12.12.24

Reductive Groups over finite fields

k a ground field ($= \bar{k}$ unless otherwise specified)   Spec $k[x_1,\dots,x_n]/I$ reduced + fin type/k

Def. A linear algebraic group is an affine variety G/k together w. a group struct.:

$$\begin{array}{ccc} R & \mapsto & G(R) = \{\vec{x} \in R^n : f(\vec{x}) = 0 \ \forall f \in I\} \\ \mathrm{Rings}/k & \longrightarrow & \mathrm{Grp} \\ & \searrow \ \mathrm{Sets} \ \swarrow & \end{array}$$

△

"notion of a homomorphism makes sense ⇝ in particular we can speak of subgroups":

$H \hookrightarrow G$ closed embedding + hom. of algebraic groups = "subgroup"

Rmk. usually identify G w. $G(\bar{k})$.

Exp. (i) $\mathbb{G}_a := \operatorname{Spec} k[t]$, $\mathbb{G}_a(R) = R^+$. additive group

(ii) $n \geq 1$, $GL_n := \operatorname{Spec} k[(x_{ij})_{1 \leq i,j \leq n}, \det(x_{ij})^{-1}]$.

$R \mapsto GL_n(R)$ usual general linear group

(iii) $\mathbb{G}_m = GL_1$, $\mathbb{G}_m^r$ "torus" △

Thm. (Chevalley) For any lag there is an embedding $G \hookrightarrow GL_n$. △

closed subvar. + subgroup   "Linear"

Some words on the proof:

$G = \operatorname{Spec} A$ ⇝ G acts on G by right translation ⇝ G acts on A ⇝ find a f. dim'nal subspace $V \subseteq A$ s.t. G acts faithfully on V ⇒ $G \hookrightarrow GL(V)$.

unipotent and reductive groups

G/k a l.a.g

Recall: $A \in GL_n(k)$ is unipotent if $\exists e \geq 1$ s.t.

$$(A - \mathbb{1})^e = 0.$$

⟺ A is conj. to a matrix of the form

$$\begin{pmatrix} 1 & * & \cdots & * \\ & 1 & \cdots & * \\ 0 & & \ddots & \vdots \\ & & & 1 \end{pmatrix}.$$

Def. $x \in G(k)$ is unipotent if for all embeddings (equiv. one (mult. Jordan decomp.: 2.4.8 + 2.4.9 Springer))

$$i : G \hookrightarrow GL_n,$$

$i(x) \in GL_n(k)$ is unipotent. G is unipotent if all $x \in G(k)$ are. △

Exp. • Let $U_n = \left\{\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}\right\}$. Then U_n is unipotent.

• $\mathbb{G}_a \hookrightarrow GL_2$, $a \mapsto \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ is unipotent.

Note: Consider the normal subgroup $\mathbb{G}_a^{n-1} \cong H_n \trianglelefteq U_n$ given by

$$H_n = \left\{\begin{pmatrix} 1 & 0 & \cdots & * \\ 0 & 1 & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}\right\}.$$

Then $U_n/H_n \cong U_{n-1}$ ⇝ inductively we see that U_n is solvable.

Def. G is reductive if there is no non-trivial connected normal unip. subgrp.

"These things have a pleasant repr. theory."

Exp. $\mathbb{G}_a$ is not reductive: $\mathbb{G}_a \hookrightarrow GL_2$, $a \mapsto \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$.

Exp. GL_n and SL_n are reductive.

⇝ "will argue verbally in a minute."

Some important subgroups of GL_n and SL_n

(i) Borel subgroups (maximal conn. solvable subgroups)

• GL_n: Let $B^+ = \left\{\begin{pmatrix} * & * & \cdots & * \\ & * & \cdots & * \\ 0 & & \ddots & \vdots \\ & & & * \end{pmatrix}\right\}$.

All subgroups of GL_n conjugate to B^+ are called Borel. In particular,

$$B^- = \left\{\begin{pmatrix} * & 0 & \cdots & 0 \\ * & * & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & * \end{pmatrix}\right\} \text{ is Borel:}$$

$$B^- = P B^+ P^{-1}, \quad P = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 1 & 0 \\ \vdots & ⋰ & & \vdots \\ 1 & \cdots & 0 & 0 \end{pmatrix}.$$

$U_n \subset B$ + U_n solvable + $B/U_n \cong \mathbb{G}_m^n$ solv.

⇒ B solv. (maximality = Lie-Kolchin Thm.)

Claim: "GL_n and SL_n are reductive."

Suppose $N \trianglelefteq GL_n$ is a normal connected nilpotent subgroup (⇝ solvable). Then $N \subset B^+ \cap B^- \cong \mathbb{G}_m^n$, which has no non-trivial nilpotent elements

⇒ $N = \{1\}$. For SL_n similar. □

(ii) Maximal tori

• GL_n: $T = \mathbb{G}_m^n \hookrightarrow GL_n$ is a maximal torus.

$$(u_1, \dots, u_n) \mapsto \begin{pmatrix} u_1 & & \\ & \ddots & \\ & & u_n \end{pmatrix}$$

The maximal tori in GL_n are the conjugate subgroups of T.   ⇝ int. of max torus in GL_n w. SL_n

(iii) Parabolic subgroups

- GL_n : "Stabilizers of flags"

$I = (a_1, \dots, a_r)$, $a_1 + \dots + a_r = n$, partition,

$$P_I = \left\{ \begin{pmatrix} A_1 & * & * \cdots * \\ & A_2 & * \cdots * \\ 0 & & \ddots \vdots \\ & & A_r \end{pmatrix} : A_i \in GL_{a_i}(k) \right\}.$$

Note: $B^+ = P_{(1,1,\dots,1)}$

the P_I + conjugates = Parabolic subgroups

(iv) Levi subgroups

- GL_n:

$$GL_{a_1} \times \dots \times GL_{a_r} \simeq L_I = \left\{ \begin{pmatrix} A_1 & & 0 \\ & A_2 & \\ 0 & & \ddots \\ & & & A_r \end{pmatrix} \right\}.$$

The L_I + conjugates = Levi subgroups.

"Relevance comes from Levi-decomposition

$P = R_u(P) \rtimes L$. In particular: $B^+ = U_n \rtimes T$

$R_u(P)$: max. unipotent; L: reductive; $T = L_{(1,\dots,1)}$

For SL_n : Bor subgrps, max. tori, ... in SL_n

= (corr. subgrps of GL_n) $\cap$ SL_n.

Weyl groups

Def. Let G be a connected reductive linear algebraic group. Let $T \subset G$ be a maximal torus of G. (they are all conjugate)

Let N_G be the normalizer of T in G. The Weyl group of G is

$$W = N_G(T)/T.$$

Exp. For GL_n : $T = \mathbb{G}_m^n \hookrightarrow GL_n$.
$(u_1, \dots, u_n) \mapsto \begin{pmatrix} u_1 & & \\ & \ddots & \\ & & u_n \end{pmatrix}$

Then $N_G(T) = \left\{ A \in GL_n(k) : A \text{ has exactly one non-zero entry in each column} \right\}$. "monomial matrices"

So $S_n \xrightarrow{\simeq} W_n$.
$\sigma \mapsto (e_i \mapsto e_{\sigma(i)})$

"There is a way to attach to a l.a.g a root system s.t. W is the weyl group of said root system."

The flag variety $V = k^{\oplus n}$ a k-vector space.

Def. A flag of V is a sequence

$$0 = W_0 \subsetneq W_1 \subsetneq \dots \subsetneq W_{n-1} \subsetneq W_n = V$$

of subspaces of V. $\triangle$

Denote by Fl_n the set of flags of V.

We have a transitive action $GL_n \curvearrowright Fl_n$. If

$$f : 0 \subset \langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \dots$$

is the standard flag, then Borel subgrp. $B^+ = \{$upper triangular matrices$\}$ is exactly the stabilizer of f.

We obtain

$$GL_n / B^+ \xrightarrow{\simeq} Fl_n.$$

Prop. GL_n/B^+ is naturally a proj. var. $/k$. $\square$

⇝ word about structure as algebraic variety below.

Exp. $Fl_2 \cong \mathbb{P}^1$.

Bruhat decomposition

$G = GL_n$ or SL_n, $T \subset B$ a maximal torus in a Borel subgroup. $W = N_G(T)/T$.

Thm. (Bruhat decomposition) We have

$$G = \coprod_{w \in W} BwB. \quad \square$$

⇝ algebraic interpretation below.

Geometric version

$$Fl_n = GL_n/B^+ = \coprod_{w \in W} BwB/B.$$
$$= \coprod_{w \in W} BwB \cdot f$$

Exp. $G = GL_2$, $\mathbb{P}^1 \cong Fl_2 \ni f : 0 \subsetneq \langle e_1 \rangle \subsetneq k^{\oplus 2}$, $\ni g : 0 \subsetneq \langle e_2 \rangle \subsetneq k^{\oplus 2}$.

For $w = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in W$ we have

$$BwBf = \{f\}.$$

For $w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in W$ we have

$$BwBf = Bwf$$
$$= Bg = \{ 0 \subsetneq \langle ae_1 + be_2 \rangle \subsetneq k^{\oplus 2} \}, \quad b \neq 0$$
$$= Fl_2 \setminus \{f\}.$$

Bruhat decomposition is just

$$\begin{matrix} Fl_2 & = & \{f\} & \amalg & Fl_2 \setminus \{f\} \\ \cong & & \cong & & \cong \\ \mathbb{P}^1 & = & pt & \amalg & \mathbb{A}^1. \end{matrix}$$

Linear algebraic groups over $\mathbb{F}_q$.

Let G_0 be a l.a.g over $\mathbb{F}_q$. $G := G_0 \otimes_{\mathbb{F}_q} \bar{\mathbb{F}}_q$.

Let $F_0 : G_0 \to G_0$, $x^q \leftarrow\!\shortmid x$, the Frobenius.

We obtain the Frobenius endomorphism

$$F = F_0 \otimes_{\mathbb{F}_q} \mathrm{id}_{\bar{\mathbb{F}}_q} : G \to G.$$

Exp. For $G_0 = GL_{n,\mathbb{F}_q}$: $F(a_{ij}) = (a_{ij}^q)$.

Note $G_0(\mathbb{F}_q) = G(\bar{\mathbb{F}}_q)^F$.

Thm. (Lang - Steinberg) Let G_0 be a connected linear algebraic group $/\mathbb{F}_q$ and let $F: G \to G$ be the Frobenius. Then the map

$$G \to G$$
$$x \mapsto x^{-1}F(x)$$

is surjective. $\square$

"$H^1(\mathbb{F}_q, G) = \{*\}$." "no non-trivial G-torsors over $\mathbb{F}_q$."

A few words about the proof:

For $a \in G$ set $L_a: G \to G$, $x \mapsto x^{-1}aF(x)$.

Then a computation shows $dL_a: \mathrm{Lie}(G) \xrightarrow{\simeq} \mathrm{Lie}(G)$

$\Rightarrow L_a(G)$ contains a nonempty dense open U_a.

$U_a \cap U_e \neq \emptyset \Rightarrow \exists x, y \in G$ s.t. $x^{-1}F(x) = y^{-1}aF(y)$

$\Rightarrow a = L(xy^{-1})$. $\boxtimes$

Fl_n as a variety:

G a l.a.g., $H < G$ a subgroup

(i) find a repr. $\rho: G \to GL(V)$ and a 1-dim subspace $W \subseteq V$ s.t. $H = \mathrm{Stab}(W)$. ($= \langle w \rangle$)

(ii) Then $G \to \mathbb{P}(V)$, $g \mapsto g \cdot \langle w \rangle$ has image $G \cdot \langle w \rangle \subset \mathbb{P}(V)$ a q.proj. variety

(iii) $\Rightarrow G/H \xrightarrow{\simeq} G \cdot \langle w \rangle$

Algebraic interpretation of Bruhat:

Let G be the general linear group $\mathbf{GL}_n$ of invertible $n \times n$ matrices with entries in some algebraically closed field, which is a reductive group. Then the Weyl group W is isomorphic to the symmetric group S_n on n letters, with permutation matrices as representatives. In this case, we can take B to be the subgroup of upper triangular invertible matrices, so Bruhat decomposition says that one can write any invertible matrix A as a product U_1PU_2 where U_1 and U_2 are upper triangular, and P is a permutation matrix. Writing this as $P = U_1^{-1}AU_2^{-1}$, this says that any invertible matrix can be transformed into a permutation matrix via a series of row and column operations, where we are only allowed to add row i (resp. column i) to row j (resp. column j) if $i > j$ (resp. $i < j$). The row operations correspond to U_1^{-1}, and the column operations correspond to U_2^{-1}.