal arrow on the bottom is given by precomposing the action on a finite set by the canonical inclusion $\pi_1(\widetilde{C},a) \to \pi_1(\widetilde{C},a) * \hat{\mathbb{Z}}$. ■

Suppose now that C has r-singularities $n_1,\dots,n_r$, all of whom are simple nodes. Let $\nu\colon C^n \to C$ denote the normalization of C. Applying the above theorem inductively, we obtain the following corollary.

Corollary 2.8. *There is an isomorphism*

$$\pi_1(C) \simeq \pi_1(C^n) * \left(\ast_{i=1}^{r} \widehat{\mathbb{Z}}\right).$$

2.2.3. *Some applications.* Suppose now that C is a proper rational curve whose only singularity is the node n. Then its normalization is $\pi\colon \widetilde{C} \simeq \mathbb{P}^1 \to C$.

Corollary 2.9. *We have an isomorphism $\pi_1(C) \simeq \widehat{\mathbb{Z}}$.*

Proof. Follows immediately from Corollary 2.8 and the fact that $\pi_1(\mathbb{P}^1) = 1$. ■

We may assume without loss of generality that neither a nor b is the point 0 or ∞. Write $\mathbb{G}_m = \mathbb{P}^1 \backslash \{0,\infty\}$ and $C^\circ = C \backslash \{\pi(0),\pi(\infty)\}$.

Corollary 2.10. *The maximal prime-to-p quotient of $\pi_1(C^\circ,n)$ is canonically isomorphic to*

$$\pi_1(C^\circ,n)^{(p')} \simeq \widehat{\mathbb{Z} * \mathbb{Z}}^{(p')},$$

and the map $\pi_*\colon \pi_1(\mathbb{G}_m,a) \to \pi_1(C^\circ,n)$ is identified with the canonical inclusion

$$\pi_1(\mathbb{G}_m,a) \to \pi_1(\mathbb{G}_m,a) * \hat{\mathbb{Z}}$$

under the isomorphism above.

Proof. The second part of the corollary is by Theorem 2.7(iii). The first part follows from Lemma 2.6 and Lemma 2.5:

$$\begin{aligned}\pi_1(C^\circ,n)^{(p')} &\simeq (\pi_1(\mathbb{G}_m,a) * \hat{\mathbb{Z}})^{(p')}\\ &\simeq \pi_1(\mathbb{G}_m,a)^{(p')} *^{(p')} \hat{\mathbb{Z}}^{(p')}\\ &\simeq \hat{\mathbb{Z}}^{(p')} *^{(p')} \hat{\mathbb{Z}}^{(p')}\\ &\simeq (\hat{\mathbb{Z}} * \hat{\mathbb{Z}})^{(p')} \simeq \widehat{\mathbb{Z} * \mathbb{Z}}^{(p')}.\end{aligned}$$

■

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