Fourier Analysis and Representation Theory: Topological Groups I

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Abstract

We introduce the concept of a topological group, and give plenty of examples to motivate the theory. A few constructions involving topological groups, such as products and quotients, are discussed. We prove the classification of closed subgroups of $\mathbb{R}^n$. The separation axioms T_0 up to T_3 are shown to be equivalent for topological groups. Finally, we prove a few statements regarding subgroups of topological groups.

1 First results and examples

For a given group G, we denote the identity element by e_G, the group operation by

$$\begin{aligned} \mu_G\colon G\times G &\to G \\ (g,h) &\mapsto g\cdot h, \end{aligned}$$

and the inversion map by

$$\begin{aligned} \iota_G\colon G &\to G \\ g &\mapsto g^{-1}. \end{aligned}$$

As already indicated above, the group operation is always denoted multiplicatively. If no confusion can arise, then we omit the subscript G and just write e, μ and ι, respectively.

Recall that for spaces X and Y, the product $X\times Y$ is naturally equipped with a topology, referred to as the *product topology*, which has a basis conisting of opens of the form $U\times V$, with $U\subset X$ and $V\subset Y$ open.

Definition 1.1. *A topological group $(G,\mathcal{T})$ is a group G equipped with a topology $\mathcal{T}$ such that the multiplication map $\mu\colon G\times G\to G$ and the inversion map $\iota\colon G\to G$ are continuous. Here $G\times G$ is equipped with the product topology.*

Remark 1.2. Topological groups are so called "group objects in the category of topological spaces". One might also consider, for example, group objects in the category of (complex) smooth manifolds or algebraic varieties over a field to obtain the notion of a (complex) Lie group or a group variety. ▲

Usually, the topology of a group is clear from the context and so we omit it from the notation and just write G for a topological group.

Proposition 1.3. *If $H \subset G$ is a subgroup of a topological group G, then H with the subspace topology inherited from G is itself a topological group.*

Proof. The product $H \times H \subset G \times G$ is again equipped with the subspace topology coming from $G \times G$. If $U \cap H \subset H$ is an open of H, with $U \subset G$ open, then its inverse image under μ_H is $H \times H \cap \mu^{-1}U$, which is open in $H \times H$. Continuity of the inversion map is shown similarly. ■

Example 1.4. (i) Let G be *any* group. Then we can turn G into a topological group by equipping it with the discrete topology or the chaotic/trivial topology (Lukas also suggested the term "coarse topology"). Indeed, if G has the discrete topology, then so does $G \times G$ and it is immediately clear that μ and ι are continuous. If G has the trivial topology, then any map into G is continuous, and so in particular μ and ι are. We denote G with the discrete topology by G_d and G with the chaotic topology by G_c.

(ii) The group of real numbers $\mathbb{R}$ under addition with the euclidean topology is a topological group. As an illustration, we prove that the addition map $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous. To show this, notice that a convergent sequence in $\mathbb{R} \times \mathbb{R}$ is given by a pair of sequences (x_n) and (y_n) in $\mathbb{R}$, both of which are convergent, say to x respectively y. We know from analysis, that the sequence $(x_n + y_n)$ is then also convergent and that it converges to $x + y$. It follows that

$$\lim_{n\to\infty} \mu(x_n, y_n) = \lim_{n\to\infty} x_n + y_n = x + y = \mu(x, y) = \mu(\lim_{n\to\infty}(x_n, y_n)),$$

and so the map μ is continuous by a basic fact from metric topology. Analogously, the group of nonzero real numbers $\mathbb{R}^\times$ under multiplication is a topological group, and so is the connected component $\mathbb{R}_{>0}$ of 1 of this group.

(iii) Similarly to the last example, we have topological groups $\mathbb{C}$ and $\mathbb{C}^\times$.

(iv) The *circle group*,

$$\mathbb{T} = \{z \in \mathbb{C}^\times : |z| = 1\}$$

is a topological subgroup of $\mathbb{C}^\times$.

(v) Lastly, the matrix groups $\mathrm{GL}_n(\mathbb{R})$ and $\mathrm{GL}_n(\mathbb{C})$ are topological groups. They inherit their topologies from $\mathbb{R}^{n^2}$ and $\mathbb{C}^{n^2}$. To see that multiplication and inversion are indeed continuous for these topologies, notice that the multiplication maps and the inversion maps can be written down as rational functions in the coefficients.

(vi) Generalizing example (ii), if k is any field with a norm $|\cdot|\colon k \to \mathbb{R}$, then k gets a topology from the metric defined by

$$d(x, y) = |x - y|.$$

Analogously to (ii), one proves that both $k^+ = k$ under addition and $k^\times$ under multiplication are topological groups. *Tate's thesis* (see [CF10, Chapter 15]) is a piece of number theory that applies the analysis from this course to such topological groups, where k is a number field (think of $\mathbb{Q}$) or a completion thereof (think of $\mathbb{R}, \mathbb{C}$ or the field of p-adic numbers $\mathbb{Q}_p$).

Definition 1.5. *Let G and G' be topological groups. A morphism of topological groups is a homomorphism $\varphi: G \to G'$ of groups that is continuous with respect to the topologies of G and G'.*

Topological groups, together with morphisms of topological groups, constitute a category that we will denote TopGrp and call *the category of topological groups.*

We remark that if $H \subset G$ is a subgroup of a topological group, then its topology is precisely such that if $G' \to G$ is a morphism of topological groups whose image lies in H, then it factors as $G' \to H \hookrightarrow G$ (universal property of $H \hookrightarrow G$, if you wish).

Example 1.6. (i) If G is any group, then $G_d \to G_c, g \mapsto g$ is continuous. Indeed, the identity $G \to G$ is a homomorphism of groups, and it is continuous, because G_c is chaotic.

(ii) Consider the exponential map

$$\begin{aligned} \varphi\colon \mathbb{R} &\to \mathbb{T} \\ t &\mapsto \exp(2\pi i t). \end{aligned}$$

It is a morphism of topological groups. That φ is a homomorphism follows from the property $\exp(a+b) = \exp(a)\exp(b)$ for complex numbers a and b. Continuity follows from continuity of exp as a map $\mathbb{C} \to \mathbb{C}$ and the fact that its restriction to $\mathbb{R}$ factors through $\mathbb{T}$.

(iii) Similarly to the last example, the exponential map gives us a homomorphism of groups

$$\exp\colon \mathbb{R} \to \mathbb{R}_{>0}.$$

Its inverse (which we also know to be continuous from analysis), is given by the logarithm

$$\log\colon \mathbb{R}_{>0} \to \mathbb{R}.$$

In particular, the topological groups $\mathbb{R}$ and $\mathbb{R}_{>0}$ are isomorphic.

(iv) Let $A \in \mathrm{Mat}(n \times m, \mathbb{R})$ be an $(n \times m)$-matrix with real coefficients. Then the multiplication-by-A map

$$A\colon \mathbb{R}^m \to \mathbb{R}^n.$$

is a morphism of topological group. It is a homomorphism by linearity, and continuity follows by the fact that linear maps are continuous. The same is true when replacing $\mathbb{R}$ with $\mathbb{C}$. If the integers m and n happen to be equal and the matrix A is invertible, then this morphism is actually an isomorphism of topological groups with inverse A^{-1}.

(v) The determinant maps $\det : \mathrm{GL}_n(\mathbb{R}) \to \mathbb{R}^\times$ and $\det : \mathrm{GL}_n(\mathbb{C}) \to \mathbb{C}^\times$ are morphisms of topological groups. Indeed, the fact that they are homomorphisms follows from multiplicativity of the determinant. Continuity follows from the fact that the determinant of $A = (a_{ij})$ is a (rather cumbersome) polynomial function in the coefficients a_{ij}.

Remark 1.7. From topology we know that a continuous bijective map of spaces $X \to Y$ is not necessarily an isomorphism. This is, unfortunately, not true for topological groups either: look at (i) of the example above. It is, however, true that a continuous bijective map $X \to Y$ is a homeomorphism, if it is in addition an *open map* (equivalently, a closed map). ▲

For a topological group G and an element $g \in G$, define the map λ_g by

$$\begin{aligned} \lambda_g \colon G &\to G \\ h &\mapsto gh. \end{aligned} \tag{1.1}$$

Define the map ρ_g by

$$\begin{aligned} \rho_g \colon G &\to G \\ h &\mapsto hg. \end{aligned} \tag{1.2}$$

Proposition 1.8. *Let G be a topological group. Then for any $g \in G$, the maps λ_g and ρ_g defined in* (1.1) *and* (1.2) *are homeomorphisms. Moreover, the inversion map $\iota \colon G \to G$ is a homeomorphism. In particular, if G is abelian, then ι is an automorphism of G as a topological group.*

Proof. We prove that λ_g is a homeomorphism. The proof for ρ_g is analogous. The map λ_g can be written a composition of maps

$$G \xrightarrow{\simeq} \{g\} \times G \hookrightarrow G \times G \xrightarrow{\mu} G,$$

all of which are continuous; hence, λ_g is continuous. Its inverse is the map $\lambda_{g^{-1}}$, which is continuous by the same argument.

The map ι is continuous by assumption. It is its own inverse, and hence ι is a homeomorphism. If G happens to be abelian, then ι is also a homomorphism. ■

Remark 1.9. The above proposition, roughly speaking, tells us that G looks "the same at every point". Indeed, for any two points $g, h \in G$, there is a homeomorphism (the map $\lambda_{hg^{-1}}$) sending g to h. This, for instance, gives us a hunch that the space $S^1 \vee S^1$ obtained by glueing two circles together at a point cannot be given the structure of a topological group: the base point (where they are glued together) really "looks different" from the other points. One can actually prove rigorously that $S^1 \vee S^1$ can never be equipped with the structure of a topological group by computing its fundamental group! It turns out that any topological group has an *abelian* fundamental group [1]. The fundamental group of $S^1 \vee S^1$ is $\mathbb{Z} * \mathbb{Z}$ by the Van Kampen Theorem, which is non-abelian.

This theme of object with group structures "looking the same everywhere" also plays a role in other parts of mathematics. For example, it proves that group varieties are smooth, and that Lie groups are orientable. ▲

[1] For those that know a little category theory: you can see this by noticing that $\pi_1 : \mathrm{Top}_* \to \mathrm{Grp}$ preserves products, and hence sends group-objects to group-objects. But the group-objects in the category of groups are precisely the abelian groups.

We end this section with a technical lemma on topological groups that will prove to be useful later on. First recall two key concepts from topology.

Definition 1.10. *A topological space X is called quasi-compact if any open covering of X admits a finite subcover. A space which is quasi-compact and Hausdorff is called compact. If $A \subset X$ is a subset of X, then it is said to be (quasi-)compact if it is (quasi-)compact when equipped with the subspace topology inherited from X.*

⚠ For many authors, the term compact means what we call quasi-compact, and thus does not contain the Hausdorff property. Almost all topological groups in this course will be Hausdorff, so don't wory about this unduly.

We list some facts about (quasi-)compact spaces below:

Proposition 1.11. *(i) Any closed subset of a compact (respectively quasi-compact) space is compact (respectively quasi-compact).*

(ii) Any compact subset of a Hausdorff space is closed.

(iii) The image of a quasi-compact subset under a continuous map is quasi-compact (be ware of the "quasi-"!).

Proof. See [Run05, Proposition 3.3.6] for part (i) and (ii) and [Run05, Proposition 3.3.8] for part (iii). Note: Runde calls quasi-compact spaces compact. ■

We introduce a little more terminology and notation. Let G be a topological group. A *unit-neighborhood* $U \subset G$ is a neighborhood of $e \in G$. For a subset $V \subset G$, we write

$$V^{-1} = \{v^{-1} : v \in V\}.$$

We say that V is *symmetric* if $V^{-1} = V$. For subsets $A, B \subset G$ we write

$$AB = \{ab : a \in A, b \in B\}.$$

We will also write A^2 for AA, and so forth.

Lemma 1.12. *Let G be a topological group, let $U \subset G$ be a unit-neighborhoud, and let $A, B \subset G$ be subsets.*

(i) There exists a symmetric unit-neighborhood V such that $V^2 \subset U$.

(ii) If A or B is open, then so is AB.

(iii) If A and B are quasi-compact, then so is AB.

(iv) If A is quasi-compact and B is closed, then AB and BA are closed.

(v) The topological closure $\overline{A} \subset G$ of A equals

$$\overline{A} = \bigcap_V AV,$$

where the intersection runs over all unit-neighborhoods $V \subset G$.

Proof. Recall that $\mu : G \times G \to G$ denotes the multiplication map and $\iota : G \to G$ the inversion map.

(i) The set $\mu^{-1}U$ is open in $G \times G$ and contains (e,e). By definition of the product topology, we can find an open $V \subset G$ such that $(e,e) \in V \times V \subset \mu^{-1}U$. Then $V^2 = \mu(V \times V) \subset U$. To see that V can be taken symmetric, replace it by $V \cap V^{-1}$.

(ii) Without loss of generality, we can assume that A is open, because $BA = (A^{-1}B^{-1})^{-1}$. Then

$$AB = \bigcup_{b \in B} Ab = \bigcup_{b \in B} \rho_b(A)$$

is a union of open sets, because ρ_b is a homeomorphism by Proposition 1.8. Hence AB is open.

(iii) The product $AB \subset G$ is the image of the quasi-compact subset $A \times B \subset G \times G$ under μ. Then apply Proposition 1.11(iii).

(iv) It suffices to prove that AB is closed by the identity $BA = (A^{-1}B^{-1})^{-1}$, and A^{-1} and B^{-1} are quasi-compact and closed, respectively. We prove that $(AB)^c := G \setminus AB$ is open. Let $x \in (AB)^c$. Then $xB^{-1} \subset G$ is closed by Proposition 1.8. We have $xB^{-1} \cap A = \emptyset$. Indeed if there is an element in the intersection, then there are $a \in A$ and $b \in B$ such that $a = xb^{-1}$, which implies that $x = ab \in AB$. We claim that there is a unit neighborhood $U \subset G$ such that $UxB^{-1} \cap UA = \emptyset$. This would show that

$$U^{-1}Ux \cap AB = \emptyset,$$

proving that x has a neighborhood, $U^{-1}Ux$, in the complement of AB, which is what we are after.

The complement of $x^{-1}B$ is open and contains A. So, by part (i), for every $a \in A$ there exists a unit-neighborhood $V \subset G$ such that $V^2 a \cap x^{-1}B = \emptyset$. By quasi-compactness of A, there exist $a_1, \ldots, a_k \in A$ and unit-neighborhoods $V_1, \ldots, V_k$ such that $V_i^2 a_i \cap x^{-1}B = \emptyset$ for all i and

$$A \subset \bigcup_{i=1}^{k} V_i a_i.$$

Let $V = \bigcap_{i=1}^{k} V_i$, which is still a unit-neighborhood. Then $VA \cap xB^{-1} = \emptyset$. Indeed, if $v \in V$ and $a \in A$, then there exists i with $a \in V_i a_i$. It follows that $va \in V_i^2 a_i$, which is disjoint from xB^{-1}. Finally, let $U \subset G$ be a symmetric unit-neighborhood such that $U^2 \subset V$ (which exists by part (i)). Then we have $U^2 \cap xB^{-1} = \emptyset$, from which it follows that $UA \cap UxB^{-1} = \emptyset$ by symmetry of U.

(v) (after [DE14, Lemma 1.1.3(f), p. 2]) We prove the inclusion from left to right. Let $x \in \overline{A}$ and let $V \subset G$ be a unit-neighborhood. Then xV^{-1} is a neighborhood of x, and so has nonempty intersection with A. Let a be an element in this intersection. Then we find $v \in V$ such that $a = xv^{-1}$, and hence $x = av \in AV$.

Conversely, let $x \in \bigcap AV$ and let W be a neighborhood of x in G. Then $V = x^{-1}W$ is a unit-neighborhood and so is V^{-1}. We find $a \in A$ and $v \in V$ such that $x = av^{-1}$, and hence $a = xv \in xV = W$.

$\blacksquare$

Remark 1.13. The proof of (iv) in the Lemma above is a little awkward. For a more conceptual proof, using *nets*, see [DE14, Lemma 1.1.5].

1.1 Constructing new topological groups from old ones

We start off by constructing (finite) products of topological groups.

Proposition 1.14. *Let G and G' be topological groups. Then $G \times G'$ equipped with the product topology is again a topological group.*

Proof. A priori, we know $G \times G'$ to at least be a group. Its mutliplication map is continuous, because it can be written as the composition

$$(G \times G') \times (G \times G') \xrightarrow{\cong} (G \times G) \times (G' \times G') \xrightarrow{\mu_G \times \mu_{G'}} G \times G'.$$

Both μ_G and $\mu_{G'}$ are continuous, because G and G' are topological groups, and hence so is their product $\mu_G \times \mu_{G'}$. As a composition of continuous maps, $\mu_{G \times G'}$ is continuous.

The inversion map $\iota_{G \times G'} \colon G \times G' \to G \times G'$ is the product of the continuous maps ι_G and $\iota_{G'}$, and hence is itself continuous. $\blacksquare$

Corollary 1.15. *For topological groups $G_1, \dots, G_k$ the product $G_1 \times \ldots \times G_k$ with the product topology is a topological group.*

Proof. Apply the previous proposition inductively. $\blacksquare$

We notice that a product of topological groups $P = G_1 \times \ldots \times G_k$ with the projection maps $\pi_i \colon P \to G_i$ now has the following univeral property. For a collection of morphisms $f_i \colon G \to G_i$, $i = 1, \dots, k$, there exists a *unique* morphism $f \colon G \to P$ such that $\pi_i \circ f = f_i$ for all i:

$$\begin{tikzcd}
 & G \arrow[d, dashed, "f"] \arrow[ddl, bend right, "f_i"'] \\
 & P \arrow[dl, "\pi_i"'] \\
G_i. &
\end{tikzcd}$$

Example 1.16. (i) For a positive integer $n > 0$, euclidean n-space $\mathbb{R}^n$ is a topological group under addition.

(ii) The torus $\mathbb{T} \times \mathbb{T}$ is a topological group.

If G is a topological group G with a subgroup $H \subset G$, then we equip G/H with the quotient topology from $G \to G/H$. Specifically, this means that a subset $U \subset G/H$ is open if and only if its preimage in G is open.

Lemma 1.17. *If G is a topological group and $H \subset G$ is a subgroup, then $G \to G/H$ is an open map.*

Proof. Denote the quotient map $G \to G/H$ by π. Let $U \subset G$ be open, then we have

$$\pi^{-1}\pi(U) = UH$$

is open by Lemma 1.12(ii). By definition of the quotient topology, $\pi(U)$ is open in G/H. ■

⚠ Quotient maps of topological spaces are in general not open. For example, consider the quotient map $[0,1] \to S^1$ identifying the points 0 and 1. Then the image of the open set $[0,1/2) \subset [0,1]$ is not open, because the preimage of this image in $[0,1]$ is $[0,1/2) \cup \{1\}$, which is not open in $[0,1]$.

Lemma 1.18. *Let $G_1, \dots, G_k$ be topological groups with subgroups $H_1 \subset G_1, \dots, H_k \subset G_k$. Then the natural map*

$$\varphi\colon (G_1 \times \ldots \times G_k)/(H_1 \times \ldots \times H_k) \to G_1/H_1 \times \ldots \times G_k/H_k$$

is a homeomorphism. If the subgroups $H_1, \dots, H_k$ happen to be normal, then this is an isomorphism of topological groups.

Proof. It is clear that φ is an isomorphism of groups. We need only prove that it is continuous and an open map. Consider the following commutative diagram

$$\begin{tikzcd} & G_1 \times \ldots \times G_k \arrow[dl, "\pi"'] \arrow[dr, "\pi_1 \times \ldots \times \pi_k"] & \\ (G_1 \times \ldots \times G_k)/(H_1 \times \ldots \times H_k) \arrow[rr, "\varphi"] & & G_1/H_1 \times \ldots \times G_k/H_k. \end{tikzcd}$$

Both diagonal maps are open by Lemma 1.17. For an open $U \subset (G_1 \times \ldots \times G_k)/(H_1 \times \ldots \times H_k)$, we have

$$\varphi(U) = (\pi_1 \times \ldots \times \pi_k)(\pi^{-1}U),$$

using the fact that the projection maps are surjective. This set is open, because π is continuous and $\pi_1 \times \ldots \times \pi_k$ is open. An analogous argument shows that the inverse of φ sends opens to opens and hence proves continuity of φ. ■

Proposition 1.19. *Let G be a topological group and let $N \subset G$ be a normal subgroup. Then G/N, equipped with the quotient topology from $G \to G/N$, is a topological group.*

Proof. We have the following commutative diagram

$$\begin{array}{ccc} G & \xrightarrow{\iota_G} & G \\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} \\ G/N & \xrightarrow{\iota_{G/N}} & G/N. \end{array}$$

Surjectivity of π shows that for $U \subset G/N$ we have

$$\iota_{G/N}^{-1} U = \pi(\iota_G^{-1} \pi^{-1} U).$$

Continuity of π and ι_G, and the fact that π is open by Lemma 1.17, shows that $\iota_{G/N}^{-1} U$ is open, and hence that $\iota_{G/N}$ is continuous.

To show continuity of the multiplication map, consider the commutative diagram

$$\begin{array}{ccc} G \times G & \xrightarrow{\mu_G} & G \\ \downarrow{\scriptstyle\pi\times\pi} & & \downarrow{\scriptstyle\pi} \\ G/N \times G/N & \xrightarrow{\mu_{G/N}} & G/N, \end{array}$$

and apply an argument analogous to the one proving continuity of $\iota_{G/N}$. ■

It is now not hard to see that for a topological group G and a normal subgroup $N \subset G$, the morphism $G \to G/N$ has the following universal property: for any morphism $\varphi\colon G \to G'$ vanishing on N (meaning $N \subset \ker\varphi$), there is a *unique* arrow $\overline{\varphi}\colon G/N \to G'$ such that

$$\begin{tikzcd} G \arrow[rr, "\varphi"] \arrow[dr] & & G' \\ & G/N \arrow[ur, dashed, "\overline{\varphi}"] & \end{tikzcd}$$

commutes.

⚠ The first isomorphism theorem is false for general topological groups. That is, if $\varphi : G \to G'$ is a morphism of topological groups, then it is not necessarily true that $G/\ker\varphi \simeq \varphi(G)$. Example 1.6(i) already shows this. It is, however, true if for example G is quasi-compact and G' is Hausdorff.

Example 1.20. Consider the topological group $\mathbb{R}$ and the (normal) subgroup of integers $\mathbb{Z}$. The map

$$\exp\colon \mathbb{R} \to \mathbb{T}$$

from Example 1.6 is surjective and has kernel $\mathbb{Z}$. It therefore induces a continuous bijection

$$\overline{\exp}\colon \mathbb{R}/\mathbb{Z} \to \mathbb{T}.$$

The map $\overline{\exp}$ can be proved to be open by proving that exp is open. Hence, we have an isomorphism of topological groups $\mathbb{R}/\mathbb{Z} \simeq \mathbb{T}$. ▲

Remark 1.21. For another nice example of quotients in the context of Galois theory, you can have a look at Example A.2. ▲

We state one more Lemma regarding quotients of topological groups, that should already be familiar to you from group theory, that will be useful in the next section.

Lemma 1.22. *(quotient trick) Let G be a topological group and let $N, N' \subset G$ be normal subgroups such that $N' \subset N$. Then the natural morphism*

$$\frac{G/N'}{N/N'} \xrightarrow{\simeq} G/N$$

is an isomorphism of topological groups.

Proof. We know from algebra that the natural map $(G/N')/(N/N') \to G/N$ is an isomorphism of groups. Consider the commutative diagram

$$\begin{tikzcd} G \arrow[d] \arrow[ddr, bend left] & \\ G/N' \arrow[d] & \\ (G/N')/(N/N') \arrow[r] & G/N. \end{tikzcd}$$

The vertical maps and the diagonal map are of course the usual quotient maps. They are open by Lemma 1.17. Now an argument analogous to that found in the proofs of Lemma 1.18 and Proposition 1.19 shows that the map $(G/N')/(N/N') \to G/N$ is open and continuous; hence, it is an isomorphism of topological groups. ■

1.2 Closed subgroups of $\mathbb{R}^n$

At first sight, studying a group with a topology, instead of just its algebraic structure, appears to make things more complicated; but, in fact, quite the opposite is true. To illustrate this, we will show that the *closed* subgroups of $\mathbb{R}^n$ are much easier to understand than general subgroups of $\mathbb{R}^n$.

First consider the case $n = 1$. We examine a few (not necessarily closed) subgroups of $\mathbb{R}$.

- We have the group of rational numbers $\mathbb{Q} \subset \mathbb{R}$. It is dense in $\mathbb{R}$.
- Consider the subgroup $\mathbb{Z}[\sqrt{2}] = \{a + b\sqrt{2} : a, b \in \mathbb{Z}\} \subset \mathbb{R}$. It is also dense in $\mathbb{R}$ (exercise!).
- Another example of a dense subgroup is $\mathbb{Z}[\pi]$ (which is isomorphic to the polynomial ring over $\mathbb{Z}$ in one variable.

Restricting our attention to the closed subgroups of $\mathbb{R}$, we see that things simplify.

Proposition 1.23. *If $H \subset \mathbb{R}$ is a proper non-trivial closed subgroup of $\mathbb{R}$, then there exists $\alpha \in \mathbb{R} \setminus \{0\}$ such that $H = \mathbb{Z}\alpha$.*

Proof. We first prove that there is an element in H that is minimal among all positive non-zero elements. Suppose this is not the case. We prove that $H = \mathbb{R}$, contradicting properness of H, by proving that H lies dense in $\mathbb{R}$. Let $x \in \mathbb{R}$ and let $\varepsilon > 0$. Let $h \in H$ be a positive element such that $h < \varepsilon$. Let $n \in \mathbb{Z}$ be an integer such that $|n - x/h| < 1$. Then $nh \in H$ and we have

$$|nh - x| = h \cdot |n - x/h| < h < \varepsilon.$$

We conclude that H lies dense in $\mathbb{R}$.

Let α be this minimal element. We prove that $H = \mathbb{Z}\alpha$. Let $\beta \in H$ and suppose $\beta \notin \mathbb{Z}\alpha$. Then $\beta/\alpha \notin \mathbb{Z}$ and hence there exists $n \in \mathbb{Z}$ such that $0 < \beta/\alpha - n < 1$. But then

$$0 < \alpha(\beta/\alpha - n) = \beta - n\alpha < \alpha$$

is a positive element of H that is smaller than α, which contradicts minimality of α. ■

Remark 1.24. Alternatively, we could have proved that $H \subset \mathbb{R}$ as in the above proposition is always discrete in $\mathbb{R}$ and applied Lemma 1.26 below. ▲

For the proof of the main theorem of this section, we will need to discuss briefly how to "topologize" an aribtrary finite dimensional $\mathbb{R}$-vector space. Let V be an n-dimensional $\mathbb{R}$-vector space. Picking a basis $v = (v_1, \dots, v_n)$ of V gives us a linear isomorphism $\varphi_v \colon \mathbb{R}^n \xrightarrow{\simeq} V$, turning V into a topological group. If $w = (w_1, \dots, w_n)$ is any other choice of basis of V, then there exists $A \in \mathrm{GL}_n(\mathbb{R})$ such that

$$\begin{tikzcd}
\mathbb{R}^n \arrow[rr, "A"] \arrow[dr, "\varphi_v"'] & & \mathbb{R}^n \arrow[dl, "\varphi_w"] \\
& V &
\end{tikzcd}$$

commutes. Since A is a homeomorphism by Example 1.6 (iv), we see that the topology of V is the same, regardless of whether it comes from v or w.

Definition 1.25. *A lattice $\Gamma \subset V$ of an n-dimensional $\mathbb{R}$-vector space V is a subgroup of the form*

$$\Gamma = \mathbb{Z}v_1 + \ldots + \mathbb{Z}v_r,$$

where $r \leq n$ is an integer and $v_1, \dots, v_r$ are $\mathbb{R}$-linearly independent vectors. The integer r is called the rank of Γ.

The following lemma is the first key observation in the classification of closed subgroups of $\mathbb{R}^n$.

Lemma 1.26. *Let $H \subset \mathbb{R}^n$ be a non-trivial discrete subgroup. Then H is a lattice.*

Proof. (after [NS99, Chapter 1, Proposition 4.2]) Notice that H is closed by Proposition 3.2(vi).

Let $h_1, \dots, h_m$ be a set of $\mathbb{R}$-linearly independent elements of H, such that they generate $H \otimes_{\mathbb{Z}} \mathbb{R}$ as an $\mathbb{R}$-subspace of $\mathbb{R}^n$. Denote by H_0 the subgroup of H generated by $h_1, \dots, h_m$. Let

$$\Phi = \{a_1 h_1 + \ldots + a_m h_m : 0 \leq a_i < 1\}.$$

Every element of H/H_0 is represented by an element of Φ. Since Φ is a bounded set, it contains at most finitely many elements of H. To see this, notice that the closure of Φ is compact, because it is closed and bounded, and that $H \cap \overline{\Phi}$ is discrete and compact, because H is closed. Now notice that compact discrecte spaces are finite. So the index $q = (H : H_0)$ is finite. We now have

$$H \subset \frac{1}{q} H_0 = \mathbb{Z}\frac{1}{q} h_1 + \ldots + \mathbb{Z}\frac{1}{q} h_m,$$

and so H is $\mathbb{Z}$-freely generated by $r \le m$ elements $v_1, \ldots, v_r$. Since $r \le m$ and $H \otimes_{\mathbb{Z}} \mathbb{R} = H_0 \otimes_{\mathbb{Z}} \mathbb{R}$, we see that the vectors $v_1, \ldots, v_r$ are also $\mathbb{R}$-linearly independent, because they span an m-dimensional subspace of $\mathbb{R}^n$. Hence, H is a lattice. ■

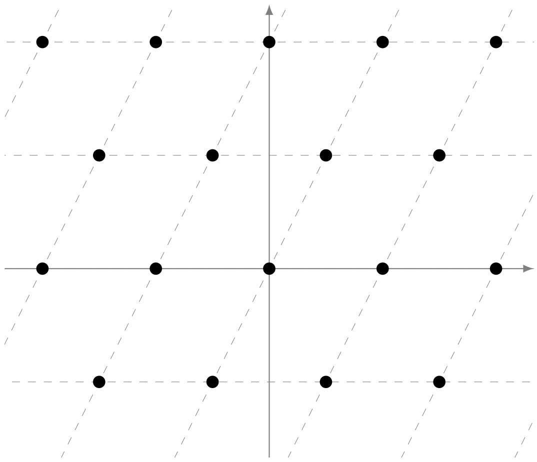

Figure 1: The lattice $\mathbb{Z}(1,2) + \mathbb{Z}(2,0)$ in $\mathbb{R}^2$.

Lemma 1.27. *Let V be an n-dimensional vector space and let $\Gamma \subset V$ be a rank r lattice of V. Then the quotient V/Γ is isomorphic to $\mathbb{T}^r \times \mathbb{R}^{n-r}$.*

Proof. We can assume $V = \mathbb{R}^n$. Let $W = \Gamma \otimes_{\mathbb{Z}} \mathbb{R} \subset \mathbb{R}^n$ be the r-dimensional subspace spanned by Γ. Let $W^\perp$ be its orthogonal complement so that $\mathbb{R}^n = W \oplus W^\perp$. By Lemma 1.18 we find

$$\mathbb{R}^n/\Gamma \simeq W/\Gamma \times W^\perp \simeq W/\Gamma \times \mathbb{R}^{n-r}.$$

Writing $\Gamma = \mathbb{Z}v_1 + \ldots + \mathbb{Z}v_r$, the same Lemma and Example 1.20(i) also gives us an isomorphism

$$W/\Gamma \simeq (\mathbb{R}v_1 \oplus \ldots \oplus \mathbb{R}v_r)/(\mathbb{Z}v_1 \oplus \ldots \oplus \mathbb{Z}v_r) \simeq \mathbb{T}^r.$$

■

Theorem 1.28. *Let $H \subset \mathbb{R}^n$ be a non-trivial proper closed subgroup. Then there exists a d-dimensional subspace $W \subset \mathbb{R}^n$ and a rank r lattice $\Gamma \subset \mathbb{R}^n/W$ such that $H = \pi^{-1}\Gamma$, where $\pi : \mathbb{R}^n \to \mathbb{R}^n/W$ is the usual quotient map. The group H is isomorphic to $\mathbb{Z}^r \times \mathbb{R}^d$ as a topological group. Furthermore, the quotient $\mathbb{R}^n/H$ is isomorphic to $\mathbb{T}^r \times \mathbb{R}^{n-d-r}$.*

Proof. We proceed by induction on the dimension n. The case $n = 1$ is treated in Proposition 1.23. If $H \subset \mathbb{R}^n$ is discrete, then we are done by Lemma 1.26 above.

If H is not discrete, then H contains a line L by Lemma 1.29 below. We now argue as follows. Let $\pi_L \colon \mathbb{R}^n \to \mathbb{R}^n/L$ denote the quotient map. Then $H/L \subset \mathbb{R}^n/L$ is again a closed subgroup by the definition of the quotient topology, because $\pi_L^{-1}(H/L) = H + L = H \subset \mathbb{R}^n$ is closed. By the induction hypothesis, there exists a subspace $W_0 \subset \mathbb{R}^n/L$ and a lattice $\Gamma \subset (\mathbb{R}^n/L)/W_0$ such that $H/L = \pi_{W_0}^{-1}\Gamma$, where $\pi_{W_0} \colon \mathbb{R}^n/L \to (\mathbb{R}^n/L)/W_0$ is the natural quotient map. Now W_0 is of the form W/L for some linear subspace $W \subset \mathbb{R}^n$ and $(\mathbb{R}^n/L)/W_0 \simeq \mathbb{R}^n/W$ by Lemma 1.22. If $\pi_W \colon \mathbb{R}^n \to \mathbb{R}^n/W$ denotes the quotient map, then we have $\pi_W = \pi_{W_0} \circ \pi_L$ under this identification and we find

$$H = \pi_L^{-1}(H/L) = \pi_L^{-1}(\pi_{W_0}^{-1}\Gamma) = \pi_W^{-1}\Gamma.$$

This argument is perhaps best visualised in the diagram below; but if you disagree, feel free to ignore it.

$$\begin{tikzcd}
H \arrow[r, hook] \arrow[d] & \mathbb{R}^n \arrow[d, "\pi_L"] \arrow[ddr, bend left, "\pi_W"] & \\
H/L \arrow[r, hook] \arrow[d] & \mathbb{R}^n/L \arrow[d, "\pi_{W_0}"] & \\
\Gamma \arrow[r, hook] & (\mathbb{R}^n/L)/W_0 \arrow[r, "\simeq"] & \mathbb{R}^n/W.
\end{tikzcd}$$

Here the squares are pullbacks.

The projection map $\mathbb{R}^n \to \mathbb{R}^n/W$ admits a section, given by sending a vector v to its orthogonal component with respect to W. This gives us a canonical (topological) splitting to the short exact sequence

$$0 \to W \to H \xrightarrow{\pi} \Gamma \to 0,$$

proving that

$$H \simeq W \times \Gamma \simeq \mathbb{R}^d \times \mathbb{Z}^r.$$

We prove the last part of the theorem regarding the quotient $\mathbb{R}^n/H$. We have

$$\mathbb{R}^n/H = \mathbb{R}^n/\pi^{-1}\Gamma \simeq \frac{\mathbb{R}^n/W}{\pi^{-1}\Gamma/W} = \frac{\mathbb{R}^n/W}{\Gamma},$$

by Lemma 1.22. Now Lemma 1.27 gives us an isomorphism

$$\mathbb{R}^n/H \simeq \frac{\mathbb{R}^n/W}{\Gamma} \simeq \mathbb{T}^r \times \mathbb{R}^{n-d-r}.$$

■

Lemma 1.29. *Let $H \subset \mathbb{R}^n$ be a closed subgroup of $\mathbb{R}^n$. If H is not discrete, then H contains a line, i.e., a 1-dimensional linear subspace.*

Proof. (after [Gar]) There exists a sequence (h_i) in H which converges to a point $h \in \mathbb{R}^n$ with none of the h_i equal to h, since H is not discrete. Since H is closed we have $h \in H$ and we can assume $h = 0$ by translating the sequence (h_i). Now consider the sequence $(h_i/|h_i|)$ on the unit-sphere. Since the sphere is compact, the sequence $(h_i/|h_i|)$ admits a convergent subsequence, converging to say $u \in S^n$. Replace the sequence $(h_i/|h_i|)$ by this convergent subsequence. We prove that H contains the line $\mathbb{R}u$. Let $t \neq 0$ be a real number, and let n_i be an integer such that $|n_i - t/|h_i|| \le 1$. Then we find

$$|n_i \cdot h_i - tu| \le \left|\left(n_i - \frac{t}{|h_i|}\right)h_i\right| + \left|\frac{th_i}{|h_i|} - tu\right| \le 1 \cdot |h_i| + |t| \cdot \left|\frac{h_i}{|h_i|} - u\right| \to 0 \quad \text{as } i \to \infty.$$

We conclude that tu is contained in the closure of $\bigcup_i \mathbb{Z}h_i$, which is contained in H. ■

Remark 1.30. In Example A.3 we briefly discuss the classification of closed subgroups of the profinite integers $\hat{\mathbb{Z}}$. The need for closed subgroups is also more evident in that example.

2 The separation axioms for topological groups

We recall the so called "seperation axioms" from topology ("Trennungsaxiome" auf Deutsch).

Definition 2.1. *Let X be a topological space. Then X is called regular if for any closed subset $A \subset X$ and any point $x \in A^c := X \setminus A$, there exist neighborhoods $A \subset U \subset X$ and $x \in V \subset X$ with $U \cap V = \emptyset$.*

Definition 2.2 (separation axioms)**.** *Let X be a topological space. Then X is called*

(i) T_0 if for any two distinct point $x, y \in X$, there exists either a neighborhood $x \in U \subset X$ with $y \notin U$, or a neighborhood $y \in V \subset X$ with $x \notin V$;

(ii) T_1 if for any two disctinct points $x, y \in X$, there exist neighborhoods $x \in U \subset X$ and $y \in V \subset X$ with $x \notin V$ and $y \notin U$;

(iii) T_2 (Hausdorff) if for any two distinct points $x, y \in X$, there exist neighborhoods $x \in U \subset X$ and $y \in V \subset X$ with $U \cap V = \emptyset$.

(iv) T_3 if it is T_0 and regular.

We have the following implications between these axioms:

Proposition 2.3. *Let X be a topological space. Then*

$$X \text{ is } T_3 \quad \Rightarrow \quad X \text{ is } T_2 \quad \Rightarrow \quad X \text{ is } T_1 \quad \Rightarrow \quad X \text{ is } T_0.$$

Proof. Only the first implication is non-trivial. Assume X to be T_0 and regular. Let $x, y \in X$ be distinct points of X. We can assume, without loss of generality, that there is an open neighborhood $U \subset X$ of x such that $y \notin U$, by the fact that X is T_0. Then x is not in the closure of $\{y\}$, and so we can separate x and $\overline{\{y\}}$ by disjoint opens using regularity. ■

We also have the following useful alternative formulation for the T_1 axiom:

Proposition 2.4. *Let X be a space. Then X is T_1 if and only if for all $x \in X$, $\{x\} \subset X$ is closed.*

Proof. See [Run05, Proposition 3.5.4]. ■

Example 2.5. We show that the implications in Proposition 2.3 really are strict in general.

(i) $T_1 \not\Leftarrow T_0$: the space $\mathbb{A}^1_{\mathbb{C}} = \operatorname{Spec}\mathbb{C}[x]$ is T_0 (in fact, the spectrum of any commutative ring is), but it is not T_1, because the zero ideal is contained in any open of $\mathbb{A}^1_{\mathbb{C}}$.

(ii) $T_2 \not\Leftarrow T_1$: let $(\mathbb{A}^1_{\mathbb{C}})^{(1)}$ be the set of closed points of $\mathbb{A}^1_{\mathbb{C}}$. It is the set $\mathbb{C}$ with topology consisting of the subsets $U \subset \mathbb{C}$ with finite complement. It is easily seen to be T_1, but it is not T_2, because any two open subsets have nonempty intersection.

(iii) $T_3 \not\Leftarrow T_2$: Let K be the set $\{1/n : n \in \mathbb{Z}_{>0}\} \subset \mathbb{R}$. The *$K$-topology* $\mathscr{T}$ on $\mathbb{R}$ is the topology generated by the sets of the form $(a,b) \setminus K \subset \mathbb{R}$, where (a,b) is an open interval. The space $(\mathbb{R}, \mathscr{T})$ is T_2, but not regular: the point 0 and the closed subset K cannot be separated by disjoint opens.

If you dislike these examples, you can find your own in [SS78]. ▲

It is remarkable that for topological groups all the separation axioms are equivalent.

Lemma 2.6. *Let G be a topological group. Then G is regular.*

Proof. Let $A \subset G$ be a closed subset of G and let $x \in U := A^c$ be a point. Let $\varphi : G \times G \to G$ be the map $(g,h) \mapsto gh^{-1}$. It is continuous, since it can be written as a composition of continuous maps $\mu \circ (\mathrm{id}_G \times \iota)$. We have $(x,e) \in \varphi^{-1}U$, and so by definition of the product topology, there exist an open neighborhood $V \subset G$ of x and an open neighborhood $W_0 \subset G$ of e such that $(x,e) \in V \times W_0 \subset \varphi^{-1}U$. Set $W = AW_0$. It is open by Lemma 1.12(ii) and contains A by the fact that $e \in W_0$. To complete the proof, we show that

$$V \cap W = \emptyset.$$

Suppose there is an element in this intersection. Then we can find $a \in A$, $v \in V$ and $w_0 \in W_0$ such that $v = aw_0$. But this would imply that

$$a = vw_0^{-1} = \varphi(v, w_0) \in U = A^c,$$

which is nonsense. ■

Proposition 2.7. *For a topological group G, all separation axioms are equivalent.*

Proof. In light of the above lemma, and Proposition 2.3, we only need to prove that $T_0 \Rightarrow T_2$. Let $x, y \in G$ be two different points of G. We can assume that there exists a unit-neighborhood $U \subset G$ not containing xy^{-1}. If this is not the case, then by the fact that we've assumed G to be T_0, we can find a neighborhood U' of xy^{-1} not containing e. Then $U := y^{-1}xU'$ is a unit-neighborhood not containing $y^{-1}x$, and so we just exchange x and y.

Now, by Lemma 1.12 (i), there exists a symmetric unit-neighborhood $V \subset G$ such that $V^2 \subset U$. We show that $Vx \cap Vy = \emptyset$, completing the proof. Indeed, if $v_1, v_2 \in V$ were such that $v_1 x = v_2 y$, then we would find $xy^{-1} = v_1^{-1} v_2 \in V^2 \subset U$, by symmetry of V. ■

3 Subgroups of topological groups

In this last section we consider some "topological algebraic" properties of subgroups of topological groups. First, we introduce one more definition.

Definition 3.1. *A topological space X is called locally compact if X is Hausdorff and every point of X admits a compact neighborhood.*

⚠ This initially looks weaker than another common definition of locally compact: for every $x \in X$ and every open $x \in U \subset X$, there exists a compact neighborhood $K \subset X$ of x such that $K \subset U$. However, by the assumption that X is Hausdorff, the two are equivalent.

Proposition 3.2. *Let G be a topological group.*

(i) If $H \subset G$ is a subgroup (respectively a normal subgroup), then so is its closure $\overline{H} \subset G$.

(ii) If G is Hausdorff and $H \subset G$ is an abelian subgroup, then so is $\overline{H} \subset G$.

(iii) Let $H = \overline{\{e\}}$. Then $H \subset G$ is a normal subgroup of G. Moreover, any closed subgroup of G contains H.

(iv) Any open subgroup of G is also closed.

(v) If G is locally compact, then so is any closed subgroup of G.

(vi) If G is Hausdorff and $H \subset G$ is a subgroup which is locally compact (in the subspace topology inherited from G), then H is also closed. In particular, any discrete subgroup of G is closed.

(vii) Let G' be a Hausdorff topological group and $\varphi : G \to G'$ be a morphism of topological groups. Then $\ker \varphi$ is a closed subgroup.

Proof. Again, the multiplication map for G is denoted $\mu : G \times G \to G$ and the inversion map $\iota : G \to G$.

(i) Let $\varphi : G \times G \to G$ denote the map $(g, h) \mapsto gh^{-1}$. It is continuous as it is a composition of continuous maps: $\varphi = \mu \circ (\mathrm{id} \times \iota)$. To show that $\overline{H}$ is again a subgroup of G, we only need to show that $\varphi(\overline{H} \times \overline{H}) = \overline{H}$. Since $\varphi^{-1}\overline{H}$ is closed, we have

$$\overline{H} \times \overline{H} = \overline{H \times H} \subset \varphi^{-1}\overline{H},$$

which proves this. The equality $\overline{H} \times \overline{H} = \overline{H \times H}$ is a little exercise in topology.

Now suppose that H is normal. For $\sigma \in G$, let $c_\sigma : G \to G$ denote the conjugation-by-σ-map $g \mapsto \sigma g \sigma^{-1}$. It is a homeomorphism by Proposition 1.8 and the fact that $c_\sigma = \lambda_\sigma \circ \rho_{\sigma^{-1}}$. We find,

$$\sigma \overline{H} \sigma^{-1} = c_\sigma(\overline{H}) = \overline{c_\sigma(H)} = \overline{H},$$

using the assumption that H is normal. So $\overline{H}$ is normal.

(ii) Consider the commutator-bracket-map $\varphi : G \times G \to G$ given by $(g,h) \mapsto [g,h] = ghg^{-1}h^{-1}$. It is a continuous map, since we can express it as a composition of continuous maps

$$\begin{aligned} &G \times G \to (G \times G) \times (G \times G) \xrightarrow{\mu \times \mu} G \times G \xrightarrow{\mu} G, \\ &(g,h) \mapsto ((g,h),(g^{-1},h^{-1})) \end{aligned}$$

where the first map comes from $\mathrm{id}\colon G \times G \to G \times G$ and $\iota \times \iota\colon G \times G \to G \times G$. Since H is abelian, we have $H \times H \subset \varphi^{-1}\{e\}$. The fiber φ^{-1} is closed by the fact that $\{e\} \subset G$ is closed (G is Hausdorff). Hence, $\overline{H} \subset \varphi^{-1}\{e\}$, and we conclude that $\overline{H}$ is also abelian.

(iii) The first part is clear by part (i). For the second part, if $H \subset G$ is a closed subgroup of G, then $\{e\} \subset H$ and hence $\overline{\{e\}} \subset H$.

(iv) Let $U \subset G$ be an open subgroup of G. Its complement is given by

$$U^c = \bigcup_{\sigma \in G \setminus U} \sigma U = \bigcup_{\sigma \in G \setminus U} \lambda_\sigma(U),$$

which is open by Proposition 1.8. Hence, U is closed.

(v) Let $H \subset G$ be a closed subgroup of G. Let $x \in H$ be a point in H. Then by the fact that G is locally compact, x admits a compact neighborhood $K \subset G$. Then $K \cap H$ is a compact neighborhood of x in H by the fact that H is closed.

(vi) (after [Mor77, Proposition 7, p.9]) Let K be a compact neighborhood of e in H. Then there exists a neighborhood $e \in U \subset G$ such that $U \cap H = K$. Let V be an neighborhood of e in G such that $V^2 \subset U$ (which exists by Lemma 1.12(i)). If $x \in \overline{H}$, then as $\overline{H}$ is a group by part (i), we have $x^{-1} \in \overline{H}$. So there is an element $y \in Vx^{-1} \cap H$. We will prove that $yx \in H$. This will prove that $H = \overline{H}$ is closed, since $x = y^{-1}(yx)$ will then be in H.

We will prove that yx is in the closure of $U \cap H$. Since $U \cap H = K$ is closed as a compact subspace of a Hausdorff space, this proves $yx \in U \cap H \subset H$, which is what we want. Let W be an arbitrary neighborhood of yx in G. Then $y^{-1}W$ is a neighborhood of x, and so $y^{-1}W \cap xV$ is a neighborhood of x. As $x \in \overline{H}$, there is an element $h \in (y^{-1}W \cap xV) \cap H$. So, $yh \in W$. We also have $yh \in (Vx^{-1})(xV) = V^2 \subset U$, and $yh \in H$. It follows that $yh \in W \cap (U \cap H)$, completing the proof.

(vii) This is clear by the fact that $\{e\} \subset G'$ is closed, since G' is Hausdorff.

■

A Some Galois theory

In this appendix we describe a few examples from Galois theory that were ommited from the talk.

Example A.1 (Galois correspondence)**.** Let L/k be a Galois extension (particularly, we're interested in the case where it is infinite). We have an isomorphism of topological groups

$$\mathrm{Gal}(L/k) \simeq \varprojlim \mathrm{Gal}(\ell/k),$$

where the limit ranges over the finite Galois extensions $k \subset \ell$ contained in L. The topology of the limit is the subspace topology inherited from $\prod \mathrm{Gal}(\ell/k)$, where all the groups $\mathrm{Gal}(\ell/k)$ are discrete. We immediately see that $\mathrm{Gal}(L/k)$ is compact by Tychonoff's theorem. Notice that this group is not discrete when L/k is infinite, because a compact discrete group is finite! Galois theory tells us that the intermediate fields of $k \subset L$ are in correspondence with the *closed* (= compact) subgroups of $\mathrm{Gal}(L/k)$. You can consult [Len] for some facts about Galois groups and their topology.

Example A.2 (quotients of Galois groups)**.** If L/k is a Galois extension and $N \subset \mathrm{Gal}(L/k) = G$ is a normal closed subgroup. Then by the correspondence from Example A.1, there is an intermediate field $k \subset \ell \subset L$ such that $N = \mathrm{Gal}(L/\ell)$. We then have a natural restriction map

$$G \to \mathrm{Gal}(\ell/k)$$

whose kernel is precisely N. This induces an isomorphism (because Galois groups are compact) $G/N \simeq \mathrm{Gal}(\ell/k)$.

Example A.3 (the group of profinite integers)**.** Consider the infinite Galois extensions $\overline{\mathbb{F}_p}/\mathbb{F}_p$, where $\mathbb{F}_p$ is the finite field with p elements and $\overline{\mathbb{F}_p}$ is an algebraic closure thereof. The Galois group of this extension is the group of *profinite integers*

$$\hat{\mathbb{Z}} = \varprojlim_{n>0} \mathbb{Z}/n\mathbb{Z}.$$

This group contains many subgroups that are not closed (there are uncountably many!). For instance, the group $\mathbb{Z}$ lies dense inside $\hat{\mathbb{Z}}$, but the two are not equal. There are, however, much fewer closed subgroups of $\mathbb{Z}$. They are in bijective correspondence with the Steinitz numbers: formal products of the form $\prod_p p^{e_p}$, where p ranges over the primes and $e_p \in \mathbb{N} \cup \{\infty\}$ for all p. See [htt] (although the proof there relies on Pontryagin duality, which we will encounter later!). From the perspective of Galois Theory (Example A.1) it is also precisely the closed subgroups of $\hat{\mathbb{Z}}$ that are of interest. ▲

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