A SPREADING ARGUMENT FOR ℓ-ADIC REPRESENTATIONS WITH FINITE DETERMINANT

HUGO ZOCK

ABSTRACT. We improve a classical spreading argument for ℓ-adic representations allowing us to preserve the property of having finite determinant. Additionally, we prove that such spreadings are unique in an appropriate sense.

1. INTRODUCTION

Let G and H be profinite groups and let G act continuously on H. Write $\widetilde{G} = H \rtimes G$ for the semi-direct product of G and H. It is again a profinite group, because the underlying topology of $\widetilde{G}$ is that of $G \times H$; hence, it is compact and totally disconnected. We think of G and H as living inside this semi-direct product. In particular, we denote the action of G on H by conjugation.

One scenario in which this situation arises naturally is described by the following standard result.

Proposition 1.1. *Suppose we have an exact sequence of profinite groups*

$$1 \to H \to K \to G \to 1$$

such that $K \to G$ admits a continuous section $G \to K$. Then G acts continuously on H by conjugation, and we have an isomorphism $K \simeq H \rtimes G$ fitting into the commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & H & \longrightarrow & K & \longrightarrow & G & \longrightarrow & 1 \\ & & \downarrow{=} & & \downarrow{\simeq} & & \downarrow{=} & & \\ 1 & \longrightarrow & H & \longrightarrow & H \rtimes G & \longrightarrow & G & \longrightarrow & 1. \end{array}$$

■

Given a representation $\rho\colon H \to \mathrm{GL}(V)$ of H and an element $g \in G$, we obtain a representation $\rho^g\colon H \to \mathrm{GL}(V)$ by precomposing ρ with the map $:\ H \to H$ obtained from the G-action on H. We consider the set of isomorphism classes of representations derived from ρ in this way:

$$[\rho] \cdot G = \{\rho^g : g \in G\} / \simeq .$$

If there exists an open subgroup $U \subset G$ such that ρ is the restriction of a representation $\widetilde{\rho}\colon H \rtimes U \to \mathrm{GL}(V)$, then clearly $[\rho] \cdot G$ must be finite: for all $g \in U$ we have

$$\rho^g = \widetilde{\rho}(g) \cdot \rho \cdot \widetilde{\rho}(g)^{-1}.$$

The first part of the main result of this paper, Theorem 2.5(i), states that finiteness of $[\rho]\cdot G$ is actually a sufficient condition if ρ is an irreducible continuous $\overline{\mathbb{Q}}_\ell$-representations of H. This is a well-known result whose proof can also be found in [Lit21, Propostion 3.1.1]. Furthermore, in Theorem 2.5(ii), we prove that if the determinant of ρ is finite, then its spreading to an open subgroup $H \rtimes U$ can be chosen to have finite determinant as well. In corollary 2.6 this additional condition is used to show that in this case spreadings are *unique* up to a diminution of U.

Finally, we apply Theorem 2.5(i) to prove a basic result about so-called *arithmetic ℓ-adic local systems* in Section 3. The relevance of Theorem 2.5(ii) comes from the fact that aritmetic ℓ-adic local systems generally have finite determinant. In [Zoc24] this is crucially exploited.

The appendix clarifies some basic results from the theory of non-abelian Galois cohomology with non-discrete coefficients. These are likely well known, but seem hard to track down in the literature.

The contents of this paper have been copied, with some minor modifications, from the author's master's thesis [Zoc24, Chapters 5 and 6]. It was written under the supervision of Moritz Kerz at the university of Regensburg. The reason for producing a separate paper is the fact that the results are likely of independent interest.

2. A SPREADING ARGUMENT FOR ℓ-ADIC REPRESENTATIONS WITH FINITE DETERMINANT

2.1. **The space of ℓ-adic representations.** Notation is as in the introduction. Throughout, we fix a prime ℓ. By an *ℓ-adic representation* of a profinite group H we mean a continuous finite-dimensional $\overline{\mathbb{Q}}_\ell$-representation. We define the set $\mathscr{R}$ to be the set of isomorphism classes of semisimple ℓ-adic representations of H:

$$\mathscr{R} = \{\text{semisimple } \ell\text{-adic representations } H \to \mathrm{GL}(V)\}/\simeq .$$

For $\rho\colon H \to \mathrm{GL}(V)$ a semisimple ℓ-adic representation, we denote by $[\rho] \in \mathscr{R}$ its isomorphism class. We have a natural right action of G on $\mathscr{R}$ defined by

$$[\rho]^g = [\rho^g] \quad (g \in G, [\rho] \in \mathscr{R}),$$

where

$$\rho^g(h) = \rho(ghg^{-1}) \quad (h \in H).$$

We intend to equip $\mathscr{R}$ with a topology such that the action described above is continuous. Denote by $\mathrm{Map}(H, \overline{\mathbb{Q}}_\ell)$ the set of continuous (set-)maps $H \to \overline{\mathbb{Q}}_\ell$. We obtain an injection

$$\begin{aligned} \mathscr{R} &\hookrightarrow \mathrm{Map}(H, \overline{\mathbb{Q}}_\ell) \\ [\rho] &\mapsto \mathrm{Tr}\,\rho := \mathrm{Tr} \circ \rho, \end{aligned}$$

where $\mathrm{Tr}\colon \mathrm{GL}(V) \to \overline{\mathbb{Q}}_\ell$ denotes the (continuous) trace map, by the following lemma.

Lemma 2.1 ([Wie12, Proposition 2.4.3])**.** *Let k be a field of characteristic* 0*, A a k-algebra, and V and V' two* semisimple *A-modules of finite k-dimension. If the characters* $\mathrm{Tr}_V\colon G \to$

k and $\mathrm{Tr}_{V'}\colon G \to k$ *obtained by sending* $g \in G$ *to* $\mathrm{Tr}(g|_V)$, *respectively* $\mathrm{Tr}(g|_{V'})$, *are equal, then* V *and* V' *are isomorphic as* A*-modules.* ■

We equip $\mathrm{Map}(H, \overline{\mathbb{Q}}_\ell)$ with the compact-open topology. Then $\mathscr{R}$ is equipped with the subspace topology inherited from $\mathrm{Map}(H, \overline{\mathbb{Q}}_\ell)$.

Proposition 2.2. *The action of G on $\mathscr{R}$ is continuous.*

Proof. By assumption, the map $G \times H \to H$ is continuous. By [Mun14, Theorem 46.11] the induced map

$$\varphi\colon G \to \mathrm{Map}(H, H)$$

is continuous if we equip $\mathrm{Map}(H, H)$ with the compact-open topology. The space H is locally compact and Hausdorff (H is profinite), and so by [Mun14, Exercise 7, §46] we find that the composition map

$$c\colon \mathrm{Map}(H, H) \times \mathrm{Map}(H, \overline{\mathbb{Q}}_\ell) \to \mathrm{Map}(H, \overline{\mathbb{Q}}_\ell)$$

is continuous. As a result, the map

$$G \times \mathrm{Map}(H, \overline{\mathbb{Q}}_\ell) \xrightarrow{\varphi\times\mathrm{id}} \mathrm{Map}(H, H) \times \mathrm{Map}(H, \overline{\mathbb{Q}}_\ell) \xrightarrow{c} \mathrm{Map}(H, \overline{\mathbb{Q}}_\ell)$$

is continuous, and hence so is

$$G \times \mathscr{R} \to \mathscr{R}.$$

■

Corollary 2.3. *Let $[\rho] \in \mathscr{R}$. Then the stabilizer $\mathrm{Stab}_{[\rho]}$ of $[\rho]$ is a closed subgroup of G. If the orbit of $[\rho]$ is finite, then $\mathrm{Stab}_{[\rho]}$ is also open.*

Proof. Write

$$\Phi\colon G \times \mathscr{R} \to \mathscr{R}$$

for the action map. We have

$$\mathrm{Stab}_{[\rho]} = \Phi^{-1}([\rho]) \cap G \times \{[\rho]\},$$

and so we only have to argue that $[\rho] \in \mathscr{R}$ is a closed point. This follows from the fact that $\mathrm{Map}(H, \overline{\mathbb{Q}}_\ell)$ is Hausdorff: $\overline{\mathbb{Q}}_\ell$ is Hausdorff so that we can apply [Mun14, Exercise 6, §46]. The second part of the corollary follows from the fact that $\mathrm{Stab}_{[\rho]}$ has finite index if the orbit of $[\rho]$ is finite. ■

Corollary 2.4. *Let $[\rho] \in \mathscr{R}$ such that orbit of $[\rho]$ in $\mathscr{R}$ is finite. Then the orbit of each of the irreducible constituents of ρ is finite.*

Proof. Let U be the stabilizer of $[\rho]$. By Corollary 2.3 it is open. The subgroup U permutes the irreducible constituents of ρ, and so there exists an open subgroup $V \subset U$ fixing all of them. ■

2.2. **Spreading ℓ-adic representations with finite determinant.** Notation is as before.

Theorem 2.5. *(i) Let $\rho\colon H \to \mathrm{GL}_r(\overline{\mathbb{Q}}_\ell)$ be an irreducible ℓ-adic representation and assume that the orbit $[\rho]\cdot G \subset \mathscr{R}$ is finite. Then there exists an open subgroup $U \subset G$ such that ρ extends to a continuous representation*

$$\widetilde{\rho}\colon \widetilde{U} := H \rtimes U \to \mathrm{GL}_r(\overline{\mathbb{Q}}_\ell).$$

(ii) Furthermore, if $\det\rho$ *is finite (i.e., the determinant character of ρ has finite image), then $\widetilde{\rho}$ can be chosen such that* $\det\widetilde{\rho}$ *is finite.*

Proof. We can find E a finite extension of $\mathbb{Q}_\ell$ such that ρ factors as $H \to \mathrm{GL}_r(E) \to \mathrm{GL}_r(\overline{\mathbb{Q}}_\ell)$. By the assumption that $[\rho]\cdot G$ is finite, and Corollary 2.3, $\mathrm{Stab}_{[\rho]}$ is open. By potentially replacing G by the open subgroup $\mathrm{Stab}_{[\rho]}$, we can assume that G acts trivially on $[\rho]$. As a result, for every $g \in G$ there is an isomorphism $\rho^g \simeq \rho$ so that ρ and ρ^g are conjugate to each other by some $A_g \in \mathrm{GL}_r(E)$:

$$\rho^g = A_g \cdot \rho \cdot A_g^{-1}.$$

We can indeed take the A_g to be defined over E, because G acts trivially on the trace character of ρ, and hence trivially on the E-isomorphism class of ρ by Lemma 2.1. We define

$$\overline{A}\colon G \to \mathrm{PGL}_r(E)$$
$$g \mapsto \overline{A}_g,$$

where $\overline{A}_g$ denotes the class of A_g in $\mathrm{PGL}_r(E)$. It is easily seen that $\overline{A}$ is a homomorphism by Schur's Lemma. We argue that $\overline{A}$ is additionally continuous by applying Lemma 2.8 below. We employ the notation introduced in that lemma. Notice first that, since ρ is irreducible over $\overline{\mathbb{Q}}_\ell$, we have

$$E[\rho(h)\colon h \in H] = \mathrm{Mat}(r \times r; E)$$

by [EG11, Theorem 3.2.2]. For $1 \le i, j \le r$ we can therefore write

$$e_{i,j} = \sum_{h\in H} \alpha_h^{i,j} \rho(h)$$

with $\alpha_h^{i,j} \in E$ zero for all but finitely many $h \in H$. Then for $g \in G$ we compute

$$\begin{aligned}(\mathrm{ev}_{i,j} \circ \overline{A})(g) &= A_g e_{i,j} A_g^{-1} \\ &= \sum_{h\in H} \alpha_h^{i,j} \rho^g(h) \\ &= \sum_{h\in H} \alpha_h^{i,j} \rho(ghg^{-1}).\end{aligned}$$

We see that $\mathrm{ev}_{i,j} \circ \overline{A}$ is a linear combination of continuous functions and hence is continuous. It follows that $\overline{A}$ is continuous by Lemma 2.8.

Our goal is to lift $\overline{A}|_U$ to a continuous homomorphism $A\colon U \to \mathrm{GL}_r(E)$ for some open subgroup $U \subset G$. To this end, we apply the theory of continuous non-abelian cohomology, which is recalled in Appendix A. Consider the strict exact sequence of topological G-groups (each with trivial G-action)

$$1 \to \mu_r \to \mathrm{SL}_r(E) \to \mathrm{PSL}_r(E) \to 1$$

from Corollary 2.11 below. By Theorem A.3, we obtain an exact sequence of pointed sets

$$H^1_{\mathrm{cont}}(G;\mathrm{SL}_r(E)) \to H^1_{\mathrm{cont}}(G;\mathrm{PSL}_r(E)) \xrightarrow{\delta} H^2_{\mathrm{cont}}(G;\mu_r).$$

By potentially shrinking G to an open subgroup we can assume that the image of $\overline{A}$ lies in $\mathrm{PSL}_r(E)$ by Lemma 2.10. Let $U \subset G$ be an open subgroup such that $\mathrm{res}^G_U(\delta(\overline{A})) = 0$. This is possible by the fact that μ_r is discrete. Then, since restriction is compatible with connecting homomorphisms, we find

$$\delta(\mathrm{res}^G_U(\overline{A})) = 0 \in H^2_{\mathrm{cont}}(U;\mu_r),$$

where now δ denotes the connecting homomorphism in the sequence

$$H^1_{\mathrm{cont}}(U;\mathrm{SL}_r(E)) \to H^1_{\mathrm{cont}}(U;\mathrm{PSL}_r(E)) \to H^2_{\mathrm{cont}}(U;\mu_r).$$

It follows that there exists $A \in H^1_{\mathrm{cont}}(U;\mathrm{SL}_r(E))$ lifting $\overline{A}|_U$[1]. We now set

$$\widetilde{\rho} = \rho \rtimes A\colon H \rtimes U \to \mathrm{GL}_r(E).$$

The last part of the proposition follows by construction since A takes values in $\mathrm{SL}_r(E)$. ■

Corollary 2.6. *With hypotheses as in the theorem above, suppose that $\rho\colon H \to \mathrm{GL}_r(\overline{\mathbb{Q}}_\ell)$ is irreducible with finite determinant. Then an extension $\widetilde{\rho}\colon H \rtimes U \to \mathrm{GL}_r(\overline{\mathbb{Q}}_\ell)$ of ρ with finite determinant is* unique *up to a diminution of U.*

Proof. Let $\widetilde{\rho}, \widetilde{\rho}'\colon H \rtimes U \rightrightarrows \mathrm{GL}_r(\overline{\mathbb{Q}}_\ell)$ be two extensions of ρ with finite determinant. Then their "projectivizations" $H \rtimes U \rightrightarrows \mathrm{GL}_r(\overline{\mathbb{Q}}_\ell) \to \mathrm{PGL}_r(\overline{\mathbb{Q}}_\ell)$ must both equal $\rho \rtimes \overline{A}$, where $\overline{A}\colon U \to \mathrm{PGL}_r(\overline{\mathbb{Q}}_\ell)$ is the unique homomorphism such that $\rho^g = \overline{A}_g \rho \overline{A}_g^{-1}$ for all $g \in U$. Therefore, $\widetilde{\rho}$ and $\widetilde{\rho}'$ differ by a character $\chi\colon H \rtimes U \to \overline{\mathbb{Q}}_\ell^\times$. By finiteness of the determinants, this must be a finite character. Since χ is trivial on H, χ will vanish after shrinking U. Hence, $\widetilde{\rho}$ and $\widetilde{\rho}'$ will coincide after shrinking U. ■

Corollary 2.7. *Let $\rho\colon H \to \mathrm{GL}_r(\overline{\mathbb{Q}}_\ell)$ be a semisimple representation such that the orbit $[\rho]\cdot G \subset \mathscr{R}$ is finite. Then there exists an open subgroup $U \subset G$ such that ρ extends to a continuous representation*

$$\widetilde{\rho}\colon H \rtimes U \to \mathrm{GL}_r(\overline{\mathbb{Q}}_\ell).$$

Proof. Each of the irreducible constituents of ρ has finite G-orbit by Lemma 2.4. Then we apply Theorem 2.5 to spread each of the irreducible constituents. After taking an appropriate direct sum, we find a spreading of ρ. ■

[1]Notice that by surjectivity of $\mathrm{SL}_r(E) \to \mathrm{PSL}_r(E)$ we can find an actual lift of $\overline{A}$ and not just of its conjugacy class in $H^1_{\mathrm{cont}}(U;\mathrm{SL}_r(E))$.

2.3. **Some auxiliary results.** Let E be a finite extension of $\mathbb{Q}_\ell$. The projective general linear group $\mathrm{PGL}_r(E)$ is equipped with the quotient topology from $\mathrm{GL}_r(E) \twoheadrightarrow \mathrm{PGL}_r(E)$. Denote by $M_r(E)$ the algebra of $r \times r$-matrices over E.

Lemma 2.8 (Topology of PGL_r)**.** *The space* $\mathrm{PGL}_r(E)$ *has the coarsest topology making each of the evaluation maps*

$$\begin{aligned} \mathrm{ev}_{i,j}\colon\ \mathrm{PGL}_r(E) &\to M_r(E) \\ M &\mapsto Me_{i,j}M^{-1} \end{aligned}$$

continuous. Here $e_{i,j} \in M_r(E)$ *denotes the matrix with a* 1 *in the* (i,j)*-th entry and zeroes everywhere else.*

Proof. By the Skölem-Noether Theorem (see [GS17, Theorem 2.7.2]), we obtain a continuous bijection

$$\begin{aligned} \mathrm{PGL}_r(E) &\to \mathrm{Aut}_E(M_r(E)), \\ M &\mapsto (\varphi_M\colon N \mapsto MNM^{-1}), \end{aligned} \tag{2.1}$$

where $\mathrm{Aut}_E(M_r(E))$ denotes the set of E-algebra automorphisms of $M_r(E)$ with the subspace topology inherited from $\mathfrak{gl}(M_r(E))$, the set of E-linear endomorphisms of $M_r(E)$. As a topological space, it is homeomorphic to $E^{\oplus r^4}$. The space $\mathrm{Aut}_E(M_r(E))$ is a locally compact Hausdorff space, because it is a subspace of $\mathfrak{gl}(M_r(E))$. As a result, it is a Baire space. By [Ser92, Part II, Chapter IV, Section 4, Lemma 1], we conclude that the map from (2.1) is a homeomorphism. It is clear that the topology on $\mathfrak{gl}(M_r(E))$ is the coarsest one for which each of the maps

$$\begin{aligned} \mathrm{ev}_{i,j}\colon\ \mathfrak{gl}(M_r(E)) &\to M_r(E) \\ \varphi &\mapsto \varphi(e_{i,j}) \end{aligned}$$

is continuous. The result follows. ■

Denote by $\mathrm{PSL}_r(E)$ the image of $\mathrm{SL}_r(E)$ in $\mathrm{PGL}_r(E)$ equipped with the subspace topology.

Lemma 2.9. *The map* $\mathrm{SL}_r(E) \twoheadrightarrow \mathrm{PSL}_r(E)$ *admits a continuous (set-theoretic) section* $\mathrm{PSL}_r(E) \to \mathrm{SL}_r(E)$.

Proof. We show that the map $\mathrm{SL}_r(E) \to \mathrm{PGL}_r(E)$ of ℓ-adic Lie groups induces an isomorphism on Lie-algebras. The Lie algebra of $\mathrm{SL}_r(E)$ is $\mathfrak{sl}_r(E)$, the $r \times r$ matrices over E with trace 0. By construction of the quotient Lie group (see [Ser92, Part II, Chapter IV, Section 5]), the Lie algebra of $\mathrm{PGL}_r(E)$ is given by

$$\mathrm{Lie}(\mathrm{PGL}_r(E)) = M_r(E)/EI_r,$$

where I_r denotes the identity matrix. The induced map

$$\mathfrak{sl}_r(E) \to \mathrm{Lie}(\mathrm{PGL}_r(E))$$

is an isomorphism of Lie algebras. By the Inverse Function Theorem (see [Ser92, Part II, Chapter 2, Section 9]), $\mathrm{SL}_r(E) \to \mathrm{PGL}_r(E)$ is a local isomorphism. Certainly then, $\mathrm{SL}_r(E) \to \mathrm{PSL}_r(E)$ admits sections locally. Since $\mathrm{PSL}_r(E)$ has a basis consisting of open (hence closed) subgroups, we can extend such local sections to the whole of $\mathrm{PSL}_r(E)$. ■

Lemma 2.10. *The subspace* $\mathrm{PSL}_r(E) \subset \mathrm{PGL}_r(E)$ *is open.*

Proof. The proof of Lemma 2.9 shows that $\mathrm{SL}_r(E) \to \mathrm{PGL}_r(E)$ is a local isomorphism, from which it follows that the image of $\mathrm{SL}_r(E)$ in $\mathrm{PGL}_r(E)$ is open. ■

Corollary 2.11. *We have a strict exact sequence of topological groups*

$$1 \to \mu_r \to \mathrm{SL}_r(E) \to \mathrm{PSL}_r(E) \to 1, \tag{2.2}$$

where $\mu_r \subset E$ denotes the set of r-th roots of unity in E. The sequence (2.2) *satisfies properties (i) and (ii) of Section A.2.*

3. Arithmetic ℓ-adic local systems

Let k be a finitely generated field of characteristic $p \geq 0$, and let ℓ be a prime different from p. Pick an algebraically closed field Ω containing k and let $\overline{k}$ be the separable closure of k in Ω. Let X be an integral separated finite type $\overline{k}$-scheme. Let $\overline{x}\colon \operatorname{Spec}\Omega \to X$ be a geometric point. An *ℓ-adic (or $\overline{\mathbb{Q}}_\ell$-) local system* $\mathbb{L}$ on X is completely determined by its monodromy representation $\pi_1^{\text{ét}}(X,\overline{x}) \to \mathrm{GL}(\mathbb{L}_{\overline{x}})$. Conversely, any ℓ-adic representation of $\pi_1(X,\overline{x})$ gives rise to a local system on X. See, for instance, [FK13, Appendix A].

Definition 3.1. An ℓ-adic local system $\mathbb{L}$ on X is said to be *arithmetic* if there exists an ℓ-adic local system $\mathbb{L}^0_{k'}$ on a spreading $X^0_{k'}$ of X to a scheme $X^0_{k'}$ over k', for some finite separable extension $k \subset_f k' \subset \overline{k}$, such that $\mathbb{L}^0_{k'}$ pulls back to $\mathbb{L}$.

Example 3.2. A *local system coming from geometry* is a local system of the form $R^i f_* \mathbb{Q}_\ell$, where f is a smooth proper morphism and X is normal. By [Del80, Corollaire 3.4.13] these are semisimple local systems. A spreading argument shows that local systems coming from geometry are arithmetic.

By potentially replacing k by a finite extension in $\overline{k}$, we can assume that X spreads to a scheme $X^0 \to \operatorname{Spec} k$. Furthermore, by enlarging k further, we can assume that X^0 admits a k-rational point $x\colon \operatorname{Spec} k \to X^0$. Let $\overline{x}\colon \operatorname{Spec}\Omega \to X$ be the induced geometric point. The rational point x now splits the homotopy exact sequence

$$1 \to \pi_1(X,\overline{x}) \to \pi_1(X^0,\overline{x}) \to \mathrm{Gal}(\overline{k}/k) \to 1 \tag{3.1}$$

from [Sza09, Proposition 5.6.1]. As a result, $\mathrm{Gal}(\overline{k}/k)$ acts continuously on $\pi_1(X^0,\overline{x})$ by conjugation.

Notice that an ℓ-adic local system $\mathbb{L}$ on X is arithmetic if and only if it spreads to X_ℓ for some finite extension $k \subset \ell \subset \overline{k}$. By Corollary 2.7 the following result is now clear.

Proposition 3.3. *A semisimple ℓ-adic local system $\mathbb{L}$ on X is arithmetic if and only if the isomorphism class of the associated monodromy representation $\rho\colon \pi_1(X,\overline{x}) \to \mathrm{GL}(\mathbb{L}_{\overline{x}})$ has finite orbit under* $\mathrm{Gal}(\overline{k}/k)$.

This result also appears in [Lit21, Propostion 3.1.1].

As already mentioned in the introduction, Theorem 2.5(ii) and the fact that arithmetic ℓ-adic local systems have finite determinant on curves are crucial ingredients in the proof of the main result of [Zoc24]. Finiteness of the determinant of an arithmetic ℓ-adic local system on a normal curve over $\overline{\mathbb{F}}_p$ is [Del80, Proposition 1.3.4]. On a curve over a general field one reduces to the case where the base field is $\overline{\mathbb{F}}_p$ by a specialization argument. See also [Zoc24, Proposition 6.9].

Appendix A. Continuous non-abelian Galois Cohomology

Throughout, G is a profinite group. We collect a few facts regarding continuous non-abelian cohomology used in Section 2. Although there are many references on both continuous cohomology and non-abelian cohomology, the author was unable to find any references regarding the cohomology of profinite groups with non-discrete and non-abelian coefficients. Many of the statements in this appendix are straightforward generalizations of well known results. In particular, we mimic [Ser79, Appendix to Chapter VII].

A.1. **The cohomology groups.**

Definition A.1. A G-group T is a topological group T (perhaps non-abelian) with a continuous action of G. A morphism of topological G-groups is a continuous homomorphism compatible with the actions of G.

If T is a G-group, $t \in T$ is an element of T and $\sigma \in G$, is an element of G, then we will denote the image of t under the action of σ by ${}^{\sigma}t$.

Definition A.2. Let T be a G-group. We define a continuous one-cocycle of G with coefficients in T to be a continuous map of spaces

$$c\colon G \to T \quad \sigma \mapsto c_\sigma$$

such that for all $\sigma, \tau \in G$ we have

$$c_{\sigma\tau} = c_\sigma {}^{\sigma}c_\tau.$$

Two one-cocyles c and b are said to be cohomologous if there is $t \in T$ such that for all $\sigma \in G$ we have

$$c_\sigma = t^{-1} b_\sigma {}^{\sigma}t.$$

For a G-group T, "being cohomologous" defines an equivalence relation $\sim$ on the set of continuous one-cocyles of G with coefficients in T. We define

$$H^1_{\mathrm{cont}}(G;T) = \{\text{continuous one-cocycles } c\colon G \to T\}/\sim.$$

Notice that $H^1_{\mathrm{cont}}(G,T)$ is equipped with a canonical basepoint: the class of the trivial one-cocycle $\sigma \mapsto 1$.

If T happens to be an abelian G-group, then $H^2_{\mathrm{cont}}(G;T)$ is defined in the usual way as continuous 2-cocyles modulo continuous 2-coboundaries; see [Tat76].

If $H \subset G$ is a closed subgroup, then we can define restriction

$$\mathrm{res}^G_U\colon\ H^i_{\mathrm{cont}}(G,T) \to H^i_{\mathrm{cont}}(U,T)$$

as usual.

A.2. **An analogue of the long exact sequence.** Suppose now that we have a *strict exact sequence* of topological G-groups

$$1 \to T' \to T \to T'' \to 1.$$

This means in particular that T' carries the subspace topology inherited from T and T'' carries the quotient topology inherited from T. Assume also that

(i) T' lands in the center of T;
(ii) we have a continuous set-theoretic section $s\colon\ T'' \to T$.

Notice that T' is abelian so that $H^2_{\mathrm{cont}}(G;T')$ is defined. We construct a boundary map

$$\delta\colon\ H^1_{\mathrm{cont}}(G;T'') \to H^2_{\mathrm{cont}}(G;T') \tag{A.1}$$

as follows: for the class of a continous one-cocyle $c\colon\ G \to T''$, we define

$$\delta(c)_{\sigma,\tau} = b_\sigma {}^\sigma b_\tau b_{\sigma\tau}^{-1} \in T' \quad (\sigma,\tau \in G), \tag{A.2}$$

where $b\colon\ G \to T$ is a continuous lift of c. Such a lift always exists, since we can compose c with the section s from (ii). As shown in [Ser79], it is a 2-cocycle. Furthermore, $\delta(c)$ is continuous by continuity of b. If we pick a different lift $\sigma \mapsto a'_\sigma b_\sigma$, with $a'_\sigma \in T'$, then $\sigma \mapsto a'_\sigma$ is continuous. Now, the two-cocycle $\delta(c)$ is replaced by $(\sigma,\tau) \mapsto a_{\sigma,\tau}\delta(c)_{\sigma,\tau}$, where

$$a_{\sigma,\tau} = (\partial a')_{\sigma,\tau} = a'_\sigma {}^\sigma a'_\tau a'^{-1}_{\sigma\tau}.$$

It follows that the class of $\delta(c)$ in $H^2_{\mathrm{cont}}(G;T')$ is independent of the choice of the lift.

We show that δ does not depend on the choice of representative for the cocycle class of c. Indeed, if c' is a continuous one-cocycle cohomologous to c, then there is $t'' \in T''$ such that

$$c'_\sigma = t''^{-1} c_\sigma {}^\sigma t'' \quad (\sigma \in G).$$

Let $t \in T$ such that $t \mapsto t''$. We can lift c' to $\sigma \mapsto t^{-1} b_\sigma {}^\sigma t$. Clearly, this is again a continuous lift. As shown in [Ser79], the resulting continuous two-cocycles of G with coefficients in T' are the same. We conclude that the map δ is well-defined.

Theorem A.3. *The sequence*

$$H^1_{\mathrm{cont}}(G;T) \to H^1_{\mathrm{cont}}(G;T'') \xrightarrow{\delta} H^2_{\mathrm{cont}}(G;T'),$$

with δ the map from (A.1)*, is an exact sequence of pointed sets.*

Proof. The fact that the composition of the two maps is the trivial map is exactly as in the classical discrete case. Suppose we have a one-cocycle $c \in H^1_{\text{cont}}(G; T'')$ such that $\delta(c) = 0 \in H^2_{\text{cont}}(G; T')$. Then there is $a \in C^1_{\text{cont}}(G; T')$ a continuous map such that

$$\delta(c)_{\sigma,\tau} = a_\sigma {}^\sigma a_\tau a_{\sigma\tau}^{-1} \quad (\sigma, \tau \in G).$$

By property (i) above we get

$$(b_\sigma a_\sigma^{-1})^\sigma (b_\tau a_\tau^{-1})(b_{\sigma\tau} a_{\sigma\tau}^{-1})^{-1} = 1 \in T' \quad (\sigma \in G). \tag{A.3}$$

Define now $b' \colon G \to T$ by $\sigma \mapsto b_\sigma a_\sigma^{-1}$. Then b' is continuous by the fact that a and b are. By (A.3), b' is a cocycle. We clearly have $b' \mapsto c$ and so we win. ■

References

[Del80] Pierre Deligne. “La conjecture de Weil : II”. fr. In: *Publications Mathématiques de l’IHÉS* 52 (1980), pp. 137–252.

[EG11] Pavel I. Etingof and Slava Gerovitch. *Introduction to representation theory.* Vol. 59. Student mathematical library. American Mathematical Society, 2011.

[FK13] Eberhard Freitag and Reinhardt Kiehl. *Etale Cohomology and the Weil Conjecture.* Vol. 13. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge / A Series of Modern Surveys in Mathematics. Springer, 2013.

[GS17] Philippe Gille and Tamás Szamuely. *Central simple algebras and Galois cohomology.* Second edition. Vol. 165. Cambridge studies in advanced mathematics. Cambridge University Press, 2017.

[Lit21] Daniel Litt. “Arithmetic representations of fundamental groups, II: Finiteness”. In: *Duke Mathematical Journal* 170.8 (2021), pp. 1851–1897. DOI: 10.1215/00127094-2020-0086.

[Mun14] James Munkres. *Topology.* 2nd ed. Pearson, 2014.

[Ser79] Jean-Pierre Serre. *Local Fields.* Vol. 67. Graduate Texts in Mathematics. Springer New York, 1979.

[Ser92] Jean-Pierre Serre. *Lie Algebras and Lie Groups.* Vol. 1500. Lecture Notes in Mathematics. Springer Berlin Heidelberg, 1992.

[Sza09] Tamás Szamuely. *Galois Groups and Fundamental Groups.* 1st ed. Vol. 117. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2009.

[Tat76] John Tate. “Relations between K2 and Galois cohomology”. In: *Inventiones mathematicae* 36.1 (1976), pp. 257–274.

[Wie12] Gabor Wiese. *Galois Representations.* 2012. URL: https://math.uni.lu/~wiese/notes/GalRep.pdf.

[Zoc24] Hugo Zock. *A Local Version of Kashiwara’s Conjecture.* 2024. URL: https://www.math.uni-duesseldorf.de/~zock/master_thesis.pdf.

MATHEMATISCHES INSTITUT, HEINRICH-HEINE-UNIVERSITÄT DÜSSELDORF, 40225 DÜSSELDORF, GERMANY
Email address: hugo.zock@uni-duesseldorf.de